Generating functions

Def.4.1: Given a collection of real numbers $(a_k)_{k\geq 0}$, the function

$$G(s) = G_a(s) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} a_k\, s^k$$

is called the *generating function* of $(a_k)_{k\geq 0}$.

If $G_a(s)$ is analytic near the origin, then

$$a_k = \frac{1}{k!}\, \frac{d^k}{ds^k}\, G_a(s)\,\Big|_{s=0} \ .$$

This result is often referred to as the **uniqueness** property of generating functions.

Def.4.2: If X is a discrete random variable with values in $\mathbb{Z}^+ \stackrel{\text{def}}{=} \{0, 1, \dots\}$, its (*probability*) *generating function*,

$$G(s) \equiv G_X(s) \stackrel{\text{def}}{=} \mathsf{E}\big(s^X\big) = \sum_{k=0}^{\infty} s^k \mathsf{P}(X = k)\,, \tag{1.1}$$

is just the generating function of the pmf $\{p_k\} \equiv \{\mathsf{P}(X = k)\}$ of X.

Recall that the moment generating function $M_X(t) \stackrel{\text{def}}{=} \mathsf{E}(e^{tX})$ of a random variable X is just $\sum_{k \geq 0} \frac{\mathsf{E}(X^k)}{k!} t^k$.

Why do we introduce both $G_X(s)$ and $M_X(t)$?

Theorem 4.3: *If X and Y are independent random variables with values in $\{0,1,2,\dots\}$ and $Z \stackrel{\text{def}}{=} X+Y$, then their generating functions satisfy*

$$G_Z(s) = G_{X+Y}(s) = G_X(s)\, G_Y(s)\,.$$

Recall: if X and Y are *discrete random variables*, and f, $g : \mathbb{Z}^+ \to \mathbb{R}$ are *arbitrary* functions, then $f(X)$ and $g(Y)$ are *independent random variables* and $\mathsf{E}\big[f(X)g(Y)\big] = \mathsf{E}f(X)\cdot\mathsf{E}g(Y)$.

Example 4.4: *If X_1, X_2, ..., X_n are i.i.d.r.v. with values in* $\{0,1,2,\dots\}$ *and if $S_n = X_1 + \dots + X_n$, then*

$$G_{S_n}(s) = G_{X_1}(s) \dots G_{X_n}(s) \equiv \big[G_X(s)\big]^n .$$

Example 4.5: *Let X_1, X_2, ..., X_n be i.i.d.r.v. with values in* $\{0,1,2,\dots\}$ *and let $N \geq 0$ be an integer-valued random variable independent of $\{X_k\}_{k\geq 1}$. Then $S_N \stackrel{\text{def}}{=} X_1 + \dots + X_N$ has generating function*

$$G_{S_N}(s) = G_N\big(G_X(s)\big) .$$

Example 4.6: [Renewals] *Imagine a diligent janitor who replaces a light bulb the same day as it burns out. Suppose the first bulb is put in on day* 0 *and let* X_i *be the lifetime of the* i*th light bulb. Let the individual lifetimes* X_i *be i.i.d.r.v.'s with values in* $\{1, 2, \dots\}$ *and have a common distribution with generating function* $G_f(s)$*. Define* $r_n \stackrel{\text{def}}{=} \mathsf{P}\big(\text{a light bulb was replaced on day } n\big)$ *and* $f_k \stackrel{\text{def}}{=} \mathsf{P}\big(\text{the first light bulb was replaced on day } k\big)$*. Then* $r_0 = 1$*,* $f_0 = 0$*, and for* $n \geq 1$*,*

$$r_n = f_1 r_{n-1} + f_2 r_{n-2} + \cdots + f_n r_0 = \sum_{k=1}^{n} f_k r_{n-k} \,.$$

A standard computation implies that $G_r(s) = 1 + G_f(s)\, G_r(s)$ *for all* $|s| \leq 1$*, so that* $G_r(s) = 1/(1 - G_f(s))$*.*

In general, we say a sequence $(c_n)_{n\geq 0}$ is the convolution of $(a_k)_{k\geq 0}$ and $(b_m)_{m\geq 0}$ (write $c = a \star b$), if

$$c_n = \sum_{k=0}^{n} a_k \, b_{n-k} \,, \qquad n \geq 0 \,.$$

Theorem 4.7: [Convolution thm] *If $c = a \star b$, then the generating functions $G_c(s)$, $G_a(s)$, and $G_b(s)$ satisfy*

$$G_c(s) = G_a(s) \, G_b(s) \,.$$

Example 4.8: *Let $X \sim \mathrm{Poi}(\lambda)$ and $Y \sim \mathrm{Poi}(\mu)$ be independent. Then $Z = X + Y$ is $\mathrm{Poi}(\lambda + \mu)$.*

Solution. A straightforward computation gives $G_X(s) = e^{\lambda(s-1)}$, and Theorem 4.3 implies $G_Z(s) = G_X(s) \, G_Y(s) \equiv e^{(\lambda+\mu)(s-1)}$, so that the result follows by uniqueness.

Notice: if X_1, X_2, ..., X_n are independent Bernoulli rv's with parameter $p \in [0,1]$, each X_k has the generating function $G(s) = qs^0 + ps^1 = q + ps$. If $S_n = \sum_{k=1}^n X_k$, then

$$G_{S_n}(s) = G_{X_1}(s)\, G_{X_2}(s) \dots G_{X_n}(s) = \big(G(s)\big)^n = (q+ps)^n\,.$$

By the Binomial theorem, this reads

$$G_{S_n}(s) = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}\, s^k\,,$$

ie., $S_n \sim \mathrm{Bin}(n,p)$.

Example 4.9: *If $X \sim \mathrm{Bin}(n,p)$ and $Y \sim \mathrm{Bin}(m,p)$ are independent, then $X+Y \sim \mathrm{Bin}(n+m,p)$.*

Computation of moments

Theorem 4.10: *If a rv X has generating function $G(s)$, then*

a) $\mathsf{E}(X) = G'(1)$,

b) *more generally,* $\mathsf{E}\big[X(X-1)\dots(X-k+1)\big] = G^{(k)}(1)$*;*

here, $G^{(k)}(1)$ *is the shorthand for* $\lim_{s\uparrow 1} G^{(k)}(s)$*, the limiting value of the* k*th left derivative of* $G(s)$ *at* $s = 1$*.*

The quantity $\mathsf{E}\big[X(X-1)\dots(X-k+1)\big]$ is known as the kth **factorial moment** of X.

It is a straightforward **exercise** to show that

$$\mathsf{Var}(X) = G''_X(1) + G'_X(1) - \big(G'_X(1)\big)^2 .$$

Proof:

b) Fix $s \in (0,1)$ and differentiate $G(s)$ k times to get

$$G^{(k)}(s) = \mathsf{E}\big[s^{X-k} X(X-1)\dots(X-k+1)\big]\,.$$

Taking the limit $s \uparrow 1$ and using the Abel theorem, we obtain

$$G^{(k)}(s) \to \mathsf{E}\big[X(X-1)\dots(X-k+1)\big]\,. \qquad \blacksquare$$

Notice also that

$$\lim_{s \nearrow 1} G_X(s) \equiv \lim_{s \nearrow 1} \mathsf{E}[s^X] = \mathsf{P}(X < \infty)\,.$$

This allows us to check whether a variable is finite, if we do not know this *apriori*.

Recurrences

Generating functions are very useful in solving **recurrences**, especially when combined with the following algebraic fact.

Lemma 4.12: *Let $f(x) = g(x)/h(x)$ be a ratio of two polynomials without common roots. Let $\deg(g) < \deg(h) = m$ and suppose that the roots $a_1, \ldots, a_m$ of $h(x)$ are all distinct. Then $f(x)$ can be decomposed into a sum of partial fractions, ie., for some constants $b_1, b_2, \ldots, b_m$,*

$$f(x) = \frac{b_1}{a_1 - x} + \frac{b_2}{a_2 - x} + \cdots + \frac{b_m}{a_m - x}\,.$$

Since

$$\frac{b}{a-x} = \frac{b}{a}\sum_{k\geq 0}\left(\frac{x}{a}\right)^k = \sum_{k\geq 0}\frac{b}{a^{k+1}}x^k\,,$$

a partial fraction sum can be easily written as a power series.

Example 4.13: *Let a_n be the probability that n independent Bernoulli trials (with success probability p) result in an* **even number of successes***. Find the generating function of a_n.*

Solution. The event under consideration occurs if an **initial failure** at the first trial is followed by an even number of successes or if an **initial success** is followed by an odd number of successes. Therefore, $a_0 = 1$ and

$$a_n = (1-p)\, a_{n-1} + p\,(1 - a_{n-1})\,, \qquad n \geq 1\,.$$

Multiplying these equalities by s^n and adding we get (with $q = 1-p$)

$$G_a(s) - 1 = qs\, G_a(s) + p \sum_{n \geq 1} s^n - ps\, G_a(s) = (q-p)s\, G_a(s) + \frac{ps}{1-s}\,,$$

and after rearranging,

$$G_a(s) = \Bigl(1 + \frac{ps}{1-s}\Bigr) / \bigl(1 - (q-p)s\bigr) = \frac{1}{2}\Bigl(\frac{1}{1-s} + \frac{1}{1-(q-p)s}\Bigr)\,.$$

As a result, $a_n = \bigl(1 + (q-p)^n\bigr)/2$.

Example 4.14: *A biased coin is tossed repeatedly; on each toss, it shows a "head" with probability p. Let r_n be the probability that a sequence of n tosses never has two "heads" in a row. Show that $r_0 = 1$, $r_1 = 1$, and for all $n > 1$, $r_n = qr_{n-1} + pqr_{n-2}$, where $q = 1 - p$. Find the generating function of the sequence $(r_n)_{n\geq 0}$.*

Solution. Every sequence of n tosses contributing to r_n starts either with T or with HT; therefore, for all $n > 1$, $r_n = qr_{n-1} + pqr_{n-2}$ (where $q = 1 - p$). Multiplying these equalities by s^n and summing, we get

$$G_r(s) = \sum_{n\geq 0} r_n s^n = 1 + s + qs \sum_{n\geq 2} r_{n-1} s^{n-1} + pqs^2 \sum_{n\geq 2} r_{n-2} s^{n-2}$$

$$= (qs + pqs^2) G_r(s) + 1 + ps\,,$$

so that

$$G_r(s) = \frac{1 + ps}{1 - qs - pqs^2}\,.$$

Continuity theorem

Theorem 4.15: *Let for every fixed n the sequence $a_{0,n}, a_{1,n}, \ldots$ be a* **probability distribution**, *ie., $a_{k,n} \geq 0$ and $\sum_{k\geq 0} a_{k,n} = 1$, and let $G_n(s) = \sum_{k\geq 0} a_{k,n} s^k$ be the corresponding generating function. Then*

$$^{\forall} k \geq 0 \quad a_{k,n} \to a_k \quad \Longleftrightarrow \quad ^{\forall} s \in [0,1) \quad G_n(s) \to G(s)\,,$$

where $G(s) = \sum_{k\geq 0} a_k s^k$, the generating function of the limiting sequence $(a_k)_{k\geq 0}$.

The convergence $\left\{a_{k,n}\right\}_{k\geq 0} \to \left\{a_k\right\}_{k\geq 0}$ is known as the **convergence in distribution**!

Example 4.16: *If* $X_n \sim \mathrm{Bin}(n, p)$ *with* $p = p_n$ *satisfying* $n \cdot p_n \to \lambda$ *as* $n \to \infty$, *then*

$$G_{X_n}(s) \equiv \big(1 + p_n(s-1)\big)^n \to \exp\{\lambda(s-1)\}\,,$$

so that the distribution of X_n *converges to that of* $X \sim \mathrm{Poi}(\lambda)$.

R script!

Generating functions

By the end of this section you should be able to:

- define ordinary, probability, and moment generating functions;
- derive the value of the nth term of a sequence from the corresponding generating function;
- state and apply the theorem about generating functions of convolutions;
- use probability generating functions to compute moments of random variables;
- state and apply the uniqueness and continuity theorems for generating functions;
- use generating functions in solving recurrent relations;
- explain the role of generating functions in proving convergence in distribution.

Problem GF-17

Two players play a game called *heads or tails*. In this game, a coin coming up heads with probability p is tossed consecutively. Each time a head comes up Player I wins 1 pence, otherwise she loses 1 pence. Let $X_k \in \{-1, 1\}$ denote the outcome of the kth trial, and let S_n, $n \geq 0$, be the total gain of Player I after n trials, $S_n = X_1 + \cdots + X_n$, where different outcomes are assumed independent. Let T be the moment when Player I is first in the lead, ie.,

$$S_k \leq 0 \quad \text{for } k < T, \qquad \text{and} \qquad S_T = 1\,.$$

(A) find the generating function $G_T(s) = \mathsf{E}(s^T)$ of T;

(B) for which values of p is T a proper random variable, ie., when $\mathsf{P}(T < \infty) \equiv \lim_{s \nearrow 1} G_T(s) = 1$?

(C) compute the expectation $\mathsf{E}\,T$ of T.

Solution

Let $f_{ij}^{(k)} = \mathsf{P}(S_1 \neq j, \ldots, S_{k-1} \neq j, S_k = j \mid S_0 = i)$ be the probability to **first** hit state j after k jumps, starting from i. Then

$$\mathsf{P}(T = k) = f_{01}^{(k)} \qquad \text{and} \qquad G_T(s) = \sum_{k=0}^{\infty} f_{01}^{(k)} s^k \,.$$

On the event $\{X_1 = 1\}$ we have $T = 1$, and on the event $\{X_1 = -1\}$ we have $T = 1 + T_2$, where T_2 is the time to **first** hit state 1 starting from -1. By the partition theorem,

$$G_T(s) = \mathsf{E}(s^T \mid X_1 = 1)\mathsf{P}(X_1 = 1) + \mathsf{E}(s^T \mid X_1 = -1)\mathsf{P}(X_1 = -1)$$

or (with $q = 1 - p$)

$$G_T(s) = ps + qs\mathsf{E}(s^{T_2}) = ps + qs\, G_{T_2}(s)\,. \qquad (*)$$

To get from state -1 to state 1 we have to pass through 0, so that

$$\mathsf{P}(T = m) = f^{(m)}_{-1,1} = \sum_{\ell=1}^{m-1} f^{(\ell)}_{-1,0}\, f^{(m-\ell)}_{0,1}\,,$$

where, by homogeneity, $f^{(\ell)}_{-1,0} \equiv f^{(\ell)}_{0,1}$; the convolution theorem now implies $G_{T_2}(s) = G_T(s)G_T(s) = \big(G_T(s)\big)^2$. Substituting this into (*) and solving for $G_T(s)$, we get (since $G_T(s) \to 0$ if $s \to 0$!)

$$G_T(s) = \frac{1-\sqrt{1-4pqs^2}}{2qs} = \frac{2ps}{1+\sqrt{1-4pqs^2}}\,.$$

Now,

$$\mathsf{P}(T < \infty) = G_T(1) = \frac{1-|p-q|}{2q} = \begin{cases} 1\,, & p \ge q\,,\\ p/q\,, & p < q\,. \end{cases}$$

Similarly (**do** the computation!),

$$\mathsf{E}T = G_T'(1) = \begin{cases} 1/(p-q)\,, & p > q\,,\\ +\infty\,, & p = q\,. \end{cases}$$

Also, for $p < q$ we have $\mathsf{P}(T = \infty) = 1 - \frac{p}{q} > 0$, so that $\mathsf{E}T = \infty$.