Random Digits

- Generate a sequence $a_1 a_2 \dots a_{120}$ of 120 independent "random" digits from $\{1, 2, \dots, 9\}$;
- choose one of the first 10 digits $\{a_1, a_2, \dots, a_{10}\}$;
- move along the sequence $a_1 a_2 \dots a_{120}$ according to the following rule: if your current position is k, jump to $k + a_k$;
- stop when you cannot jump forward, and record your answer.

Using the same sequence of digits, repeat the game several times from different starting positions.

Any explanations?

lose+lose = win?

Consider the following coin flipping games:

- Game **A**: you flip a coin with winning probability .495;
- Game **B**: if your current fortune is a multiple of $M = 3$, flip a coin with winning probability .095; otherwise flip a coin with winning probability .745.

Easy to check: **both** games **A** and **B** are **losing** games.

However:
many (deterministic or random!) combinations of these games (like **AABB**, **ABBAB**, play **A** with probability 0.4, play **A** with probability 0.5 etc.) are in fact **winning** games!!!

Any explanations?

To play or not to play?

Imagine: To thank you for your loyalty, your banker offers you an interest-free three-day loan of £10K for the coming bank holiday weekend.

By pure coincidence, a casino opens in the nearby big city during the same bank holiday weekend, with the introductory offer for the first three days advertised as follows:

- Saturday, day of **responsible gambling**: you play as many times as you wish, but you only collect your winnings on exiting the casino;
- Sunday, day of **realistic gambling**: as before, but each game costs you £1;
- Monday, day of **adventurous gambling**: you play as many times as you wish, receive your winnings on the spot, but you can only play on a single machine at a time.

A friend of yours, who works for the casino, claims that during the introductory three-day period all machines in the casino will be programmed to independently flip biased coins, allowing customers to win with probability 0.55.

Should you take the offer and try your luck?

Waiting for a double

In a sequence of coin-flipping experiments (in which H is seen with probability p), let T_1 be the time until we observe the first H,

$$T_1 = \min\left\{ k \geq 1 : X_k = H \right\}, \qquad \text{where } X_k \text{ is the } k\text{th result}.$$

Similarly, let T_2 be the time when the word HH is first observed,

$$T_2 = \min\left\{ k \geq 2 : X_k = X_{k-1} = H \right\}, \qquad \text{etc.}$$

Let $G_m(s) \equiv G_{T_m}(s)$ be the generating function of T_m.

a) Show that $G_1(s) = \dfrac{ps}{1-qs}$, where $q = 1 - p$; hence, find $\mathsf{E}(T_1)$.

b) Show that $G_2(s) = \dfrac{G_1(s)ps}{1 - G_1(s)qs}$ and deduce the value of $\mathsf{E}(T_2)$.

c) Guess the answers for $G_m(s)$ and $\mathsf{E}(T_m)$ with $m \geq 3$.

Quadruple six

A standard fair die is tossed repeatedly. Let Q be the number of tosses until the first occasion when four consecutive tosses have shown 6.

a) Find the probability generating function of the random variable Q and compute its expectation $\mathsf{E}Q$.

b) Relate Q to the hitting time of '6666' starting from '*' for a suitably defined Markov chain with the state space $S = \{*, 6, 66, 666, 6666\}$ and find its expectation.

Making big profit

Consider the Markov chain X_n on $S = \{0, 1, 2, \dots\}$ with $X_0 = 1$ and the following jump probabilities: $p_{01} = 1$, and for every $k > 0$

$$p_{k,k+1} = p\,, \qquad p_{k,k-1} = q = 1 - p\,.$$

Fix a large $M > 0$, and let T be the hitting time of M.

a) compute the expectation $\mathsf{E}_1 T$;

b) compute the generating function of T; check that it gives the same value of $\mathsf{E}_1 T$;

c) under the realistic condition $p < q$, approximate $\mathsf{E}_1 T$;
what can you say about the distribution of the rescaled random variable $T/\mathsf{E}_1 T$?

Which comes sooner – ‘AB’ or ‘AA’?

A coin showing ‘A’ with probability p and ‘B’ with probability $1-p$ is tossed repeatedly, with individual results R_k being independent. Let T_{AB} (resp. T_{AA}) be the moments when the words 'AB' (resp. 'AA') are first observed,

$$T_{AB} = \min_{k\geq 2}\{R_{k-1} = A, R_k = B\}, \quad T_{AA} = \min_{k\geq 2}\{R_{k-1} = A, R_k = A\}.$$

Which expectation is smaller, $\mathsf{E}\,T_{AB}$ or $\mathsf{E}\,T_{AA}$?

Any explanations?

You might wish to use generating functions or a suitably defined Markov chain with state space $\{*, A, B, AA, AB, BA, BB\}$.

Waiting for an ‘ABBA’

A coin showing ‘A’ with probability p and ‘B’ with probability $1 - p$ is tossed repeatedly, with individual results R_k being independent. Let T be the moment the word 'ABBA' is first observed,

$$T = \min\{k \geq 4 : R_{m-3} = A, R_{m-2} = B, R_{m-1} = B, R_m = A\}.$$

a) Use an appropriate Markov chain to find the expectation of T.

b) Compute the generating function of the random time T and find its expectation.