L^r convergence

Def.2.1: Let $r > 0$ be fixed. A sequence $(X_n)_{n\geq 1}$, of random variables *converges* to a random variable X *in* L^r as $n \to \infty$ (write $X_n \xrightarrow{\mathrm{L^r}} X$), if

$$\mathrm{E}\big|X_n - X\big|^r \to 0 \text{ as } n \to \infty.$$

Example 2.2: Let $\big(X_n\big)_{n\geq 1}$ be a sequence of random variables such that for some real numbers $(a_n)_{n\geq 1}$, we have

$$\mathrm{P}\big(X_n = a_n\big) = p_n\,, \qquad \mathrm{P}\big(X_n = 0\big) = 1 - p_n\,. \tag{2.1}$$

Then $X_n \xrightarrow{\mathrm{L^r}} 0$ iff $\mathrm{E}\big|X_n\big|^r \equiv |a_n|^r\, p_n \to 0$ as $n \to \infty$.

L^2-WLLN

Theorem 2.3: Let X_j, $j \geq 1$, be a sequence of uncorrelated random variables with

$$\mathrm{E}X_j = \mu \qquad \text{and} \qquad \mathrm{Var}(X_j) \leq C < \infty\,.$$

Denote $S_n = X_1 + \cdots + X_n$. Then $\dfrac{1}{n} S_n \stackrel{L^2}{\to} \mu$ as $n \to \infty$.

Proof: Indeed,

$$\mathrm{E}\Big(\frac{1}{n}S_n - \mu\Big)^2 = \mathrm{E}\frac{(S_n - n\mu)^2}{n^2} = \frac{\mathrm{Var}(S_n)}{n^2} \leq \frac{Cn}{n^2} \to 0$$

as $n \to \infty$.

Convergence in probability

Def.2.4: A sequence $(X_n)_{n\geq1}$, of random variables *converges in probability* as $n \to \infty$ to a random variable X (write $X_n \stackrel{\mathsf{P}}{\to} X$), if for *every fixed* $\varepsilon > 0$

$$\mathsf{P}\big(|X_n - X| \geq \varepsilon\big) \to 0 \qquad \text{as } n \to \infty\,.$$

Example 2.5: Let the sequence $\big(X_n\big)_{n\geq1}$ be as in (2.1). Then for every $\varepsilon > 0$

$$\mathsf{P}\big(|X_n| \geq \varepsilon\big) \leq \mathsf{P}\big(X_n \neq 0\big) = p_n\,,$$

so that $X_n \stackrel{\mathsf{P}}{\to} 0$ if $p_n \to 0$ as $n \to \infty$.

Theorem 2.6: Let X_j, $j \geq 1$, be a sequence of uncorrelated random variables with

$$\mathsf{E}X_j = \mu \qquad \text{and} \qquad \mathrm{Var}(X_j) \leq C < \infty\,.$$

Denote $S_n = X_1 + \cdots + X_n$. Then

$$\frac{1}{n} S_n \xrightarrow{\mathsf{P}} \mu \qquad \text{as } n \to \infty\,.$$

Exercise 2.7: Derive this theorem from the Chebyshev inequality!

Theorem 2.6 follows directly from the following observation:

Lemma 2.8: Let $(X_n)_{n\geq 1}$, be a sequence of random variables. If $X_n \stackrel{\mathrm{L^r}}{\to} X$ for some fixed $r > 0$, then $X_n \stackrel{\mathrm{P}}{\to} X$ as $n \to \infty$.

Proof: Indeed, for every fixed $\varepsilon > 0$,

$$\mathrm{P}\big(|X_n - X| \geq \varepsilon\big) \equiv \mathrm{P}\big(|X_n - X|^r \geq \varepsilon^r\big) \leq \frac{\mathrm{E}|X_n - X|^r}{\varepsilon^r} \to 0$$

as $n \to \infty$.

A high dimensional cube $\approx$ a sphere

One can use convergence in probability to argue that, for large n, most points of the n-dimensional cube $[-1,1]^n$ are located near the sphere of radius $\sqrt{n/3}$!

If interested, see **Example 2.9** in the notes.

Theorem 2.10: Let random variables $(S_n)_{n\geq 1}$, have two finite moments,

$$\mu_n \equiv \mathsf{E} S_n\,, \qquad \sigma_n^2 \equiv \mathsf{Var}(S_n) < \infty\,.$$

If, for some sequence b_n, we have $\sigma_n / b_n \to 0$ as $n \to \infty$, then

$$\frac{S_n - \mu_n}{b_n} \to 0 \qquad \text{as } n \to \infty$$

both in L^2 and in probability.

Proof: Indeed,

$$\mathsf{E}\Big(\frac{(S_n - \mu_n)^2}{b_n^2}\Big) = \frac{\mathsf{Var}(S_n)}{b_n^2} \to 0 \qquad \text{as } n \to \infty\,.$$

Example 2.11: In the coupon collector's problem (problem R 4), let T_n be the time to collect all n coupons. We know that

$$\mathrm{E}\,T_n = n\sum_{m=1}^{n}\frac{1}{m} \sim n\log n\,, \qquad \mathrm{Var}(T_n) \le n^2\sum_{m=1}^{n}\frac{1}{m^2} \le \frac{\pi^2 n^2}{6}\,,$$

so that

$$\frac{T_n - \mathrm{E}\,T_n}{n\log n} \to 0 \quad \text{ie.,} \quad \frac{T_n}{n\log n} \to 1$$

as $n \to \infty$ both in L^2 and in probability.

Almost sure convergence

Recall: **WLLN** is just $\stackrel{\mathsf{P}}{\to}$

Let $X_1, X_2, \ldots$ be i.i.d. r.v. with $\mathsf{E}X_1 = \mu$ and $\mathsf{Var}X_1 < \infty$. Denote $S_n \stackrel{\text{def}}{=} \sum_{k=1}^n X_k$.

By the usual (*weak*) *law of large numbers* (**WLLN**): for **every** $\delta > 0$

$$\mathsf{P}\Big(\big|n^{-1}S_n - \mu\big| > \delta\Big) \to 0 \qquad \text{as } n \to \infty. \tag{2.2}$$

I.e., **WLLN** states: $n^{-1}S_n \stackrel{\mathsf{P}}{\to} X \equiv \mu = \mathsf{E}(X_1)$, as $n \to \infty$.

Notice: In general

$$X_n \xrightarrow{\mathrm{P}} X \text{ is **not** related to } X_n(\omega) \to X(\omega) \text{ for **fixed} } \omega \in \Omega$$

ie., the **pointwise** convergence:

Example 2.13: Let $\Omega = [0, 1]$, let $\mathcal{F} = \sigma\big((a, b) : 0 \le a \le b \le 1\big)$, and let, for $A = [a, b] \subseteq [0, 1]$, $\mathrm{P}(A) = b - a$.

$$^{\forall}A \in \mathcal{F} \quad \rightsquigarrow \quad \mathbb{1}_A(\omega) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } \omega \in A\,, \\ 0, & \text{if } \omega \notin A\,. \end{cases} \tag{2.3}$$

Consider $X_n \stackrel{\text{def}}{=} \mathbb{1}_{A_n}$, where, for $n \ge 1$ such that $2^m \le n < 2^{m+1}$,

$$A_n = \left[\tfrac{n-2^m}{2^m}, \tfrac{n+1-2^m}{2^m}\right] \subseteq \big[0, 1\big]\,.$$

Then $X_n \xrightarrow{\mathrm{P}} X \equiv 0$, but

$$\big\{\omega \in \Omega : X_n(\omega) \to X(\omega) \equiv 0 \text{ as } n \to \infty\big\} = \varnothing\,,$$

ie., there is **no point** $\omega \in \Omega$ for which $X_n(\omega) \to X(\omega)$. R script!

Almost sure convergence

Def.2.14: A sequence $X_1, X_2, \ldots$ of r.v. in $(\Omega, \mathcal{F}, \mathrm{P})$ converges, as $n \to \infty$, to a random variable X **with probability one** or **almost surely** (write $X_n \stackrel{\text{a.s.}}{\to} X$) if

$$\mathrm{P}\Big(\big\{\omega \in \Omega : X_n(\omega) \to X(\omega) \text{ as } n \to \infty\,\big\}\Big) = 1\,. \qquad (2.4)$$

Remark 2.14.1: For $\varepsilon > 0$, let $A_n(\varepsilon) = \big\{\omega : |X_n(\omega) - X(\omega)| > \varepsilon\big\}$. Then the definition (2.4) is equivalent to saying that for every $\varepsilon > 0$

$$\mathrm{P}\big(\big\{A_n(\varepsilon) \text{ finitely often}\,\big\}\big) = 1\,. \qquad (2.5)$$

This is why the Borel-Cantelli lemma is so useful in studying almost sure limits.

Example 2.13 [cont'd]: Let $(\Omega, \mathcal{F}, \mathsf{P})$ be as before; consider $Y_n \stackrel{\text{def}}{=} \mathbb{1}_{(0,1/n]}$ and $Z_n \stackrel{\text{def}}{=} \mathbb{1}_{[0,1/n]}$. Then

$$\bigl\{\omega \in \Omega : Y_n(\omega) \to 0 \text{ as } n \to \infty\bigr\} \equiv [0,1]\,,$$

$$\bigl\{\omega \in \Omega : Z_n(\omega) \to 0 \text{ as } n \to \infty\bigr\} \equiv (0,1]\,,$$

so that $Y_n \stackrel{\text{a.s.}}{\to} 0$ and $Z_n \stackrel{\text{a.s.}}{\to} 0$ as $n \to \infty$.

Example 1.5 [cont'd]:

Let X be a finite random variable, $\mathsf{P}(|X| < \infty) = 1$.

Then the sequence $(X_k)_{k \geq 1}$ defined via $X_k \stackrel{\text{def}}{=} \frac{1}{k} X$ converges to zero with probability one.

Indeed, the event $\{\omega : X_k(\omega) \not\to 0\} = \{\omega : |X(\omega)| = \infty\}$ has probability zero.

Lemma 2.15: Let X_1, X_2, ... and X be r.v^s. If, for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathsf{P}\big(|X_n - X| > \varepsilon\big) < \infty \,, \qquad (2.6)$$

then X_n converges to X almost surely.

This is **NOT** the definition of the almost sure convergence, but only a **sufficient condition** for it!

Notice: Let X_1, X_2, ... and X be r.v^s. If $X_n \stackrel{\mathsf{P}}{\to} X$, then there exists a **non-random** sequence of integers n_1, n_2, ... such that

$$X_{n_k} \stackrel{\text{a.s.}}{\to} X \qquad \text{as } n \to \infty \,.$$

☞ see Problem C11 !

L^4 Strong Law of Large Numbers

Theorem 2.16 (SLLN, Borel): Let $X_1, X_2, \dots$ be i.i.d. r.v. with

$$\mathsf{E}(X_k) = \mu \qquad \text{and} \qquad \mathsf{E}\big((X_k)^4\big) < \infty\,.$$

If $S_n \stackrel{\text{def}}{=} X_1 + X_2 + \cdots + X_n$, then

$$S_n/n \stackrel{\text{a.s.}}{\to} \mu \qquad \text{as} \quad n \to \infty.$$

L^1 Strong Law of Large Numbers

Theorem 2.17 (SLLN, Kolmogorov): Let $X_1, X_2, \ldots$ be i.i.d. r.v. with

$$\mathsf{E}|X_k| < \infty\,.$$

If $\mathsf{E}(X_k) = \mu$ and $S_n \stackrel{\text{def}}{=} X_1 + X_2 + \cdots + X_n$, then

$$S_n/n \stackrel{\text{a.s.}}{\to} \mu \qquad \text{as} \quad n \to \infty.$$

Relations between $\xrightarrow{L^r}$, $\xrightarrow{P}$, and $\xrightarrow{a.s.}$

We know that (**Lemma 2.8**)

$$X_n \xrightarrow{L^r} X \qquad \Longrightarrow \qquad X_n \xrightarrow{P} X\,;$$

one can also show (we shall not do it here!)

$$X_n \xrightarrow{a.s.} X \qquad \Longrightarrow \qquad X_n \xrightarrow{P} X\,.$$

In addition, according to **Example 2.13**,

$$X_n \xrightarrow{P} X \qquad \not\Rightarrow \qquad X_n \xrightarrow{a.s.} X\,,$$

and the same construction shows that

$$X_n \xrightarrow{L^r} X \qquad \not\Rightarrow \qquad X_n \xrightarrow{a.s.} X\,.$$

$$X_n \xrightarrow{L^r} X \not\Rightarrow X_n \xrightarrow{a.s.} X$$

Example 2.18: Let X_n be a sequence of *independent* random variables such that

$$P(X_n = 1) = p_n, \qquad P(X_n = 0) = 1 - p_n.$$

Then, with $X \equiv 0$,

$$X_n \xrightarrow{P} X \iff p_n \to 0 \iff X_n \xrightarrow{L^r} X \qquad \text{as } n \to \infty,$$

whereas

$$X_n \xrightarrow{a.s.} X \iff \sum_n p_n < \infty.$$

Taking $p_n = 1/n$, we get $X_n \xrightarrow{L^r} X$ but **not** $X_n \xrightarrow{a.s.} X$.

Notice that this example also shows that $X_n \xrightarrow{P} X \not\Rightarrow X_n \xrightarrow{a.s.} X$.

$$X_n \xrightarrow{\mathsf{P}} X \not\Rightarrow X_n \xrightarrow{\mathsf{L}^r} X$$

Example 2.19: Let $(\Omega, \mathcal{F}, \mathsf{P})$ be the canonical probability space (recall Example 2.13). For $n \geq 1$, define

$$X_n(\omega) \stackrel{\text{def}}{=} e^n \cdot \mathbb{1}_{[0,1/n]}(\omega) \equiv \begin{cases} e^n, & 0 \leq \omega \leq 1/n \\ 0, & \omega > 1/n. \end{cases}$$

Clearly, $X_n \xrightarrow{\text{a.s.}} 0$, and $X_n \xrightarrow{\mathsf{P}} 0$ as $n \to \infty$; however, for every $r > 0$

$$\mathsf{E}|X_n|^r = \frac{e^{nr}}{n} \to \infty, \qquad \text{as } n \to \infty, \quad \text{ie.,} \quad X_n \stackrel{\mathsf{L}^r}{\not\to} 0.$$

Notice that this example also shows that $X_n \xrightarrow{\text{a.s.}} X \not\Rightarrow X_n \xrightarrow{\mathsf{L}^r} X$.

Convergence of random variables

By the end of this section you should be able to:

- define convergence in L^r, verify whether a given sequence of random variables converges in L^r;
- define convergence in probability, verify whether a given sequence of random variables converges in probability;
- explain the relation between convergence in L^r and convergence in probability (Lem 2.8);
- state and apply the sufficient condition for convergence in L^2 (Thm 2.10);
- define almost sure convergence, verify whether a given sequence of random variables converges almost surely;
- state and apply the sufficient condition for almost sure convergence (Lem 2.15);
- state and apply the Strong Laws of Large Numbers.