A Some useful mathematical facts

This (growing) section is aimed to help you by listing various basic facts used in the course. Feel free to suggest any further additions you find useful!

A.1 Sets and their properties

If A, B are sets (events), we write $A \subset B$ to indicate that every $x \in A$ satisfies $x \in B$. We also write $A = B$ if both $A \subset B$ and $B \subset A$, or, using the equivalence symbol $\Leftrightarrow$,

$$A = B \qquad \Leftrightarrow \qquad A \subset B \quad \text{and} \quad B \subset A\,.$$

It is useful to remember that (exercise!)

$$A \subset B \qquad \Leftrightarrow \qquad B^{\mathsf{c}} \subset A^{\mathsf{c}} \qquad \Leftrightarrow \qquad A \cap B^{\mathsf{c}} = \varnothing\,,$$

where A^{c} denotes the complement of the set (event) A. If $(A_\alpha)_{\alpha \in \mathcal{A}}$ is any collection of events, then

$$\Bigl(\bigcup_{\alpha\in\mathcal{A}} A_\alpha\Bigr)^{\mathsf{c}} = \bigcap_{\alpha\in\mathcal{A}} A_\alpha^{\mathsf{c}}\,, \qquad \Bigl(\bigcap_{\alpha\in\mathcal{A}} A_\alpha\Bigr)^{\mathsf{c}} = \bigcup_{\alpha\in\mathcal{A}} A_\alpha^{\mathsf{c}}\,.$$

Lemma A.1. *If $(A_n)_{n\ge0}$ is an increasing sequence of sets and $(B_n)_{n\ge0}$ is a decreasing sequence of sets, then*

$$A_0 \subset \bigcap_{n\ge1} A_n\,, \qquad \text{and} \qquad \bigcup_{n\ge1} B_n \subset B_0\,.$$

Lemma A.2. *Let $\bigl(A_\alpha\bigr)_{\alpha\in\mathcal{A}}$ and $\bigl(B_\beta\bigr)_{\beta\in\mathcal{B}}$ be arbitrary[1] collections of sets, such that* for every A_α *there* exists B_β *with $A_\alpha \subset B_\beta$, then*

$$\bigcup_{\alpha\in\mathcal{A}} A_\alpha \subset \bigcup_{\beta\in\mathcal{B}} B_\beta\,.$$

Similarly, if for every B_β *there* exists A_α *with $A_\alpha \subset B_\beta$, then*

$$\bigcap_{\alpha\in\mathcal{A}} A_\alpha \subset \bigcap_{\beta\in\mathcal{B}} B_\beta\,.$$

Proof. A useful exercise! □

If A is an event, its *indicator function* $\mathbb{1}_A : \Omega \to \{0,1\}$ is defined via

$$\mathbb{1}_A(\omega) = \begin{cases} 1\,, & \omega \in A\,,\\ 0\,, & \omega \notin A\,. \end{cases}$$

Convergence of events is equivalent[2] to the point-wise convergence of the corresponding indicator functions: $A_n \to A$, iff $\mathbb{1}_{A_k}(\omega) \to \mathbb{1}_A(\omega)$ for every $\omega \in \Omega$.

Notice that for a sequence $(A_k)_{k\ge1}$ of events, the random variable

$$N(\omega) = \textstyle\sum_{k=1}^\infty \mathbb{1}_{A_k}(\omega)$$

counts how many of the events A_k occur. One can show that $\mathsf{E}N = \sum_{k=1}^\infty \mathsf{P}(A_k)$.

[1] with $\mathcal{A}$ and $\mathcal{B}$ countable or even uncountable!

[2] The general theory of set convergence is the subject of 'pure' courses such as set theory or (real) analysis/measure theory; if interested, have a look at problems E26–E28 and/or get in touch!

A.2 Real sequences and their limits

The following observation explains why the Borel-Cantelli lemma (Lemma 1.6) is so useful in studying limits of random variables: If $(x_n)_{n\geq 1}$ is a sequence of real numbers, we say that it converges to a limit a (and write $a = \lim_{n\to\infty} x_n$) if for every $\varepsilon > 0$ the set $\{n \geq 1 : |x_n - a| > \varepsilon\}$ is finite,

$$\bigl|\{n \geq 1 : |x_n - a| > \varepsilon\}\bigr| < \infty .$$

Not every sequence $(x_n)_{n\geq 1}$ has a limit, however, every such sequence has the upper and the lower limits,

$$\limsup_{n\to\infty} x_n \equiv \lim_{n\to\infty} \sup_{k\geq n} x_k , \qquad \liminf_{n\to\infty} x_n \equiv \lim_{n\to\infty} \inf_{k\geq n} x_k .$$

Of course, $\lim_{n\to\infty} x_n$ exists if and only if $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$ (and their common value gives $\lim_{n\to\infty} x_n$).

Lemma A.3. *Let $(x_n)_{n\geq 1}$ be a real sequence. Then $a = \limsup_{n\to\infty} x_n$ if and only if for every $\varepsilon > 0$*

$$\bigl|\{n \geq 1 : x_n > a + \varepsilon\}\bigr| < \infty \quad \text{and} \quad \bigl|\{n \geq 1 : x_n > a - \varepsilon\}\bigr| = \infty . \tag{A.1}$$

The following fact is useful in studying extreme values, recall Remark 1.10.1; it also highlights the usefulness of the Borel-Cantelli lemma (Lemma 1.6).

Lemma A.4. *Let $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ be real sequences, where, for every $n \geq 1$, one has $y_n = \max(x_1, \dots, x_n)$. Then for every non-decreasing sequence $(b_n)_{n\geq 1}$ (ie., $b_n \leq b_{n+1}$ for all $n \geq 1$) with $b_n \to \infty$ as $n \to \infty$, the sets*

$$\{n \geq 1 : x_n > b_n\} \quad \text{and} \quad \{n \geq 1 : y_n > b_n\}$$

are both finite or both infinite.

☞ It is instructive to write a carefull proof of the previous lemmata. In particular, you might wish to explore what happens in Lemma A.4 if the condition $b_n \to \infty$ as $n \to \infty$ is dropped.

A.3 Integral calculus of sequences

The following facts show that the "integral calculus for sequences" is basically a Core B1 material:

Lemma A.5. *Let $\mathcal{S} = \bigl(s_{m,n}\bigr)_{m,n\geq 1}$ be an increasing in both indices (m and n) collection of numbers in $\overline{\mathbb{R}} \equiv [-\infty, +\infty]$, ie., as soon as $j \leq m$ and $k \leq n$, we have $s_{j,k} \leq s_{m,n}$. Then*

$$\lim_{m\to\infty} \lim_{n\to\infty} s_{m,n} = \lim_{n\to\infty} \lim_{m\to\infty} s_{m,n} = \sup \mathcal{S} ,$$

Remark A.5.1. In other words, interchanging the order of limits does not change the result!

Proof. An easy exercise using definitions of lim and sup. □

Lemma A.6. *Let $\mathcal{A} = \big(a_{m,n}\big)_{m,n\ge 1}$ be a collection of numbers in $\overline{\mathbb{R}}^+ \equiv [0,+\infty]$. Then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{m,n} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n} = \sup \mathcal{S}\,,$$

where $\mathcal{S}$ is the set of all *sums of* finitely *many elements of $\mathcal{A}$.*

Remark A.6.1. In other words, iterated sums of non-negative numbers can be summed in any order. You had a similar statement for multiple integrals in Core A; it is often referred to as the Fubini theorem (for non-negative sums).

Proof. Just consider all sums $s_{m,n} = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j}$ and use Lemma A.5. □

Lemma A.7. *Let $\big(a_{m,n}\big)_{m,n\ge 1}$ be a collection of numbers in $\overline{\mathbb{R}}^+ \equiv [0,+\infty]$, which is increasing in the second index n, ie., for* every fixed *$m \in \mathbb{N}$, the inequality $a_{m,k} \le a_{m,n}$ holds provided $k \le n$. Then*

$$\lim_{n\to\infty} \sum_{m=1}^{\infty} a_{m,n} = \sum_{m=1}^{\infty} \lim_{n\to\infty} a_{m,n}\,.$$

Remark A.7.1. If the functions $f_n : \mathbb{N} \to \overline{\mathbb{R}}^+$ are defined via $f_n(m) = a_{m,n}$, they form a point-wise monotone sequence (ie., for every fixed $m \in \mathbb{N}$, we have $f_n(m) \le f_{n+1}(m)$ for all $n \ge 1$); the statement above says that the limit of the sum (integral) equals the sum (integral) of limits. In other words, the statement above is the Monotone Convergence Theorem for sequences.

Proof. Put $s_{m,n} = \sum_{l=1}^{m} a_{l,n}$ and use Lemma A.5. □

Lemma A.8. *Let $\big(a_{m,n}\big)_{m,n\ge 1}$, $\big(a_m\big)_{m\ge 1}$ and $\big(b_m\big)_{m\ge 1}$ be collections of numbers such that for* every fixed *$m \in \mathbb{N}$, we have $\lim_{n\to\infty} a_{m,n} = a_m$, $\big|a_{m,n}\big| \le b_m$, and $\sum_m b_m < \infty$. Then*

$$\lim_{n\to\infty} \sum_{m=1}^{\infty} a_{m,n} = \sum_{m=1}^{\infty} a_m = \sum_{m=1}^{\infty} \lim_{n\to\infty} a_{m,n}\,.$$

Remark A.8.1. This is just the Dominated Convergence Theorem for sequences!

Proof. Fix arbitrary $\varepsilon > 0$. By assumption, choosing M large enough, we can get $\sum_{m>M}\big|a_{m,n} - a_m\big| \le \sum_{m>M} b_m < \varepsilon/2$. For a finite M with this property, we can find n large enough so that $\sum_{m=1}^{M}\big|a_{m,n} - a_m\big| < \varepsilon/2$. Since $\varepsilon > 0$ is arbitrary, the result follows. □

Notice that for non-negative functions on $\mathbb{N}$, the sum is linear, monotone and respects limits; in other words, in this case the integral calculus reduces to a calculus of sums!

A.4 Some properties of power series

For a real sequence $(a_k)_{k\ge 0}$, consider the power series

$$G(s) = G_a(s) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} a_k\, s^k\,. \tag{A.2}$$

Theorem A.9 (Radius of convergence)**.** *There exists a number $R \geq 0$ such that the series in (A.2) converges absolutely if $|s| < R$ and diverges if $|s| > R$. This value of R is called the* radius of convergence*, and the series (A.2) converges uniformly on sets of the form $\{s : |s| \leq R'\}$ for any $R' < R$.*

Clearly, if $G(s)$ is a probability generating function, then $|s| \leq 1$ implies $|G(s)| \leq G(1) = 1$, so that $R \geq 1$.

Remark A.9.1. The radius of convergence of the g.f. $G(s) = \sum_{k=0}^{\infty} a_k \, s^k$ can be obtained from the so-called root test; it gives[3]

$$R = \frac{1}{\limsup_{k\to\infty} \sqrt[k]{|a_k|}} \,. \tag{A.3}$$

Theorem A.10 (Differentiation)**.** *If the radius of convergence of a generating function $G_a(s)$ equals $R > 0$, then $G_a(s)$ may be differentiated or integrated term by term any number of times when $|s| < R$.*

✍ **Theorem A.11** (Uniqueness)**.** *If two generating functions, $G_a(s)$ and $G_b(s)$, are finite and coincide on a disk of radius $R' > 0$, ie., for all $|s| < R'$ we have $G_a(s) = G_b(s)$, then $a_n = b_n$ for all n. Furthermore,*

$$a_n \equiv \frac{1}{n!} \frac{d^n}{ds^n}\Big|_{s=0} G_a(s) \,.$$

Theorem A.12 (Abel's theorem)**.** *If $a_k \geq 0$ for all k and $G_a(s)$ is finite for $|s| < 1$, then*

$$\lim_{s \nearrow 1} G_a(s) = \sum_{k=0}^{\infty} a_k \,, \tag{A.4}$$

whether this sum is finite or equals $+\infty$.

[3] alternatively, if the limit $q = \lim_{n\to\infty} |a_{n+1}/a_n|$ exists, then $R = 1/q$.