Mixed Multiscale Methods for Heterogeneous Elliptic Problems

Part 2: Mixed Multiscale Numerics

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Outline

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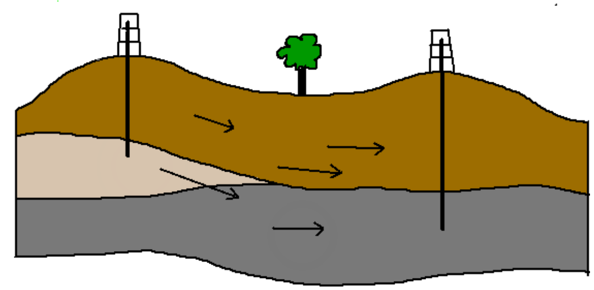
The differential problem:

$$\begin{cases} \mathbf{u} = -a_\epsilon \nabla p & \text{in } \Omega \\ \nabla \cdot \mathbf{u} = f & \text{in } \Omega \\ \mathbf{u} \cdot \nu = 0 & \text{on } \partial\Omega \end{cases}$$



Flow in porous media

The mixed variational problem:

Find $p \in W = L^2/\mathbb{R}$ and $\mathbf{u} \in \mathbf{V} = H_0(\text{div})$ such that

$$(a_\epsilon^{-1}\mathbf{u}, \mathbf{v}) = (p, \nabla \cdot \mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{V} \quad \text{(Darcy's law)}$$

$$(\nabla \cdot \mathbf{u}, w) = (f, w) \quad \forall\, w \in W \quad \text{(conservation)}$$

Remark: The mixed form preserves the conservation equation, and so allows locally conservative approximations. This is a critical property in many applications.

Define

$\mathcal{T}_h$ a reasonable finite element partition of Ω
$\mathcal{E}_h$ the set of edges of the finite elements
h the maximal element diameter
$W_h \times \mathbf{V}_h$ any inf-sup stable mixed finite element spaces in $W \times \mathbf{V}$

Find $p \in W_h \subset W$ and $\mathbf{u} \in \mathbf{V}_h \subset \mathbf{V}$ such that

$$(a_\epsilon^{-1}\mathbf{u}_h, \mathbf{v}) = (p_h, \nabla \cdot \mathbf{v}) \quad \forall\ \mathbf{v} \in \mathbf{V}_h \quad \text{(Darcy's law)}$$

$$(\nabla \cdot \mathbf{u}_h, w) = (f, w) \quad \forall\ w \in W_h \quad \text{(conservation)}$$

Difficulty: Fine-scale variation in a_ϵ (the *permeability*) leads to fine-scale variation in the solution $(\mathbf{u}, p)$.

Solution: Define $\mathbf{V}_h \times W_h$ to respect the scales:

- Multiscale finite elements (Babuška & Osborn 1983; Hou & Wu 1997; Chen & Hou 2003)
- Variational multiscale method (Hughes 1995, Arbogast, Minkoff & Keenan 1998, Arbogast & Boyd 2006)

Variational Multiscale Method

A Two-Scale Expansion

We base our expansion on local mass conservation.

Define a coarse computational grid $\mathcal{T}_h$ on Ω.

Pressure space: $W = \bar{W} \oplus W'$

$$\bar{W} = \{\bar{w} \in W : \bar{w} \text{ is constant on each coarse element } E\}$$
$$W' = \bar{W}^{\perp}$$

Velocity space: $\mathbf{V} = \bar{\mathbf{V}} \oplus \mathbf{V}'$

$$\mathbf{V}' = \{\mathbf{v}' \in \mathbf{V} : \nabla \cdot \mathbf{v}' \in W',\ \mathbf{v}' \cdot \nu = 0 \text{ on } \partial E\ \forall\ E\} \quad \text{(locality)}$$
$$\bar{\mathbf{V}} = \mathbf{V}/\mathbf{V}' \quad \text{(conservation)}$$

Then

$$(a) \quad \nabla \cdot \bar{\mathbf{V}} = \bar{W} \qquad \text{(coarse conservation)}$$
$$(b) \quad \nabla \cdot \mathbf{V}' = W' \qquad \text{(fine subgrid conservation)}$$
$$(c) \quad \bar{\mathbf{V}} \simeq \{\mathbf{v} \cdot \nu \text{ on } \partial E : E \in \mathcal{T}_h\}$$

Remark: To obtain subgrid locality in $\mathbf{V}'$, $\bar{\mathbf{V}}$ has full normal velocity coupling on the coarse edges $e \in \mathcal{E}_h$.

Separate scales uniquely via the direct sum as

$$\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}' \in \bar{\mathbf{V}} \oplus \mathbf{V}'$$
$$p = \bar{p} + p' \in \bar{W} \oplus W'$$

Coarse:

$$(a_\epsilon^{-1}(\bar{\mathbf{u}} + \mathbf{u}'), \bar{\mathbf{v}}) = (\bar{p}, \nabla \cdot \bar{\mathbf{v}}) \qquad \forall\ \bar{\mathbf{v}} \in \bar{\mathbf{V}}$$
$$(\nabla \cdot \bar{\mathbf{u}}, \bar{w}) = (f, \bar{w}) \qquad \forall\ \bar{w} \in \bar{W}$$

Subgrid:

$$(a_\epsilon^{-1}(\bar{\mathbf{u}} + \mathbf{u}'), \mathbf{v}') = (p', \nabla \cdot \mathbf{v}') \qquad \forall\ \mathbf{v}' \in \mathbf{V}'$$
$$(\nabla \cdot \mathbf{u}', w') = (f, w') \qquad \forall\ w' \in W'$$

Lemma. The inf-sup condition holds over both $\bar{W} \times \bar{\mathbf{V}}$ and $W' \times \mathbf{V}'$, with constants independent of the coarse mesh and ϵ.

Theorem. Given $\bar{\mathbf{u}} \in \bar{\mathbf{V}}$, there exists a unique solution $(p', \mathbf{u}') \in W' \times \mathbf{V}'$. Moreover,

$$\|p'\| + \|\mathbf{u}'\| \leq C\{\|f\| + \|\bar{\mathbf{u}}\|\}$$

Constant part: Define $(\tilde{p}', \tilde{\mathbf{u}}') \in W' \times \mathbf{V}'$ by

$$(a_\epsilon^{-1}\tilde{\mathbf{u}}', \mathbf{v}') = (\tilde{p}', \nabla \cdot \mathbf{v}') \qquad \forall\ \mathbf{v}' \in \mathbf{V}'$$

$$(\nabla \cdot \tilde{\mathbf{u}}', w') = (f, w') \qquad \forall\ w' \in W'$$

Linear part: For $\bar{\mathbf{v}} \in \bar{\mathbf{V}}$, define $(\hat{p}', \hat{\mathbf{u}}') \in W' \times \mathbf{V}'$

$$(a_\epsilon^{-1}(\bar{\mathbf{v}} + \hat{\mathbf{u}}'), \mathbf{v}') = (\hat{p}', \nabla \cdot \mathbf{v}') \qquad \forall\ \mathbf{v}' \in \mathbf{V}'$$

$$(\nabla \cdot \hat{\mathbf{u}}', w') = 0 \qquad \forall\ w' \in W'$$

Then

$$p' = \hat{p}'(\bar{\mathbf{u}}) + \tilde{p}'$$
$$\mathbf{u}' = \hat{\mathbf{u}}'(\bar{\mathbf{u}}) + \tilde{\mathbf{u}}'$$

Lemma. The operator $\hat{\mathbf{u}}' : \bar{\mathbf{V}} \to \mathbf{V}'$ is bounded and linear.

The coarse scale equation, in symmetric form, is:

Find $(\bar{p}, \bar{\mathbf{u}}) \in \bar{W} \times \bar{\mathbf{V}}$ such that

$$\begin{aligned} (a_\epsilon^{-1}(\bar{\mathbf{u}}+\hat{\mathbf{u}}'(\bar{\mathbf{u}})), (\bar{\mathbf{v}}+\hat{\mathbf{u}}'(\bar{\mathbf{v}}))) \\ = (\bar{p}, \nabla \cdot \bar{\mathbf{v}}) - (a_\epsilon^{-1}\tilde{\mathbf{u}}', \bar{\mathbf{v}}) \qquad & \forall\ \bar{\mathbf{v}} \in \bar{\mathbf{V}} \\ (\nabla \cdot \bar{\mathbf{u}}, \bar{w}) = (f, \bar{w}) \qquad & \forall\ \bar{w} \in \bar{W} \end{aligned}$$

Full solution:

$$\begin{aligned} p &= \bar{p} + \hat{p}'(\bar{\mathbf{u}}) + \tilde{p}' \\ \mathbf{u} &= \bar{\mathbf{u}} + \hat{\mathbf{u}}'(\bar{\mathbf{u}}) + \tilde{\mathbf{u}}' \end{aligned}$$

Remarks:

- No approximation has been made yet.
- The equations maintain strict local conservation on both scales.
- The subgrid and upscaled problems are well posed.
- Because $\mathbf{V}' \cdot \nu = 0$ on each ∂E, $\hat{\mathbf{u}}'$ is locally defined:

$$\hat{\mathbf{u}}'(\bar{\mathbf{u}})|_E = \hat{\mathbf{u}}'(\bar{\mathbf{u}}|_E)$$

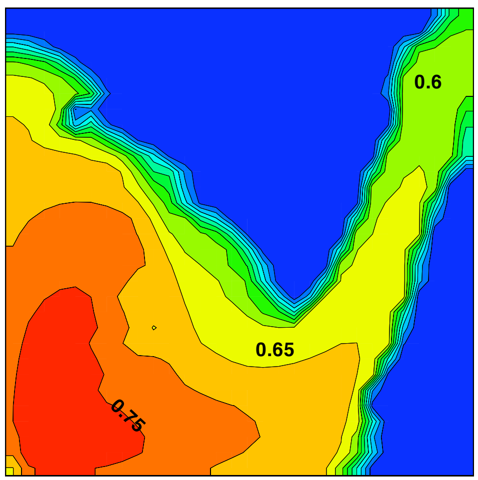
We can also rewrite the problem as

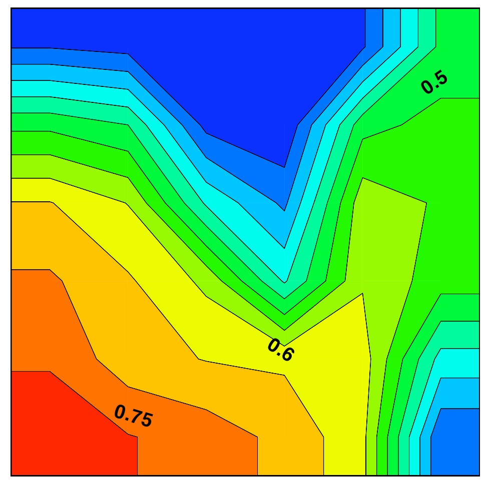
Find $(\bar{p}, \bar{\mathbf{u}}) \in \bar{W} \times \bar{\mathbf{V}}$ such that

$$(a_\epsilon^{-1}\bar{\mathbf{u}}, \bar{\mathbf{v}}) - (a_\epsilon^{-1}\hat{\mathbf{u}}'(\bar{\mathbf{u}}), \hat{\mathbf{u}}'(\bar{\mathbf{v}})) = (\bar{p}, \nabla \cdot \bar{\mathbf{v}}) - (a_\epsilon^{-1}\tilde{\mathbf{u}}', \bar{\mathbf{v}}) \qquad \forall\ \bar{\mathbf{v}} \in \bar{\mathbf{V}}$$

$$(\nabla \cdot \bar{\mathbf{u}}, \bar{w}) = (f, \bar{w}) \qquad \forall\ \bar{w} \in \bar{W}$$

Thus the subscale correction is antidiffusive on the coarse scale.





Fine 30×30 Average a coarse 6×6

Remark: This is the main reason effective parameters *cannot* work. Multiscale ideas are needed.

Let $G_x(y)$ be the Greens function on a coarse element E

$$\begin{cases} -\nabla \cdot a_\epsilon \nabla G_x = \delta_x - 1/|E| & \text{in } E \\ -a_\epsilon \nabla G_x \cdot \nu = 0 & \text{on } \partial E \end{cases}$$

Then

$$\mathbf{u}(x) = (\bar{\mathbf{u}} \cdot \nu, a_\epsilon(x) \nabla_x G_x)_{\partial E} - (f, a_\epsilon(x) \nabla_x G_x)_E$$

and

$$(a_\epsilon^{-1} \mathbf{u}, \bar{\mathbf{v}})_E = \int_E \int_E \bar{\mathbf{u}} \cdot \nabla_x \nabla_y G_x \cdot \bar{\mathbf{v}} \, dx \, dy - \int_E f' \nabla_x G_x \cdot \bar{\mathbf{v}} \, dx$$

So the upscaled permeability tensor is a nonlocal (but confined to E) operator

$$\widehat{a}_\epsilon^{-1}(x, y) = \nabla_x \nabla_y G_x(y)$$

Moreover, there is an affine correction term related to f'.

Choose any inf-sup stable mixed space $\bar{\mathbf{V}}_H \times \bar{W}_H$ on the coarse mesh.

Formulation 1: Find $(\bar{\mathbf{u}}_H, \bar{p}_H) \in \bar{\mathbf{V}}_H \times \bar{W}_H$ such that

$$\begin{aligned}\Big(a_\epsilon^{-1}(\bar{\mathbf{u}}_H + \hat{\mathbf{u}}'(\bar{\mathbf{u}}_H)), \bar{\mathbf{v}}_H + \hat{\mathbf{u}}'(\bar{\mathbf{v}}_H)\Big)&\\ = (\bar{p}_H, \nabla\cdot\bar{\mathbf{v}}_H) - (a_\epsilon^{-1}\tilde{\mathbf{u}}', \bar{\mathbf{v}}_H) &\qquad \forall\ \bar{\mathbf{v}}_H \in \bar{\mathbf{V}}_H\\ (\nabla\cdot\bar{\mathbf{u}}_H, \bar{w}_H) = (f, \bar{w}_H) &\qquad \forall\ \bar{w}_H \in \bar{W}_H\end{aligned}$$

Then

$$\begin{aligned}\mathbf{u} \approx \mathbf{u}_H &= \bar{\mathbf{u}}_H + \hat{\mathbf{u}}'(\bar{\mathbf{u}}_H) + \tilde{\mathbf{u}}'\\ p \approx p_H &= \bar{p}_H + \hat{p}'(\bar{\mathbf{u}}_H) + \tilde{p}'\end{aligned}$$

Formulation 2: Define

$$\hat{\mathbf{V}}_H = \{\,\bar{\mathbf{v}}_H + \hat{\mathbf{u}}'(\bar{\mathbf{v}}_H)\ :\ \bar{\mathbf{v}}_H \in \bar{\mathbf{V}}_H\} \subsetneq \bar{\mathbf{V}}_H + \mathbf{V}'$$

Find $\mathbf{u}_H \in \hat{\mathbf{V}}_H + \tilde{\mathbf{u}}'$ and $\bar{p}_H \in \bar{W}_H$ such that

$$\begin{aligned}(a_\epsilon^{-1}\mathbf{u}_H, \hat{\mathbf{v}}_H) = (\bar{p}_H, \nabla\cdot\hat{\mathbf{v}}_H) &\qquad \forall\ \hat{\mathbf{v}}_H \in \hat{\mathbf{V}}_H\\ (\nabla\cdot\mathbf{u}_H, \bar{w}_H) = (f, \bar{w}_H) &\qquad \forall\ \bar{w}_H \in \bar{W}_H\end{aligned}$$

Remark: We have some multiscale finite elements!

Some (Multiscale) Mixed Finite Elements

Pressure Spaces: In all cases, we take

$$W_h = \{\bar{w} \in L^2(\Omega) : \bar{w} \text{ is constant on each coarse element } E\}$$

We deal with the fact that $W_h \notin W = L^2(\Omega)/\mathbb{R}$ in the usual way.

Velocity Space: Since

$$\bar{\mathbf{V}} \simeq \{\mathbf{v} \cdot \nu \text{ on } \partial E : E \in \mathcal{T}_h\}$$

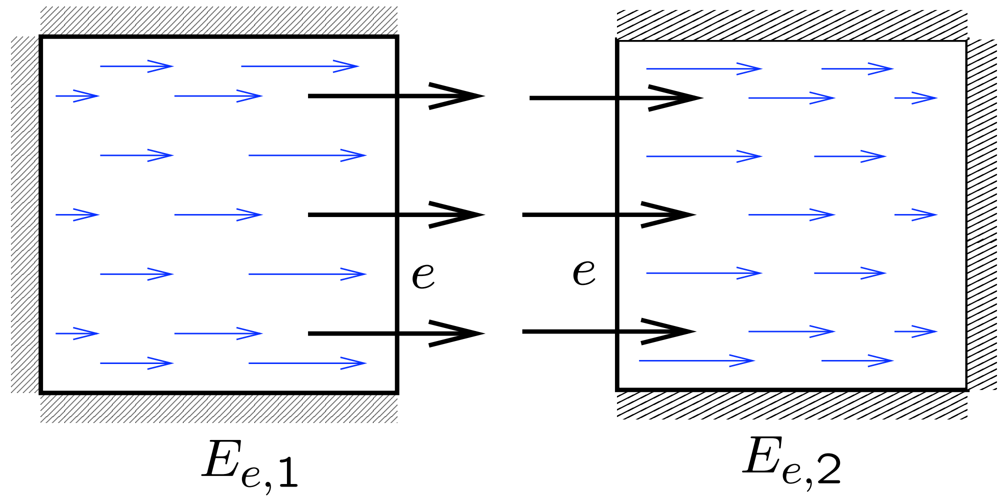
we need only specify $\bar{\mathbf{v}} \in \bar{\mathbf{V}}$ on coarse element edges $e \in \mathcal{E}_h$. We obtain the corresponding multiscale finite element $\mathbf{v}_h$ by solving the local Neumann problem.

Define $\mathbf{v}_e^{\mathrm{RT0}} \in V_h^{\mathrm{RT0}}$ for each coarse element edge $e \in \mathcal{E}_h$.

Element definition:

For each edge $e \subset \partial E$, solve

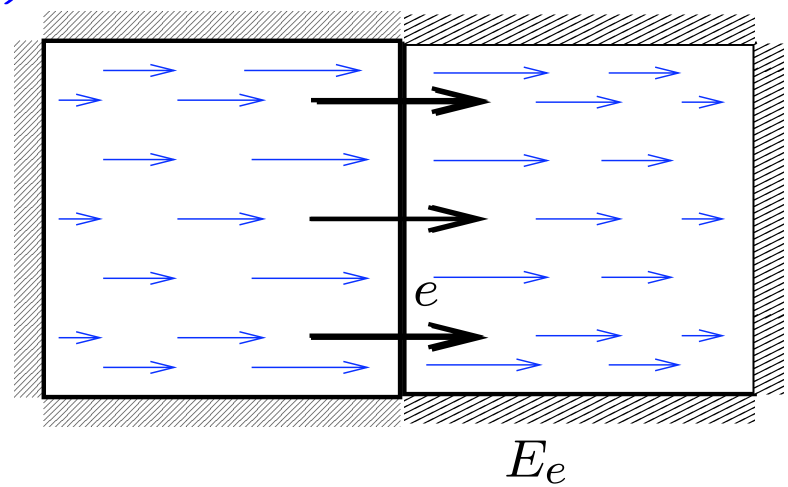
$$\begin{cases} \mathbf{v}_e^{\mathrm{RT0}} = -\nabla \phi_e^{\mathrm{RT0}} & \text{in } E, \\ \nabla \cdot \mathbf{v}_e^{\mathrm{RT0}} = \pm |e|/|E| & \text{in } E, \\ \mathbf{v}_e^{\mathrm{RT0}} \cdot \nu = \begin{cases} 0 & \text{on } \partial E \setminus e, \\ 1 & \text{on } e, \end{cases} \end{cases}$$



Dual-support definition (rectangular case):

For each edge $e \in \mathcal{E}_h$, solve

$$\begin{cases} \mathbf{v}_e^{\mathrm{RT0}} = -\nabla \phi_e^{\mathrm{RT0}} & \text{in } E_e, \\ \nabla \cdot \mathbf{v}_e^{\mathrm{RT0}} = \pm |e|/|E_{e,i}| & \text{in } E_{e,i}, \ i = 1, 2, \\ \mathbf{v}_e^{\mathrm{RT0}} \cdot \nu = 0 & \text{on } \partial E_e. \end{cases}$$



Theorem: (Raviart & Thomas, 1977)

$$\|\mathbf{u} - \mathbf{u}_h^{\mathsf{RT0}}\|_0 \leq C\|\mathbf{u}\|_1 h = \mathcal{O}\left(\frac{h}{\epsilon^2}\right)$$

Remark: These elements have no dependence on the scale ϵ. They are accurate only when $h < \epsilon$, i.e., h resolves the fine-scale heterogeneity.

Elements Based on the Heterogeneity

The main idea of multiscale finite elements is to use a_ϵ in their definition. In the boundary value problems used to define $\mathbf{v}_e^{\mathsf{RT0}} \in \mathbf{V}_h^{\mathsf{RT0}}$, simply insert the coefficient a_ϵ.

Variational Multiscale Element (ME0) Based on RT0

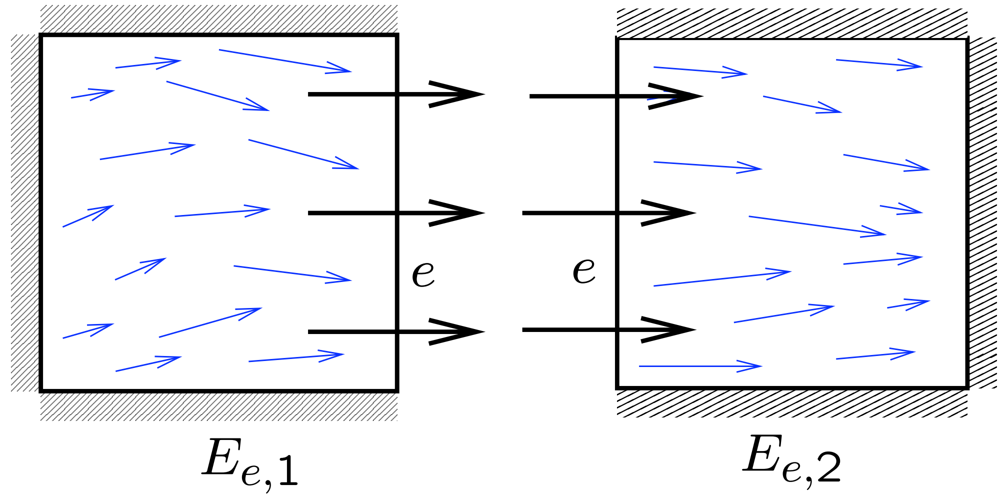
(Arbogast, Minkoff & Keenan 1998, Chen & Hou 2003)

Define $\mathbf{v}_e^{\text{ME0}} \in V_h^{\text{ME0}}$ for each coarse element edge $e \in \mathcal{E}_h$.

Element definition:

For each edge $e \subset \partial E$, solve

$$\begin{cases} \mathbf{v}_e^{\text{ME0}} = -a_\epsilon \nabla \phi_e^{\text{ME0}} & \text{in } E, \\ \nabla \cdot \mathbf{v}_e^{\text{ME0}} = \pm |e|/|E| & \text{in } E, \\ \mathbf{v}_e^{\text{ME0}} \cdot \nu = \begin{cases} 0 & \text{on } \partial E \setminus e, \\ 1 & \text{on } e, \end{cases} \end{cases}$$



Theorem: (Arbogast '04; Chen & Hou '03; Arbogast & Boyd '06)

$$\|\mathbf{u} - \mathbf{u}_h^{\text{ME0}}\|_0 \le C \|\mathbf{u}\|_1 h,$$

$$\|\mathbf{u} - \mathbf{u}_h^{\text{ME0}}\|_0 \le C \Big\{ h \|\mathbf{u}_0\|_1 + \epsilon \|\mathbf{u}_0\|_0 + \sqrt{\epsilon/h} \|\mathbf{u}_0\|_{0,\infty} \Big\},$$

where $\mathbf{u}_0$ is a smooth function independent of ϵ. Thus,

$$\|\mathbf{u} - \mathbf{u}_h^{\text{ME0}}\|_0 = \mathcal{O}\left(\min \left\{ \frac{h}{\epsilon}, \; h + \epsilon + \sqrt{\frac{\epsilon}{h}} \right\} \right)$$

Multiscale Dual-Support (MD) Elements

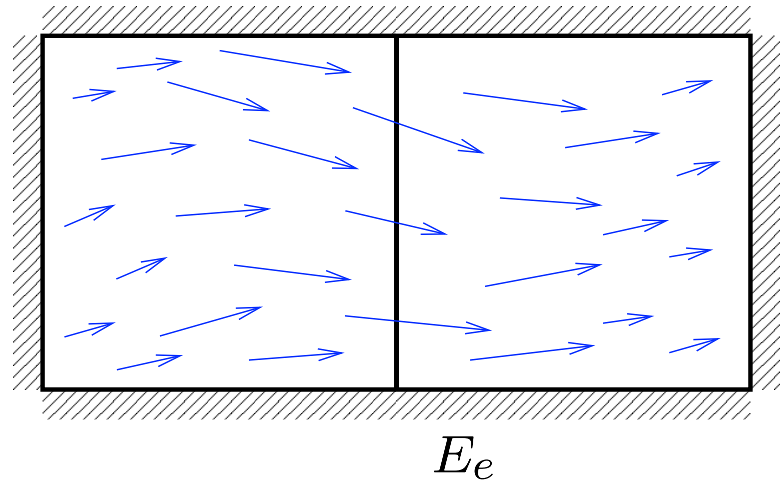
(Aarnes, 2004; Aarnes, Krogstad, Lie, 2006)

Define $\mathbf{v}_e^{\mathrm{MD}} \in V_h^{\mathrm{MD}}$ for each coarse element edge $e \in \mathcal{E}_h$.

Dual support definition (rectangular case):

For each edge $e \in \mathcal{E}_h$, solve

$$\begin{cases} \mathbf{v}_e^{\mathrm{MD}} = -a_\epsilon \nabla \phi_e^{\mathrm{MD}} & \text{in } E_e, \\ \nabla \cdot \mathbf{v}_e^{\mathrm{MD}} = \pm |e|/|E_{e,i}| & \text{in } E_{e,i},\ i = 1, 2, \\ \mathbf{v}_e^{\mathrm{MD}} \cdot \nu = 0 & \text{on } \partial E_e. \end{cases}$$



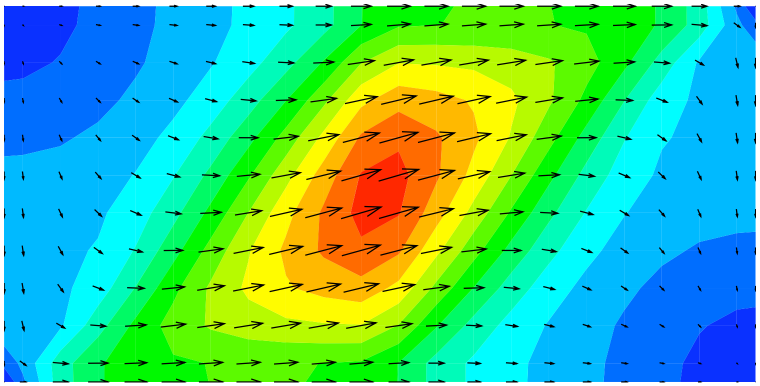
A problem: Anisotropy!

Counterexample to Convergence of MD

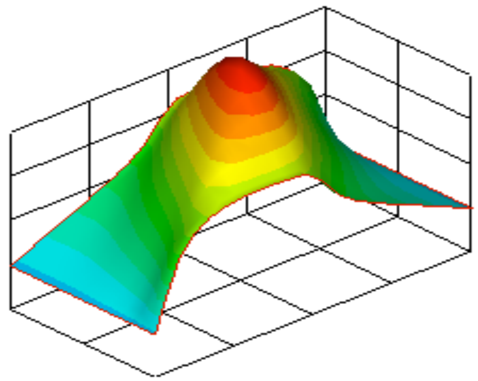
Take a constant

$$a_\epsilon(x) = a = Q\Lambda Q^T \quad \text{with } \Lambda = \begin{pmatrix} 100 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } Q = 30^\circ \text{ rotation.}$$

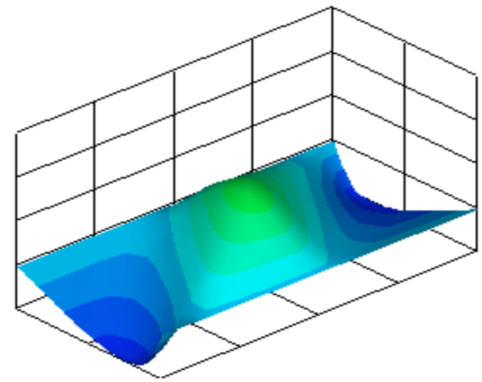
We have a genuine anisotropy, but no microstructure.



Velocity and Speed

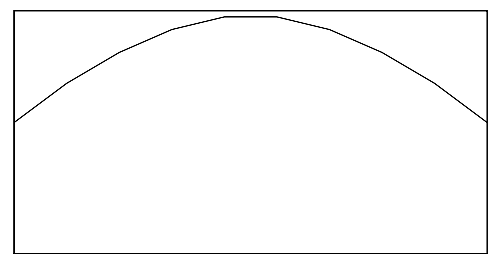


x-velocity



y-velocity

The space $\mathbf{V}_h^{\mathsf{MD}}$ cannot reproduce constants, so the method cannot converge in any reasonable sense as $h \to 0$.



Normal trace on e

(Brezzi, Douglas, Marini 1985; Arbogast 2000)

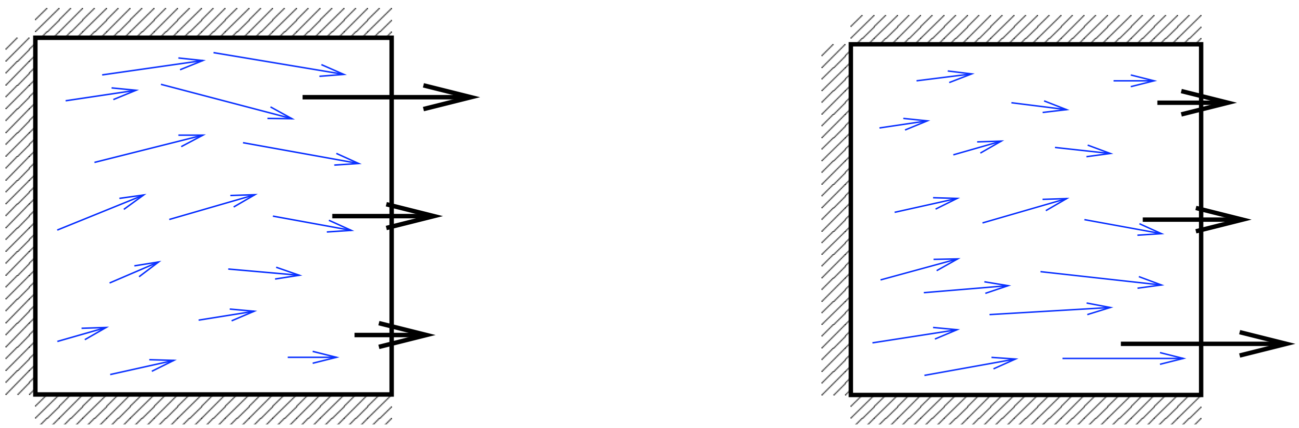
Standard BDM1 Elements: The BDM1 elements have two degrees of freedom per element edge. That is

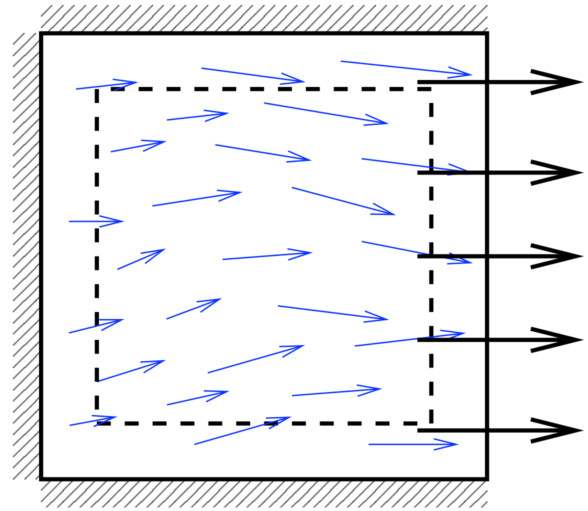
$$\mathbf{v} \cdot \nu|_e \text{ is a linear function for each edge } e \in \partial E$$

Moreover

$$\nabla \cdot \mathbf{v}|_E \text{ is a constant on each element } E \quad \Longrightarrow \quad \nabla \cdot \mathbf{v} \in W_h$$

Multiscale ME1 Elements:





Oversampled elements (OS) (Hou et al., 1997, 2003)
Solve on a larger domain and restrict back to E. Leads to a nonconforming method.

Reduced dimension-based elements (Hou, Wu 1997)
Solve a reduced dimension problem on each edge $e \subset \partial E$ to set $\mathbf{v} \cdot \nu$ on e.

Generalized finite elements and partition of unity methods (Babuška et al. 1983, 1994, 2001)
Create a multiscale finite element basis from a partition of unity modified by local multiscale functions.

Local eigenfunction-based elements (Efendiev, Galvis 2009; Hetmaniuk, Lehoucq 2010)
Base $\mathbf{v} \cdot \nu$ on solutions to local eigenfunction problems.

Microscale Structure from Homogenization and a New Mixed Multiscale Finite Element

Suppose that a_ϵ is locally periodic of period ϵ. Then

$$a_\epsilon(x) = a(x, x/\epsilon)$$

where $a(x,y)$ is periodic in y of period 1 on the unit cube Y.

Let a_0 be the homogenized permeability matrix, defined by

$$a_{0,ij}(x) = \int_Y a(x,y)\left(\delta_{ij} + \frac{\partial \omega_j(x,y)}{\partial y_i}\right) dy$$

where, for fixed x, $\omega_j(x,y)$ is the Y-periodic solution of

$$-\nabla_y \cdot (a \nabla_y \omega_j) = \nabla \cdot (a \mathbf{e}_j)$$

Homogenized solution: Let $(\mathbf{u}_0, p_0)$ solve

$$\begin{cases} \mathbf{u}_0 = -a_0 \nabla p_0 & \text{in } \Omega \\ \nabla \cdot \mathbf{u}_0 = f & \text{in } \Omega \\ \mathbf{u}_0 \cdot \nu = 0 & \text{on } \partial\Omega \end{cases}$$

Then $(\mathbf{u}_0, p_0)$ is a smooth "approximation" of $(\mathbf{u}, p)$.

Theorem: Assume that $p_0 \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$. Let $\alpha_0 = a_0^{-1}$ and define the fixed tensor independent of ϵ and the domain Ω

$$\mathcal{A}_{ij}(x,y) = \sum_{k,\ell} a_{ik}(x,y)\Big(\delta_{k\ell} + \frac{\partial \omega_\ell(x,y)}{\partial y_k}\Big)\alpha_{0,\ell j} \quad \Longleftrightarrow \quad \mathcal{A} = a\,(I + D\omega)\alpha_0$$

Let

$$\mathcal{A}_\epsilon(x) = \mathcal{A}(x, x/\epsilon)$$

Then

$$\boxed{\mathbf{u}_\epsilon(x) = \mathcal{A}_\epsilon(x)\,\mathbf{u}_0(x) + \theta_\epsilon^\Omega(x)}$$

where

$$\|\theta_\epsilon^\Omega\|_0 \le C\Big\{\epsilon\|\mathbf{u}_0\|_1 + \sqrt{\epsilon|\partial\Omega|}\|\mathbf{u}_0\|_{0,\infty}\Big\} = \mathcal{O}\big(\epsilon + \sqrt{\epsilon}\big)$$

Consequence:

$$\mathbf{u}_\epsilon \approx \mathcal{A}_\epsilon \mathbf{u}_0 \qquad \Longrightarrow \qquad \mathbf{V}_h \approx \{\mathcal{A}_\epsilon \mathbf{v} : \mathbf{v} \text{ is some nice smooth function}\}.$$

However, these finite elements lie outside $H(\text{div};\Omega)$.

Homogenization-Based Multiscale (HE) Element—1

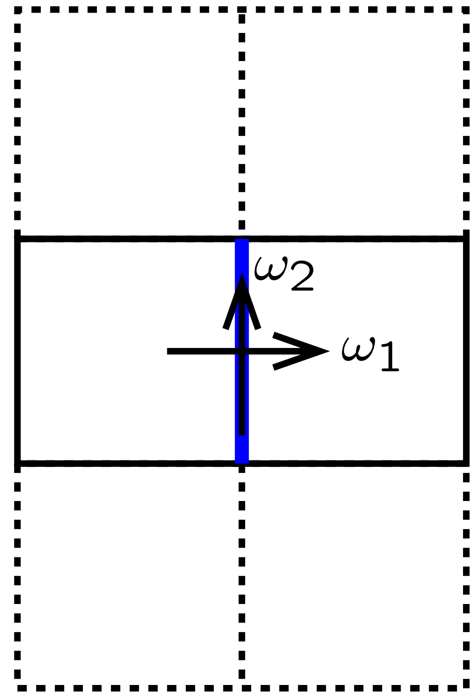
Key idea. On each edge e, we use only the normal trace and piecewise constant approximation to define $\mathbf{v}_e^{\mathsf{HE}} \in V_h^{\mathsf{HE}}$.

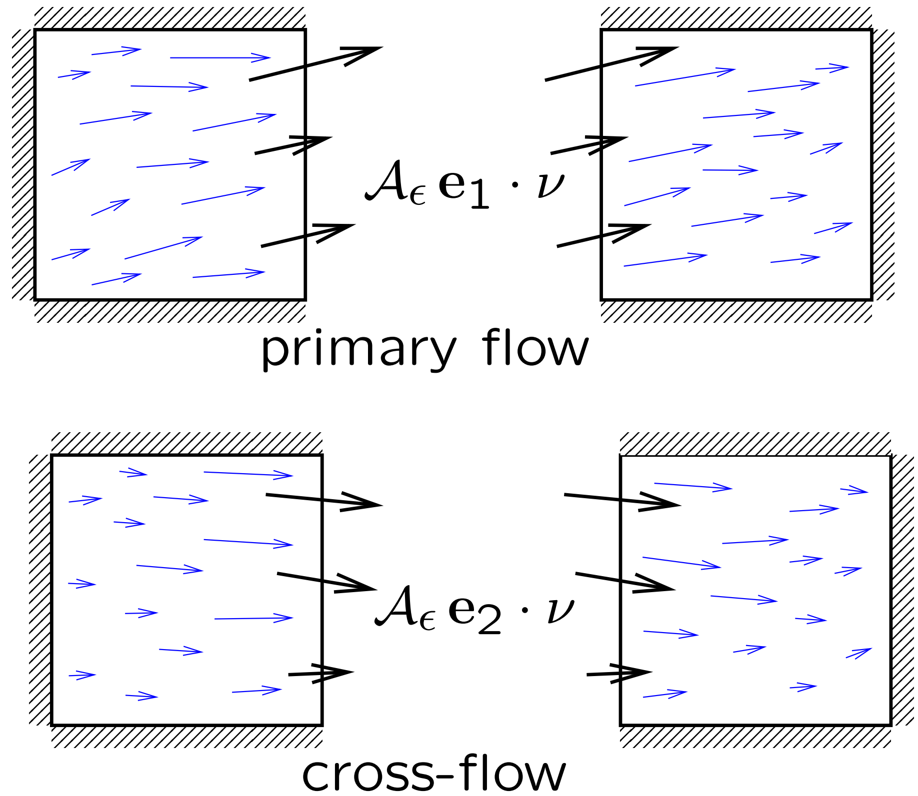
$$\mathbf{u}_\epsilon \cdot \nu \approx \mathcal{A}_\epsilon \mathbf{u}_0 \cdot \nu \sim \mathcal{A}_\epsilon \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \cdot \nu = \alpha \mathcal{A}_\epsilon \mathbf{e}_1 \cdot \nu + \beta \mathcal{A}_\epsilon \mathbf{e}_2 \cdot \nu \qquad \text{for } \alpha, \beta \in \mathbb{R}$$

Step 1: On E_e (or a larger oversampled domain), find the periodic solution $\omega_j(x)$ of

$$-\nabla \cdot (a \nabla \omega_j) = \frac{\partial a}{\partial x_j}$$

Compute $\mathcal{A}_\epsilon$ from ω_1 and ω_2, and extract $\mathcal{A}_\epsilon \mathbf{e}_1 \cdot \nu$ and $\mathcal{A}_\epsilon \mathbf{e}_2 \cdot \nu$ on e.





Step 2: On each E, $\partial E \supset e$,

$$\begin{cases} \mathbf{v}_e^{\mathsf{HE},i} = -a_\epsilon \nabla \phi_e^{\mathsf{HE},i} & \text{in } E, \\ \nabla \cdot \mathbf{v}_e^{\mathsf{HE},i} = \pm |e|/|E| & \text{in } E, \\ \mathbf{v}_e^{\mathsf{HE},i} \cdot \nu = \begin{cases} 0 & \text{on } \partial E \backslash e, \\ \mathcal{A}_\epsilon \mathbf{e}_i \cdot \nu & \text{on } e, \end{cases} \end{cases}$$

Remarks:

- This is *not* a dual-support element, but we use E_e in the definition.
- We sample the microstructure a_ϵ more thoroughly than ME0.
- It has twice the number of degrees of freedom as MD (same as ME1).

An Error Analysis

Theorem (Arbogast 2004) $\mathbf{u} \approx \mathbf{u}_h = \bar{\mathbf{u}}_h + \hat{\mathbf{u}}'(\bar{\mathbf{u}}_h) + \tilde{\mathbf{u}}'$

$$\|a_\epsilon^{-1/2}(\mathbf{u} - \mathbf{u}_h)\|_0 \leq \inf_{\substack{\mathbf{v}_h \in \bar{\mathbf{V}}_h + \mathbf{V}' \\ \nabla\cdot\mathbf{v}_h = f}} \|a_\epsilon^{-1/2}(\mathbf{u} - \mathbf{v}_h)\|_0$$

$$\nabla \cdot \mathbf{u}_h = f$$

Remarks:

1. We have assumed that the upscaling operator is solved exactly, since it can be well resolved on a fine grid.
2. The method is locally conservative on the (fully resolved) fine scale.
3. We can show optimal polynomial convergence rates in
   - h for the coarse part
   - h_f for the fine part
   - h_f for the divergence
4. Optimality is over the large space $\bar{\mathbf{V}}_h + \mathbf{V}'$, so the best approximation has an energy minimizing fine part with respect to the coarse part. That is, the optimal solution is in $\{\bar{\mathbf{v}}_h + \hat{\mathbf{u}}'(\bar{\mathbf{v}}_h) : \bar{\mathbf{v}}_h \in \bar{\mathbf{V}}_h\} \subsetneq \bar{\mathbf{V}}_h + \mathbf{V}'$

Multiscale Convergence

We present a multiscale error analysis for ME0.

- We quantify the error in terms of h and ϵ.
- The proofs are based on comparison to the homogenized solution.
- The style of proof is due to Hou, Wu, and Cai 1999. See also
  - Efendiev, Hou, and Wu 2000
  - Chen and Hou 2003 (mixed case)
  - Arbogast and Boyd 2006 (mixed case)

We present a simplified proof involving

- certain projection operators
- four key results
- we saw the first key result from homogenization theory

$$\mathbf{u}_\epsilon(x) = \mathcal{A}_\epsilon(x)\,\mathbf{u}_0(x) + \mathcal{O}(\sqrt{\epsilon}) \tag{1}$$

- a one line proof

We show where MD fails, and conjecture that HE works.

Assume $a_\epsilon(x)$ is smooth and positive definite:

$$a_*|\xi|^2 \le \xi^T \alpha_\epsilon(x)\, \xi \le a^*|\xi|^2 \quad \forall x \in \Omega.$$

Let $\mathcal{P}_{W_h}$ denote L^2-projection into W_h.

Lemma: (Quasi-optimality) If $\nabla \cdot \mathbf{V}_h \subset W_h$, then

$$\text{(2)} \qquad \|\mathbf{u}_\epsilon - \mathbf{u}_h\|_0 \le \sqrt{\frac{a^*}{a_*}} \|\mathbf{u}_\epsilon - \mathbf{v}\|_0$$

for any $\mathbf{v} \in \mathbf{V}_h$ such that $\nabla \cdot \mathbf{v} = \mathcal{P}_{W_h} \nabla \cdot \mathbf{u}_\epsilon$.

Goal: Find any $\mathbf{v}_\epsilon \approx \mathbf{u}_\epsilon$ in $\mathbf{V}_h^{\mathsf{M}}$ with $\nabla \cdot \mathbf{v}_\epsilon = \mathcal{P}_{W_h} \nabla \cdot \mathbf{u}_\epsilon$.

Key idea: To deal with the ϵ scale of our finite elements, define corresponding homogenized finite elements.

Replace the true coefficient in the definition of the finite elements with the corresponding homogenized one.

$$\text{ME0}: \quad a_\epsilon \longmapsto a_0$$

$$\mathbf{V}_{0,h}^{\text{ME0}} = \underset{e\in\mathcal{E}_h}{\text{span}}\{\mathbf{v}_{0,e}^{\text{ME0}}\}$$

Since our finite elements are defined by boundary value problems, the homogenization theorem applies.

Lemma: For each $e \in \mathcal{E}_h$

$$\mathbf{v}_e^{\mathsf{ME0}} = \mathcal{A}_\epsilon \mathbf{v}_{0,e}^{\mathsf{ME0}} + \theta_\epsilon^{E_e,\mathsf{ME0}}$$

where

$$\|\theta_\epsilon^{E_e,\mathsf{ME0}}\|_{0,E_e} \le C\Big\{\epsilon\|\mathbf{v}_{0,e}^{\mathsf{ME0}}\|_{1,E_e} + \sqrt{\epsilon|\partial E_e|}\|\mathbf{v}_{0,e}^{\mathsf{ME0}}\|_{0,\infty,E_e}\Big\}$$
$$= \mathcal{O}\bigg(\bigg\{\frac{\epsilon}{h} + \sqrt{\frac{\epsilon}{h}}\bigg\}h^{d/2}\bigg)$$

Remark: We see numerical resonance (i.e., factors of ϵ/h) here in the estimate. These terms come from localizing to the element E_e.

The average normal flux across $e \in \mathcal{E}_h$ is

$$\gamma_e = \frac{1}{|e|} \int_e \mathbf{v} \cdot \nu_e \, ds$$

The Raviart-Thomas projection is

$$\pi^{\mathsf{RT0}} \mathbf{v} = \sum_{e \in \mathcal{E}_h} \gamma_e \mathbf{v}_e^{\mathsf{RT0}} \in \mathbf{V}_h^{\mathsf{RT0}}$$

Similarly, define

$$\pi_\epsilon^{\mathsf{ME0}} \mathbf{v} = \sum_{e \in \mathcal{E}_h} \gamma_e \mathbf{v}_e^{\mathsf{ME0}} \in \mathbf{V}_h^{\mathsf{ME0}} \quad \text{and} \quad \pi_0^{\mathsf{ME0}} \mathbf{v} = \sum_{e \in \mathcal{E}_h} \gamma_e \mathbf{v}_{0,e}^{\mathsf{ME0}} \in \mathbf{V}_{0,h}^{\mathsf{ME0}}$$

Lemma:

$$\nabla \cdot \pi_\epsilon^{\mathsf{ME0}} \mathbf{v} = \nabla \cdot \pi_0^{\mathsf{ME0}} \mathbf{v} = \nabla \cdot \pi^{\mathsf{RT0}} \mathbf{v} = \mathcal{P}_{W_h} \nabla \cdot \mathbf{v}$$

Lemma:

$$(3) \qquad \|\pi_\epsilon^{\mathrm{ME0}}\mathbf{v} - \mathcal{A}_\epsilon \pi_0^{\mathrm{ME0}}\mathbf{v}\|_0 \le C\|\mathbf{v}\|_1\left(\epsilon/h + \sqrt{\epsilon/h}\right)$$

Proof:

$$\pi_\epsilon^{\mathrm{ME0}}\mathbf{v} - \mathcal{A}_\epsilon \pi_0^{\mathrm{ME0}}\mathbf{v} = \sum_{e\in\mathcal{E}_h} \gamma_e(\mathbf{v}_e^{\mathrm{ME0}} - \mathcal{A}_\epsilon \mathbf{v}_{0,e}^{\mathrm{ME0}}) = \sum_{e\in\mathcal{E}_h} \gamma_e \theta_e^{E_e,\mathrm{ME0}}$$

$\Longrightarrow$

$$\begin{aligned}
\|\pi_\epsilon^{\mathrm{ME0}}\mathbf{v} - \mathcal{A}_\epsilon \pi_0^{\mathrm{ME0}}\mathbf{v}\|_{0,E} &\le \sum_{e\subset\partial E} |\gamma_e|\,\|\theta_e^{E_e,\mathrm{ME0}}\|_{0,E} \\
&\le C \sum_{e\subset\partial E} \left(h^{-d/2}\|\mathbf{v}\|_{1,E_e}\right)\left(\left\{\frac{\epsilon}{h} + \sqrt{\frac{\epsilon}{h}}\right\}h^{d/2}\right) \\
&= C \sum_{e\subset\partial E} \|\mathbf{v}\|_{1,E_e}\left(\frac{\epsilon}{h} + \sqrt{\frac{\epsilon}{h}}\right) \qquad \square
\end{aligned}$$

Lemma: If $\mathbf{v}_0 = -a_0 \nabla \phi_0$, then

$$(4) \qquad \|\mathbf{v}_0 - \pi_0^{\mathrm{ME0}} \mathbf{v}_0\|_0 \le C \|\mathbf{v}_0\|_1 h$$

Proof:

$$\psi = \mathbf{v} - \pi_0^{\mathrm{ME0}} \mathbf{v} = -a_0 \nabla \Big(\phi_0 - \sum_{e \subset \partial E} \gamma_e \phi_{0,e}^{\mathrm{ME0}} \Big) \quad \text{in } E$$

is a potential field satisfying the Neumann problem

$$\begin{aligned} \nabla \cdot \psi &= \nabla \cdot \mathbf{v}_0 - \mathcal{P}_{W_h} \nabla \cdot \mathbf{v}_0 && \text{in } E \\ \psi \cdot \nu_e &= \mathbf{v}_0 \cdot \nu_e - \gamma_e && \text{on } e \subset \partial E \end{aligned}$$

The standard energy estimate gives the result. □

Remarks:

- The counterexamples show that similar results cannot hold for MD.
- Conjecture: A similar result holds for HE, since

$$\mathcal{A}_\epsilon \longrightarrow I \quad \text{as } \epsilon \to 0$$

Convergence Theorem

Theorem: If Ω has elliptic regularity and $p_0 \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$, then

$$\|\mathbf{u}_\epsilon - \mathbf{u}_h^{\mathrm{ME0}}\|_0 + \|\mathcal{P}_{W_h} p_\epsilon - p_h\|_0 \leq C\Big\{\Big(\epsilon + \epsilon/h + \sqrt{\epsilon/h} + h\Big)\|\mathbf{u}_0\|_1 + \sqrt{\epsilon}\,\|\mathbf{u}_0\|_{0,\infty}\Big\}$$

$$\nabla \cdot \mathbf{u}_h^{\mathrm{ME0}} = \mathcal{P}_{W_h} f \quad \text{and} \quad \|\nabla \cdot (\mathbf{u}_\epsilon - \mathbf{u}_h^{\mathrm{ME0}})\|_0 \leq C\|f\|_1 h$$

Proof:

$$\boxed{\mathbf{u}_\epsilon \approx \pi_\epsilon^{\mathrm{ME0}} \mathbf{u}_0 \in \mathbf{V}_h^{\mathrm{ME0}}} \quad \text{and} \quad \nabla \cdot \pi_\epsilon^{\mathrm{ME0}} \mathbf{u}_0 = \mathcal{P}_{W_h} \nabla \cdot \mathbf{u}_0 = \mathcal{P}_{W_h} \mathbf{u}_\epsilon$$

$$\|\mathbf{u}_\epsilon - \mathbf{u}_h^{\mathrm{ME0}}\|_0 \leq C\|\mathbf{u}_\epsilon - \pi_\epsilon^{\mathrm{ME0}} \mathbf{u}_0\|_0$$

(2) Quasi-optimality

$$\leq C\Big\{\|\mathbf{u}_\epsilon - \mathcal{A}_\epsilon \mathbf{u}_0\|_0 + \|\mathcal{A}_\epsilon(\mathbf{u}_0 - \pi_0^{\mathrm{ME0}} \mathbf{u}_0)\|_0 + \|\mathcal{A}_\epsilon \pi_0^{\mathrm{ME0}} \mathbf{u}_0 - \pi_\epsilon^{\mathrm{ME0}} \mathbf{u}_0\|_0\Big\}$$

(1) Homogenization (4) Smooth Proj. (3) Multiscale Proj.

Divergence result follows trivially from the definitions.
Pressure result follows from the inf-sup condition. □

Remark: A similar proof holds for ME1.

Corollary: If Ω has elliptic regularity, then there is some $\beta > 0$, independent of ϵ, such that

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h^{\mathrm{ME0}}} \frac{(w_h, \nabla\cdot\mathbf{v}_h)}{\|\mathbf{v}_h\|_0 + \|\nabla\cdot\mathbf{v}_h\|_0} \geq \beta\|w_h\|_0 \quad \forall w_h \in W_h$$

Proof: Solve

$$\begin{cases} \nabla\cdot\mathbf{v}_0 = w_h & \text{in } \Omega \\ \mathbf{v}_0 = -a_0\nabla\phi_0 & \text{in } \Omega \\ \mathbf{v}_0\cdot\nu = 0 & \text{on } \partial\Omega \end{cases} \implies \|\mathbf{v}_0\|_1 \leq C\|w_h\|_0$$

Take

$$\mathbf{v}_h = \pi_\epsilon^{\mathrm{ME0}}\mathbf{v}_0 \in \mathbf{V}_h^{\mathrm{ME0}} \implies \nabla\cdot\mathbf{v}_h = \mathcal{P}_{W_h}\nabla\cdot\mathbf{v}_0 = w_h$$

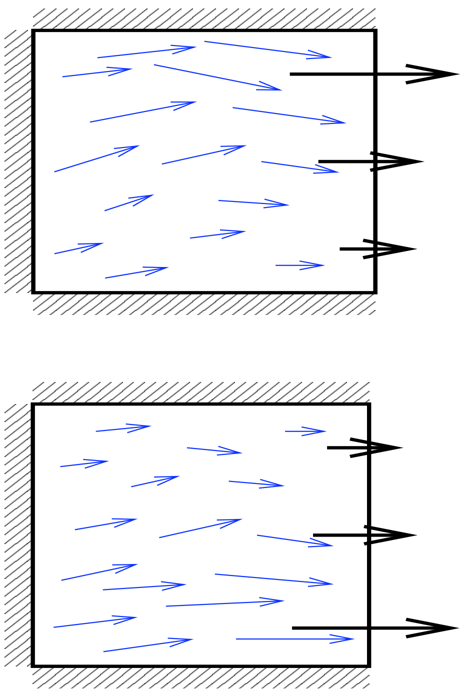
Then

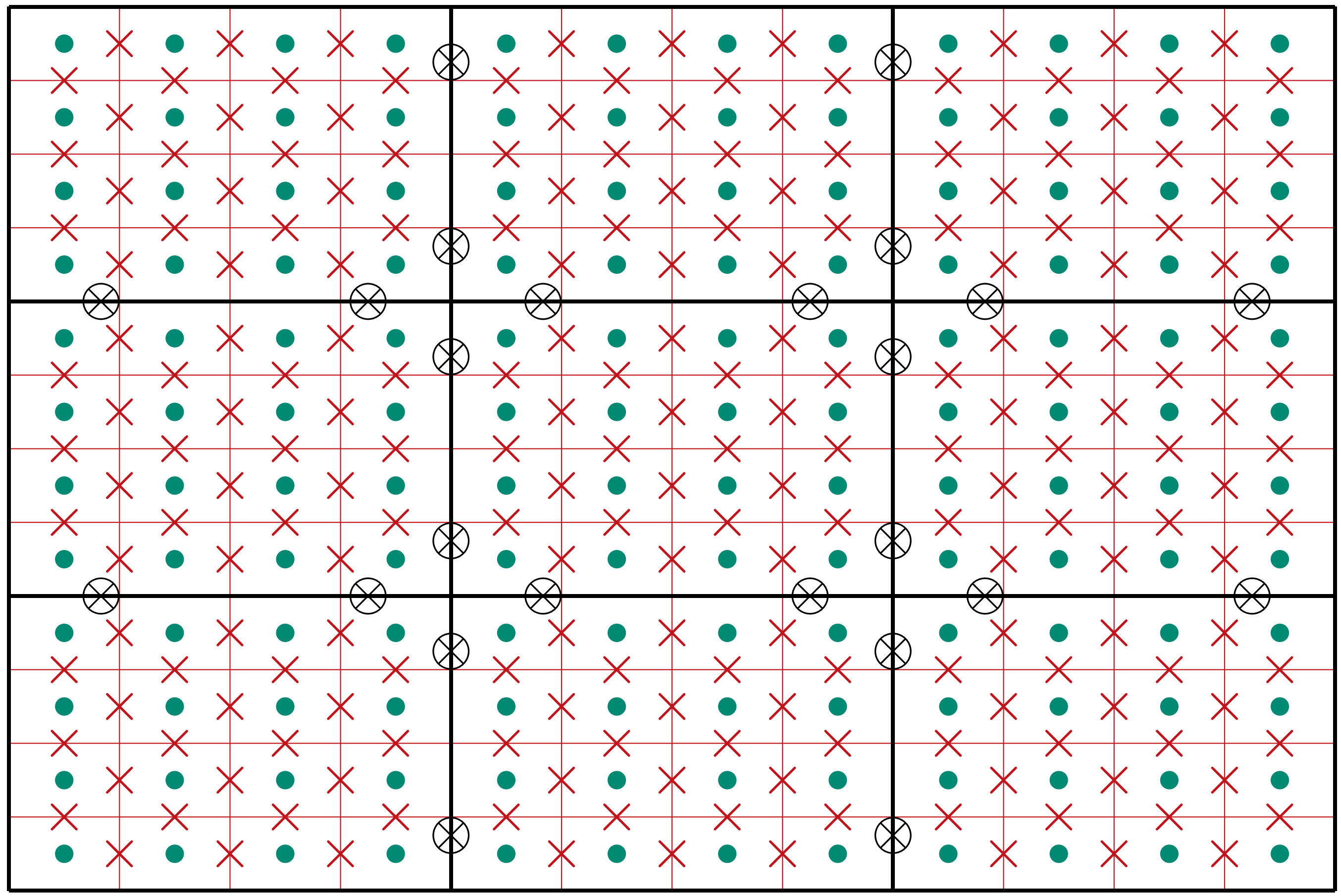
$$\|\mathbf{v}_h\|_0 \leq \|\underbrace{\pi_\epsilon^{\mathrm{ME0}}\mathbf{v}_0 - \mathcal{A}_\epsilon\pi_0^{\mathrm{ME0}}}_{(3)}\mathbf{v}_0\|_0 + \|\mathcal{A}_\epsilon(\underbrace{\pi_0^{\mathrm{ME0}}\mathbf{v}_0 - \mathbf{v}_0}_{(4)})\|_0 + \|\mathcal{A}_\epsilon\mathbf{v}_0\|_0$$

$$\leq C\|\mathbf{v}_0\|_1 \leq C\|w_h\|_0 \qquad \square$$

Some Numerical Results

We use RT0 for the fine scales in all cases.





• Pressure ⊗ Coarse velocity (linear) × Subgrid velocity

We fully resolve a and f (using the variational multiscale correction affine correction term), but we only partially couple the dynamics.

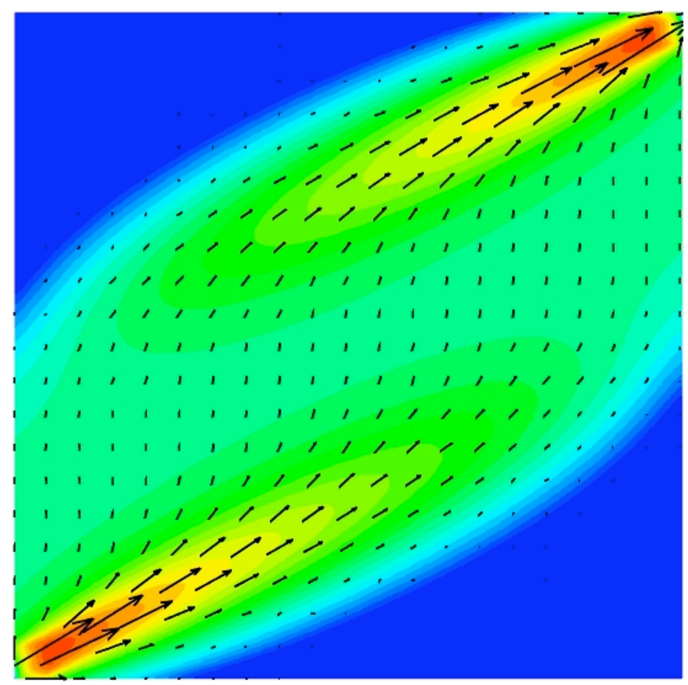
- Constant

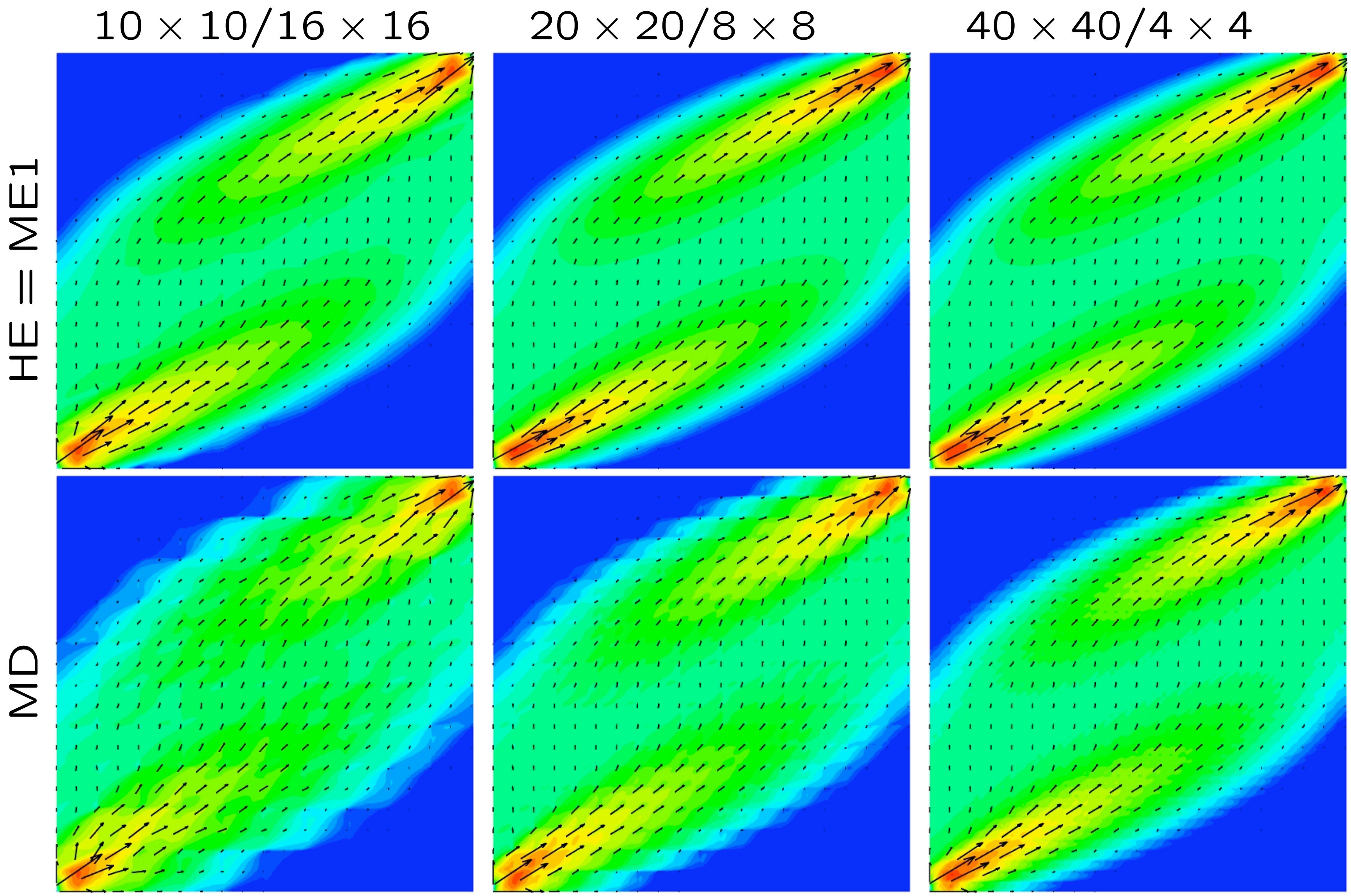
$$a = \begin{pmatrix} 80.8 & 39.4 \\ 39.4 & 21.7 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 100.5 & 0 \\ 0 & 1.99 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

- In this case, $\omega_j = 0$ for each j.
  - Half the basis functions have constant fluxes across coarse edges
  - Half the "basis functions" vanish, so HE = ME0. Reset to linear

Then HE = ME1 (and oversampling does not help).

- Unit square $\Omega = (0,1)^2$ with an injection well in the lower left corner and a production well of opposite strength in the upper right corner.
- We take BDM1 on a 160×160 grid as the exact solution.
- Color depicts speed and the arrows show the velocity direction (and speed).





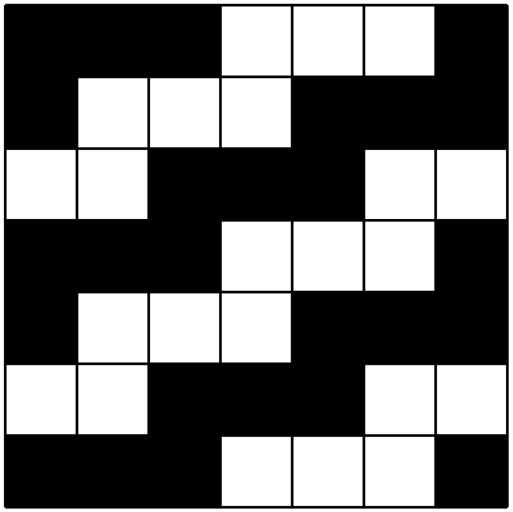
- Solution using fixed resolution $1/h = 160$
- MD exhibits a fluctuation of period H

A Constant, Anisotropic Permeability—3

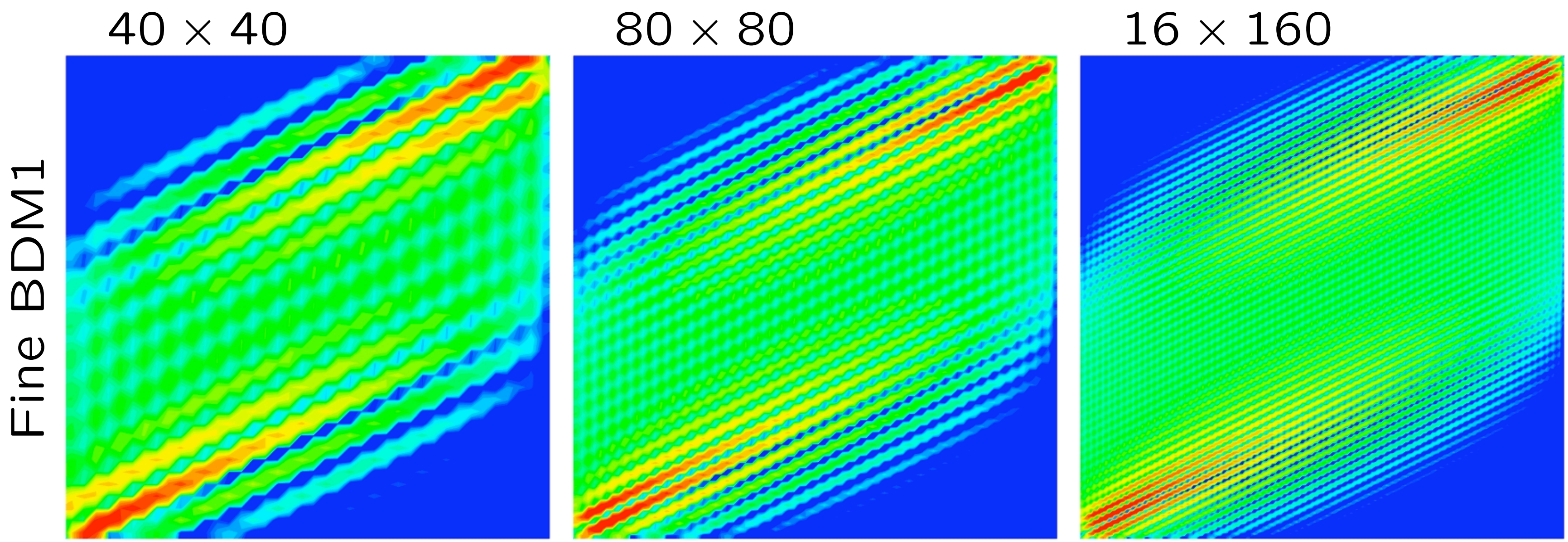
Method	Coarse mesh $N \times N$	Subgrid mesh $n \times n$	Pressure Error ℓ^2	Pressure Error ℓ^∞	Velocity Error ℓ^2	Velocity Error ℓ^∞
HE	10	16	0.0525	0.315	0.252	0.343
	20	8	0.0017	0.019	0.060	0.192
	40	4	0.0007	0.007	0.019	0.046
	80	2	0.0006	0.006	0.012	0.016
MD	10	16	0.0551	0.286	0.371	0.358
	20	8	0.0197	0.139	0.264	0.545
	40	4	0.0077	0.060	0.144	0.357
	80	2	0.0016	0.014	0.055	0.134

Permeability pattern illustrated on 7×7 grid.

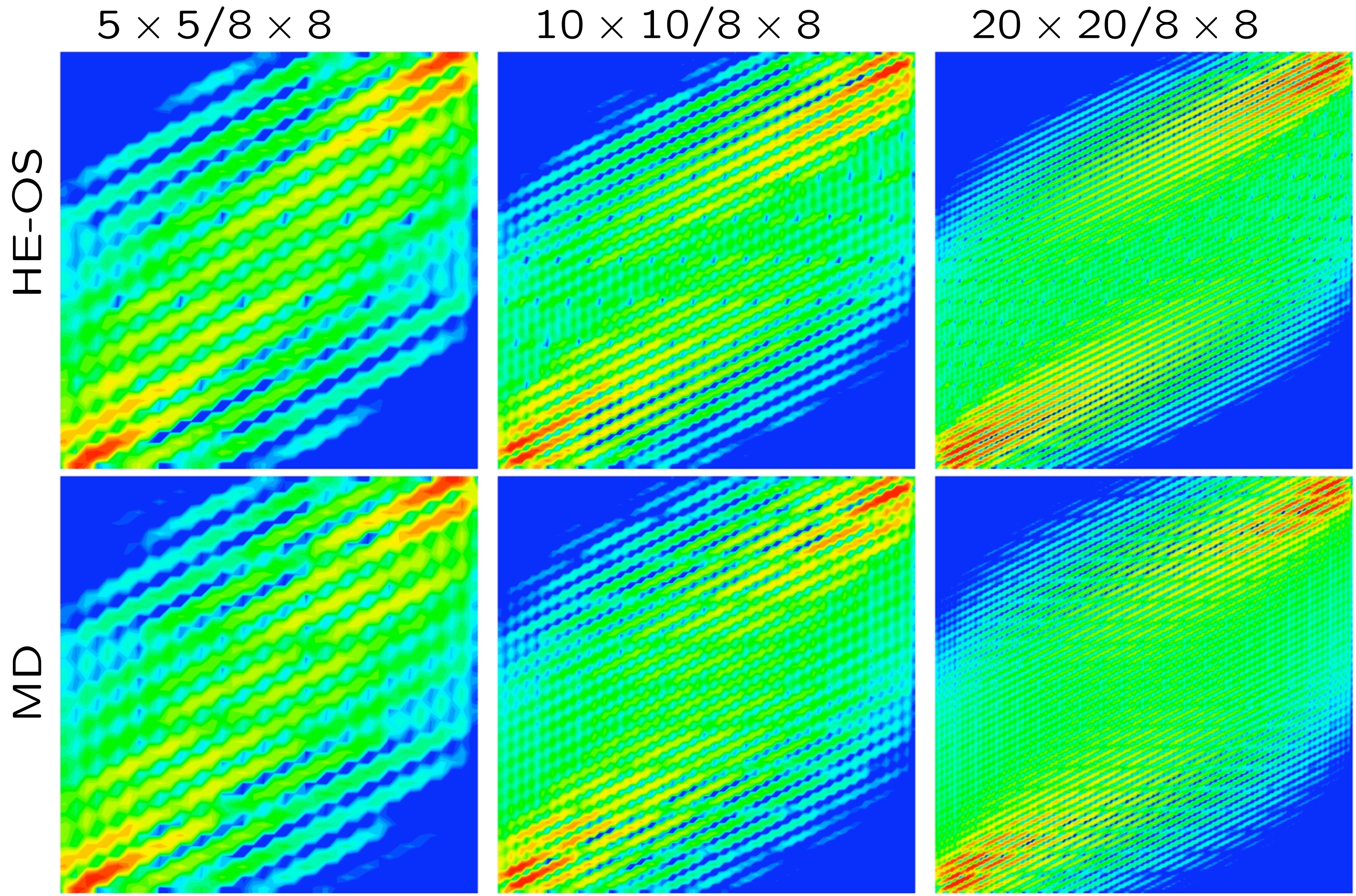
- One cell wide streaks
- Angle $\tan\theta = 1/2$ ($\theta = 26.565$)
- Alternate permeability 200 (black) and 1 (white)
- The arithmetic and harmonic means of 1 and 200 are 100.5 and 1.99



Up to the stair-step nature of the permeability streaks, these fields homogenize into the anisotropic tensor treated in the previous tests.



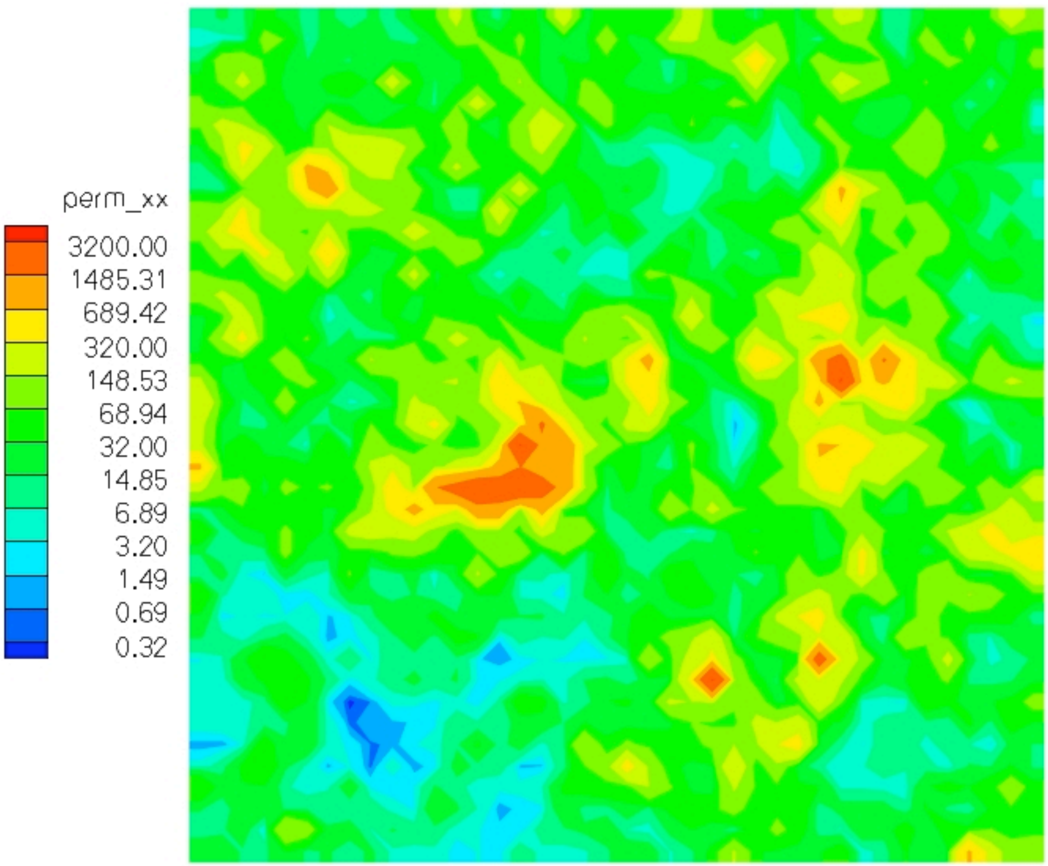
A Streaked Permeability—2

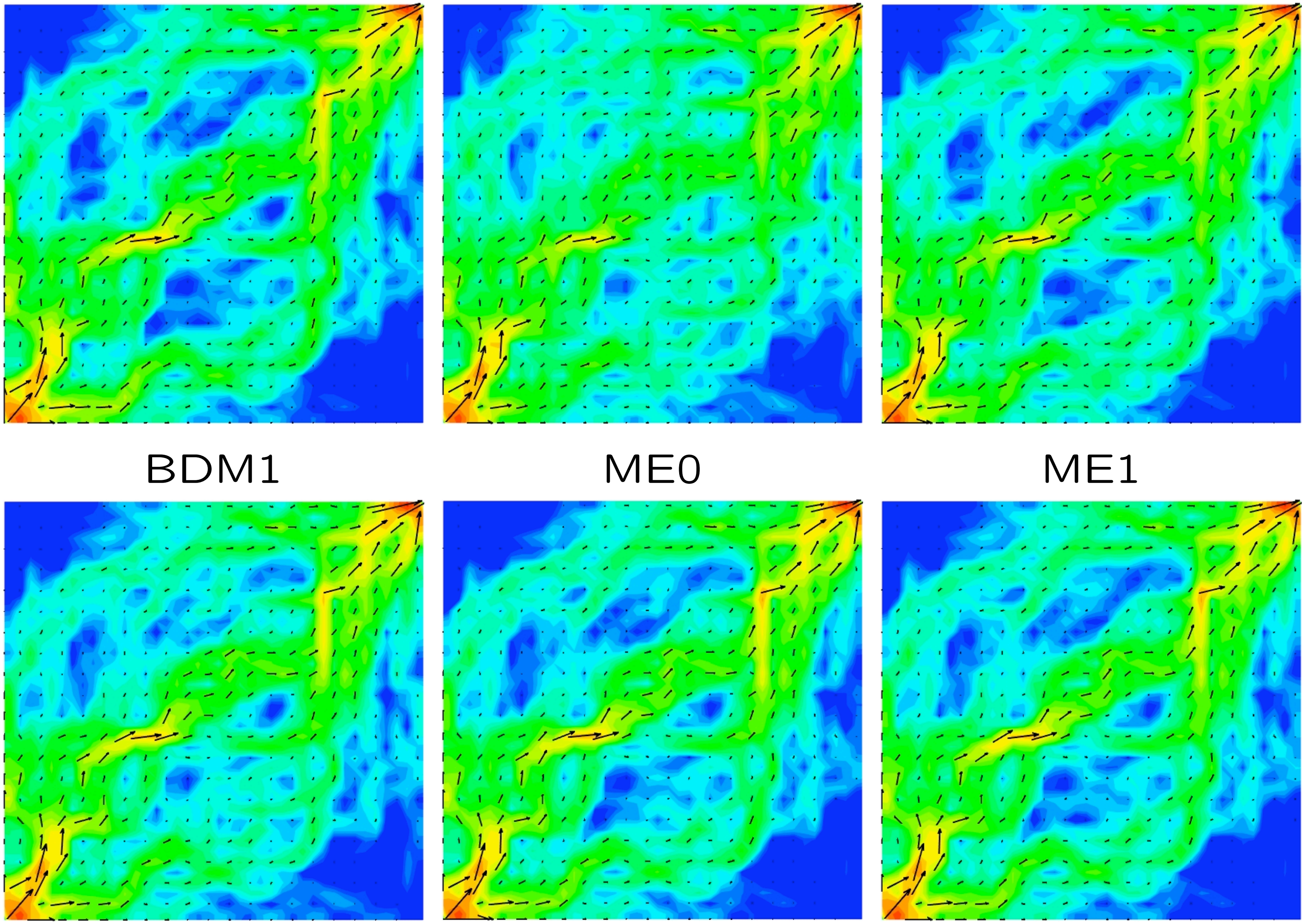


- MD is more numerically diffusive and the speed is disjointed
- MD exhibits a fluctuation across the domain of period H
- MD has difficulty since there is an induced anisotropy from the subgrid

Moderately Heterogeneous Permeability—1

- uniform 40×40 m^2 grid
- geostatistically generated permeability, mildly correlated, locally isotropic
- Permeability shown on a log scale, varies from 0.32 to 3200 millidarcy





BDM1 ME0 ME1

MD HE HE-OS

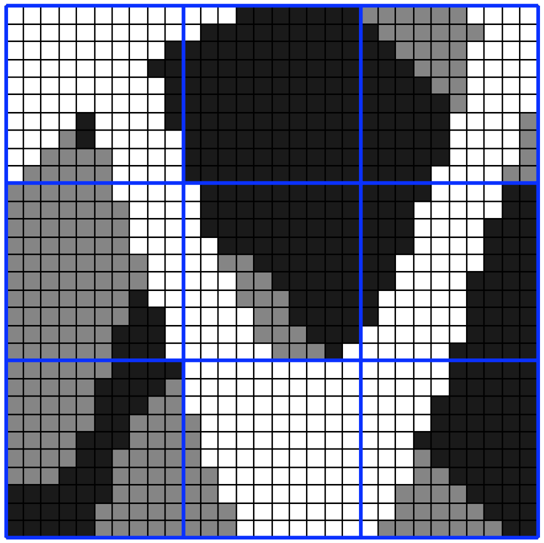
- 40×40 fine grid, 4×4 coarse grid with 10×10 subgrid
- Color depicts speed, on a log scale (arrows show velocity)

Method	Pressure Error ℓ^2	Pressure Error ℓ^∞	Velocity Error ℓ^2	Velocity Error ℓ^∞
RT0	0.04	0.03	0.03	0.03
ME0	0.16	0.21	0.29	0.26
ME1	0.10	0.16	0.19	0.17
MD	0.14	0.24	0.16	0.12
HE	0.13	0.20	0.14	0.14
HE-OS	0.14	0.19	0.12	0.11

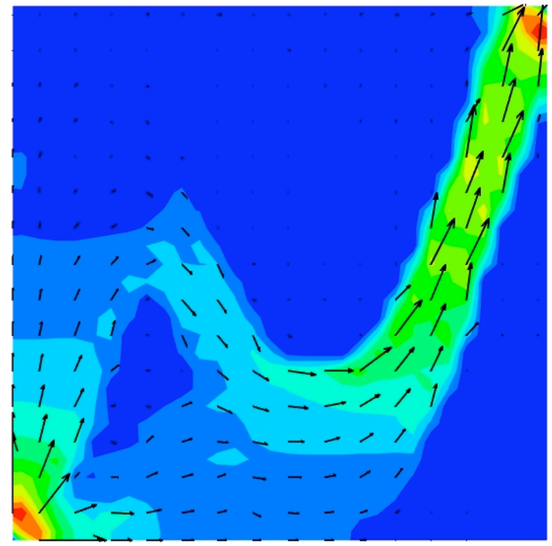
Some Channelized Flows

These are far from periodic!

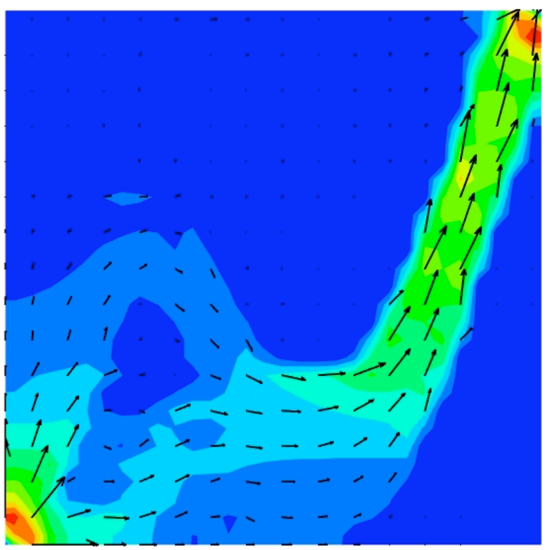
Local methods have difficulty with long-range correlations.



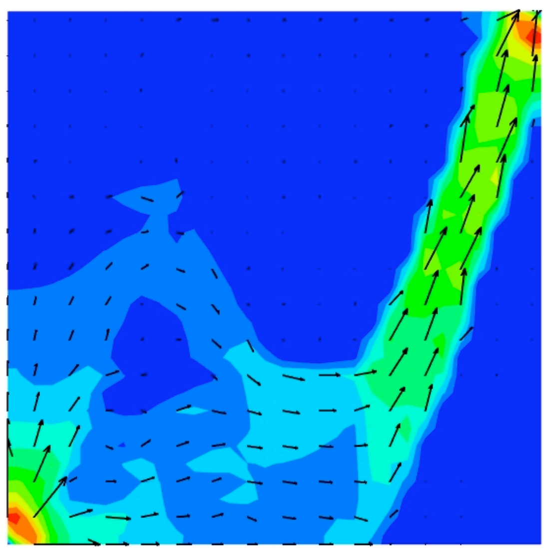
Perm: 10-white, 1-gray, 0.1-black



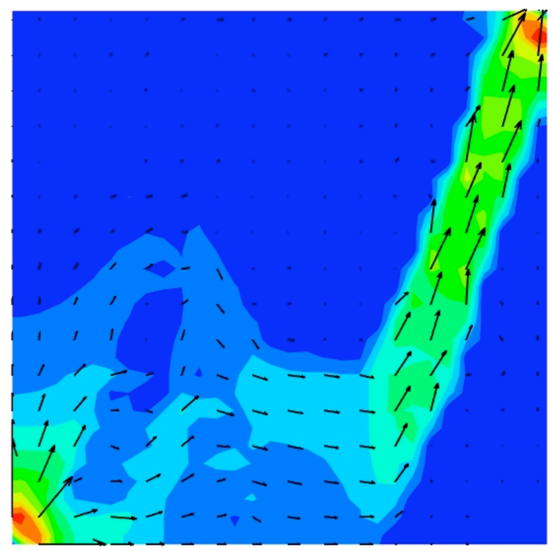
Fine BDM1 speed (log scale)



HE-OS



HE



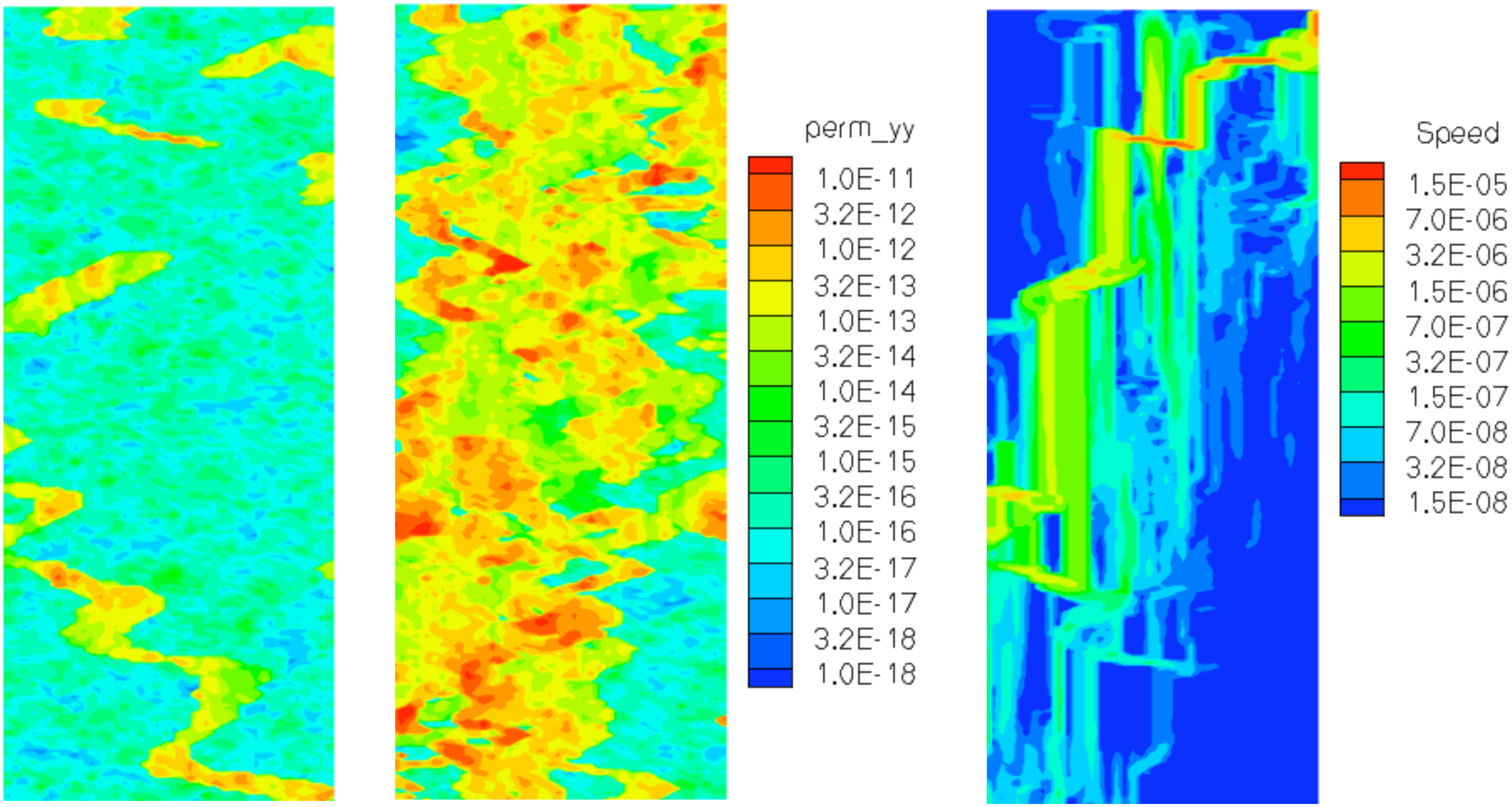
MD

Fine 30×30 grid, 3×3 coarse grid with 10×10 subgrid

Simple Channelized Permeability—2

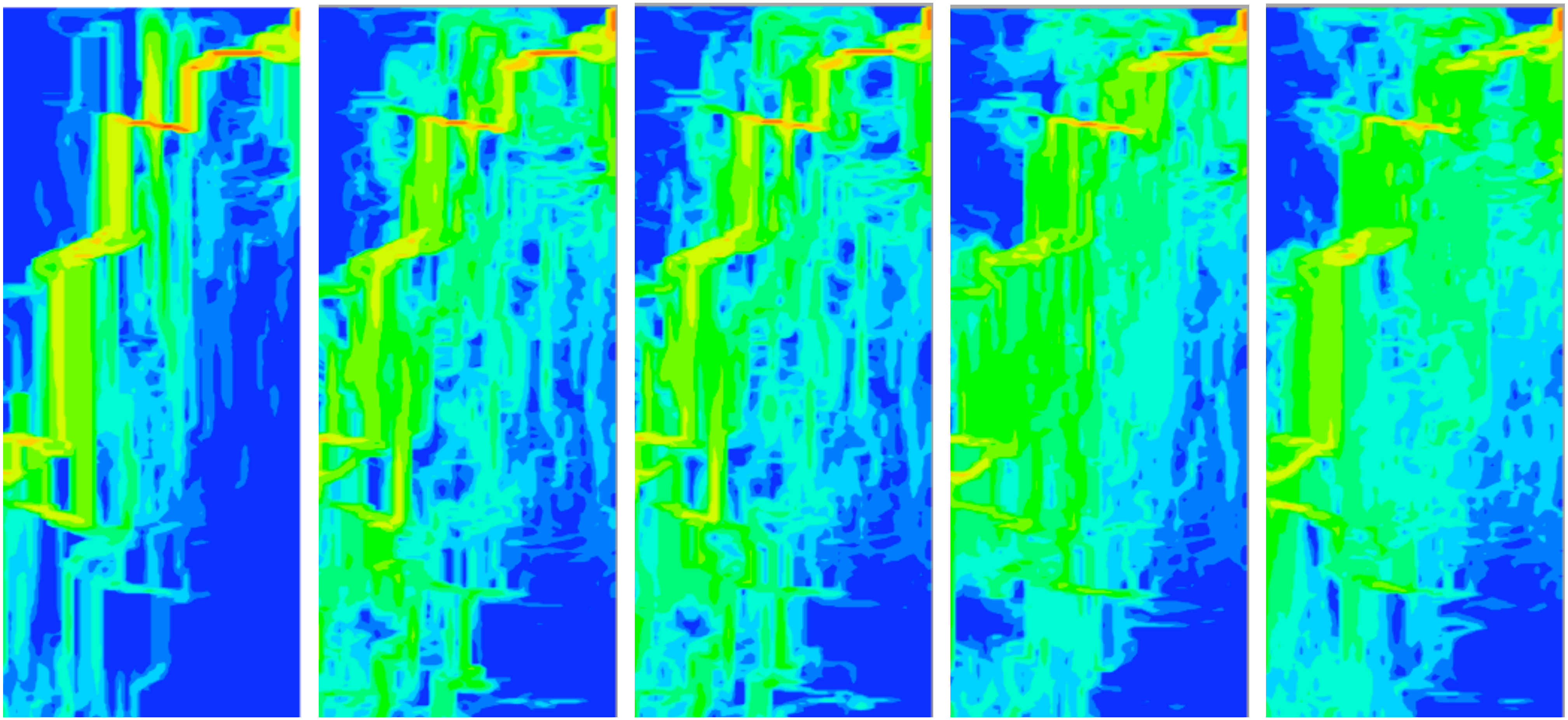
Method	Pressure Error ℓ^2	Pressure Error ℓ^∞	Velocity Error ℓ^2	Velocity Error ℓ^∞
RT0	0.02	0.02	0.03	0.04
ME0	1.10	0.18	0.47	0.34
ME1	0.49	0.12	0.28	0.32
MD	0.31	0.10	0.26	0.36
HE	0.28	0.07	0.20	0.31
HE-OS	0.28	0.07	0.15	0.22

SPE10 Permeability Layer 36—1



x-permeability y-permeability BDM

Grid: The grid is 60×220, upscaled to 6×22.

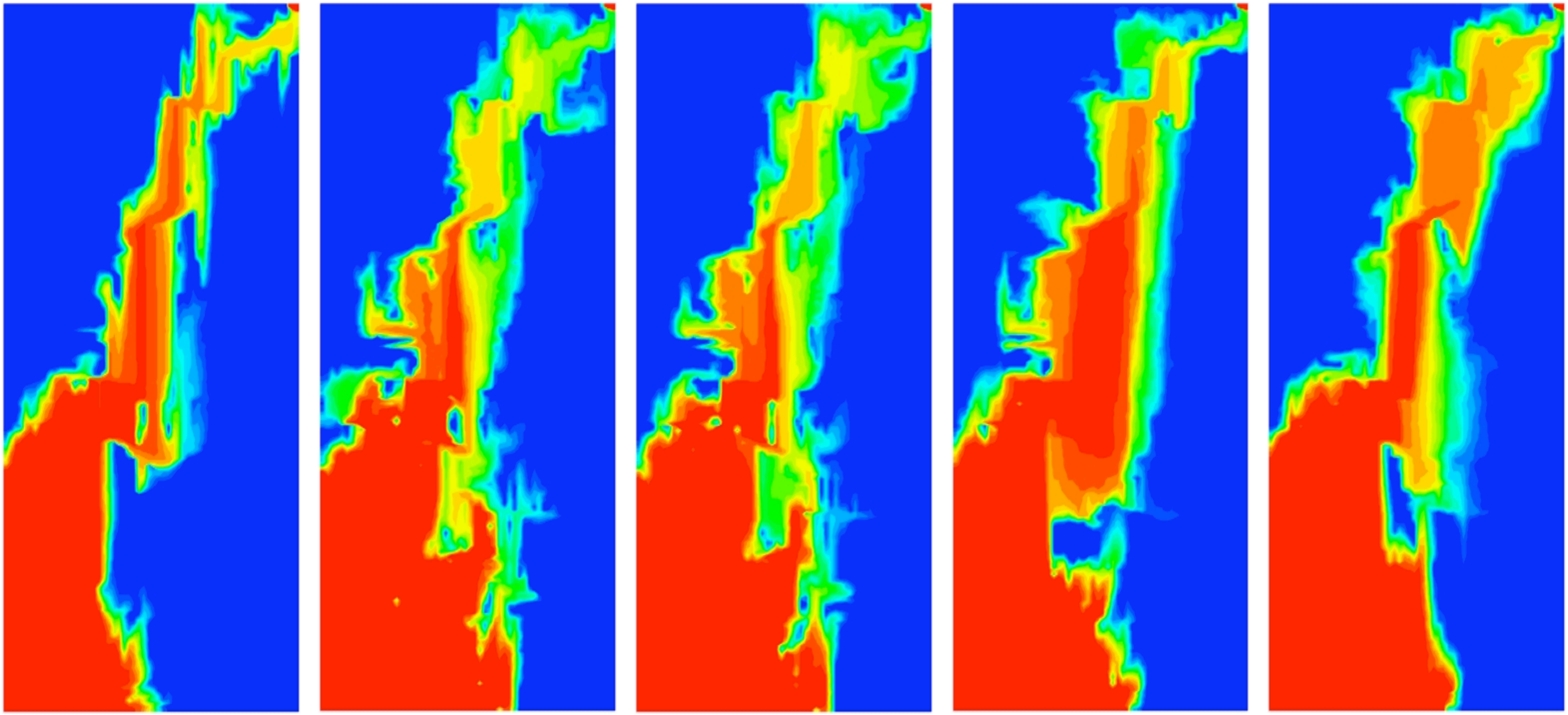


BDM HE-OS HE MD ME1

Speed

	Relative Errors with Respect to BDM Solution			
	Pressure		Velocity	
Method	L2 err	max err	L2 err	max err
RT0	0.13	0.12	0.12	0.18
ME0	1.89	1.52	0.71	0.87
ME1	1.28	1.04	0.66	0.86
MD	1.03	1.05	0.57	0.52
HE	1.50	1.27	0.48	0.50
HE-OS	1.59	1.29	0.49	0.50

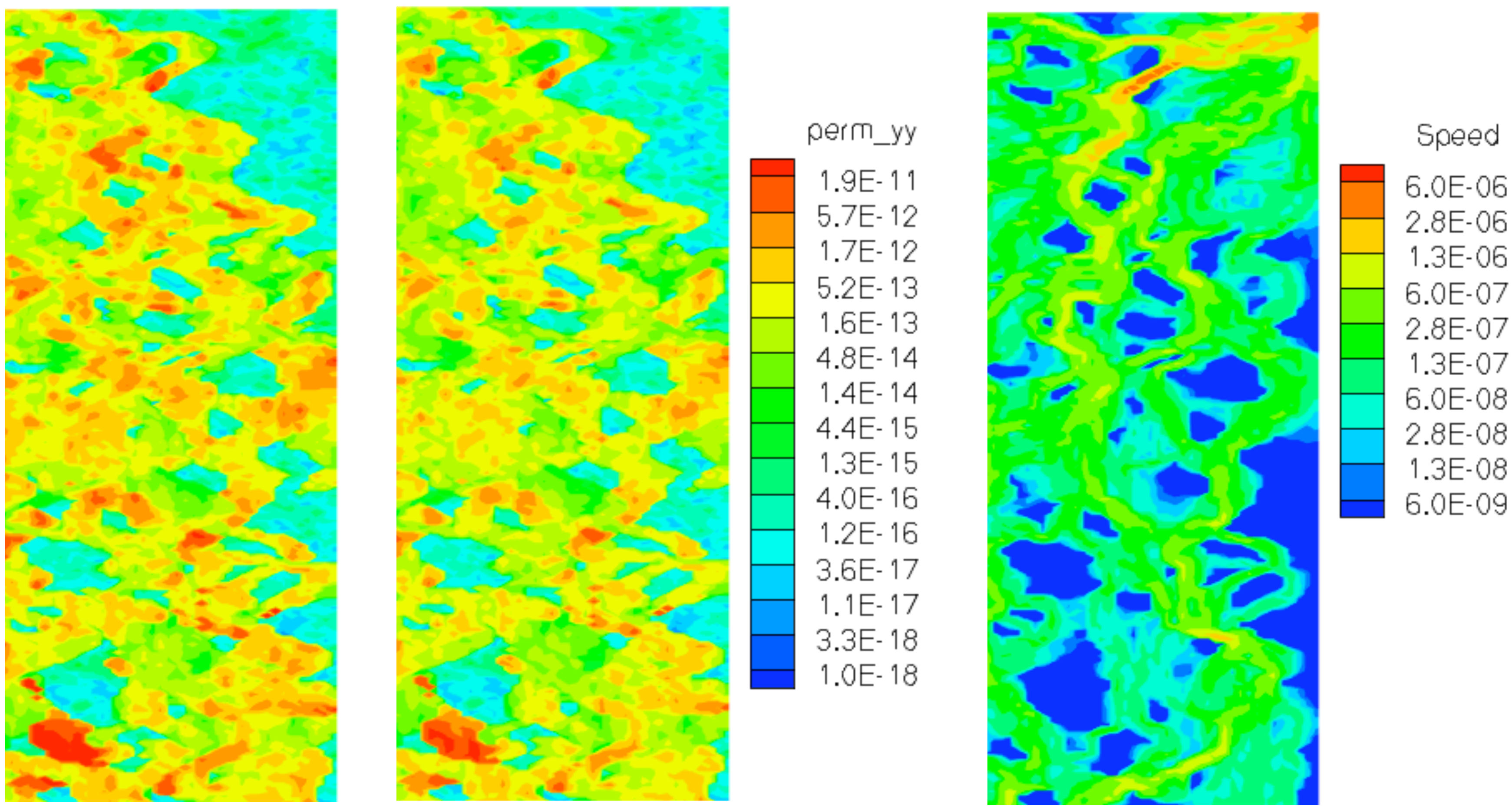
SPE10 Permeability Layer 36—4



BDM	HE-OS	HE	MD	ME1
	Err 0.57	Err 0.55	Err 0.51	Err 0.49

Tracer Concentration

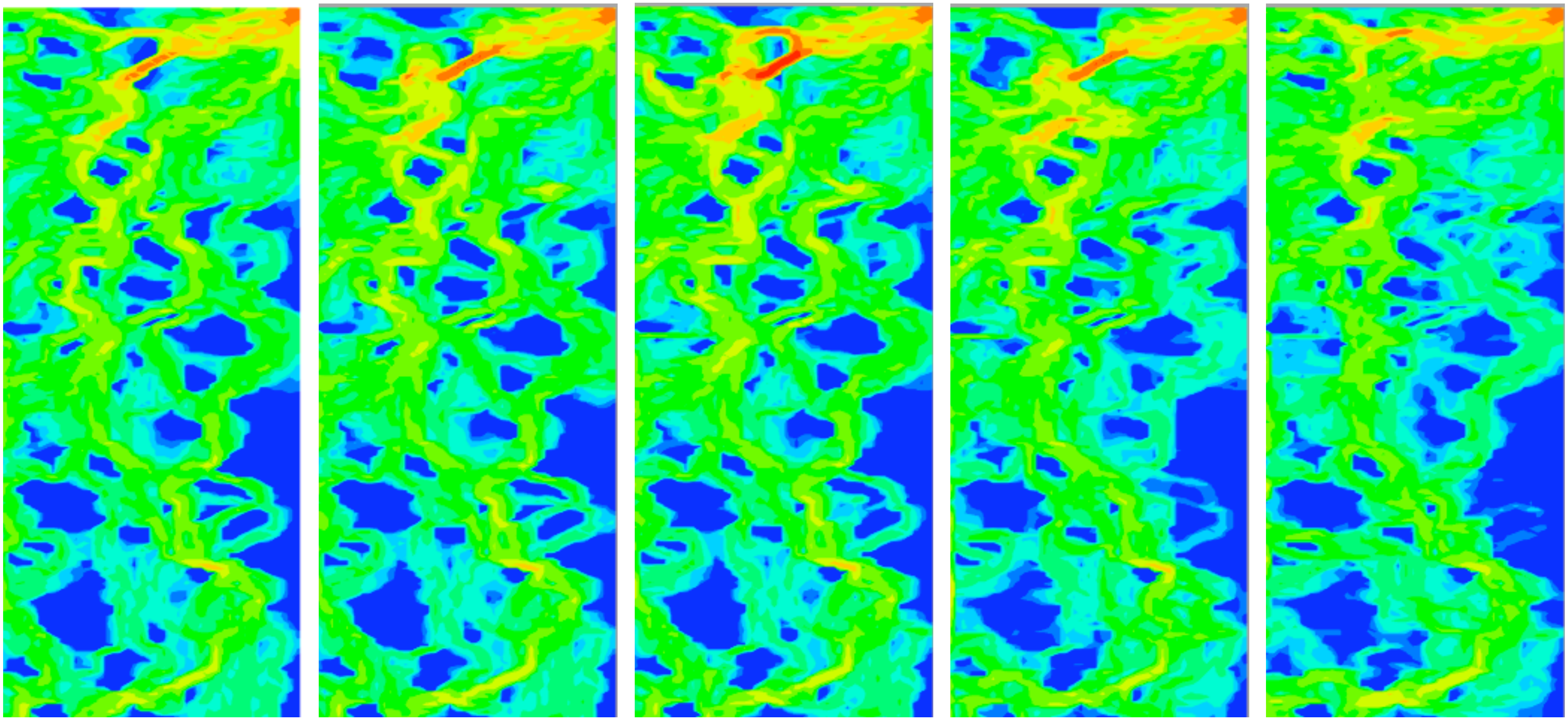
SPE10 Permeability Layer 85—1



x-permeability y-permeability   BDM

Grid: The grid is 60×220, upscaled to 6×22.

SPE10 Permeability Layer 85—2



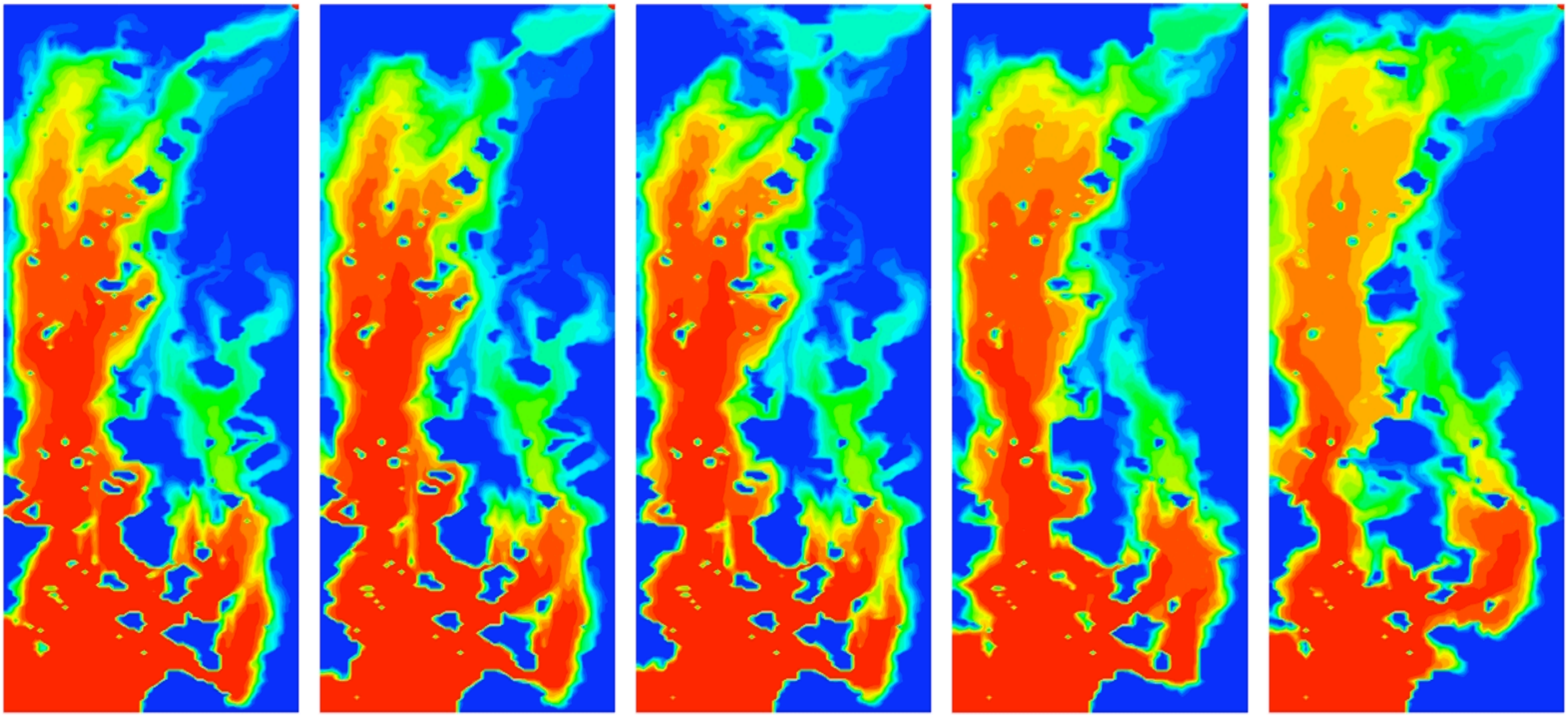
BDM HE-OS HE MD ME1

Speed

Relative Errors with Respect to BDM Solution

Method	Pressure Error		Velocity Error		Tracer Error
	L2 err	max err	L2 err	max err	L2 err
RT0	0.04	0.03	0.08	0.14	—
ME0	0.27	0.10	0.72	0.53	—
ME1	0.22	0.08	0.58	0.56	0.41
MD	0.24	0.11	0.45	0.46	0.30
HE	0.24	0.10	0.70	0.91	0.20
HE-OS	0.24	0.11	0.35	0.55	0.18

SPE10 Permeability Layer 85—4



BDM	HE-OS	HE	MD	ME1
	Err 0.18	Err 0.20	Err 0.30	Err 0.41

Tracer Concentration

Some Techniques for Controlling Errors

Limited global information (Aarnes 2004; Chen, Durlofsky 2006; Efendiev et al. 2006)

Use the solution to a full fine scale problem to set the proper BC's for $\mathbf{v} \cdot \nu$ on edges $e \subset \partial E$. This is useful for

- nonlinear problems (solve a global linear problem)
- time dependent problems
- stochastic problems

A-posteriori error estimation and control (Arbogast, Pencheva, Wheeler, Yotov 2007; Pencheva, Vohralik, Wheeler, Wildey 2010)

Include more scales where a-posteriori estimation shows high errors.

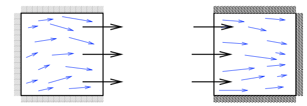
Preconditioners (Xu, Zikatanov 2004; Graham, Scheichl 2007)

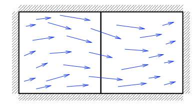
Iterate the fine scale system to convergence using multiscale ideas as a preconditioner (or in defining prolongation operators in multigrid).

Summary and Conclusions

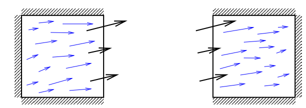
Summary and Conclusions—1

1. We presented the Variational Multiscale Method.
    - Upscaling is an affine and antidiffusive operation
    - Implicitly defines multiscale finite elements
2. We defined multiscale elements:





    - Multiscale elements (ME0 and ME1).
    - Multiscale dual-support (MD) elements.
        - These do *not* converge in the presence of anisotropy.
        - However, experience suggests they work well in a practically reasonable range of parameters ϵ and h.
    - A new homogenization-based element HE was defined from the microscale structure.



Summary and Conclusions—2

3. Error results were presented
   - Optimal approximations
   - Polynomial approximation theory
   - A simplified proof was presented for multiscale convergence of ME0:
     (2) Quasi-optimality $(\mathbf{u}_\epsilon - \pi_\epsilon^{\mathsf{ME}}\mathbf{u}_0)$
     (1) Microscale structure $(\mathbf{u}_\epsilon - \mathcal{A}_\epsilon\mathbf{u}_0)$
     (3) Multiscale projection approximation $\mathcal{A}_\epsilon(\mathbf{u}_0 - \pi_0^{\mathsf{ME}}\mathbf{u}_0)$
     (4) Smooth projection approximation $(\mathcal{A}_\epsilon\pi_0^{\mathsf{ME}}\mathbf{u}_0 - \pi_\epsilon^{\mathsf{ME}}\mathbf{u}_0)$
4. Numerical results show the methods work well
   - Except ME0
   - Perhaps HE-OS works best
   - MD works well in practice, but has some difficulty with anisotropy