Existence and stability of stationary fronts in inhomogeneous wave equations

Gianne Derks*

University of Surrey, Guildford, UK

Joint research with Arjen Doelman, Giuseppe Gaeta, Chris Knight and Hadi Susanto

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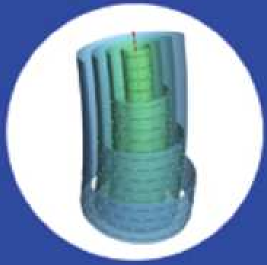
Maths research at Surrey

At the interface between pure and applied mathematics



Biosystems

- Ecology and Evolutionary Biology
- Biophysics
- Metabolism
- Statistics

Dynamical Systems and PDEs

- Nonlinear PDEs
- Dynamical Systems
- Calculus of Variations
- Geometry and Mechanics

Fields, Strings and Geometry

- Quantum Fields, String/M-Theory and Supergravity
- Dualities (AdS/CFT,...)
- Topological Strings
- Integrability, Twistor Geometry and Geometric Analysis

Fluid Mechanics and Meteorology

- Geometric Fluid Dynamics
- Mathematical Meteorology and Data Assimilation
- Water Waves
- Hydrodynamic Stability

Introduction

Homogeneous nonlinear wave equations like

- $u_{tt} - u_{xx} - V'(u) = 0$ (e.g. $V'(u) = \sin u$, sine-Gordon);
- $i\psi_t + \psi_{xx} - f'(|\psi|^2)\psi = 0$ (e.g. $f'(|\psi|^2) = a + b|\psi|^2$, NLS);

often have (families of) travelling/stationary solitary waves or kinks.

Introduction

Homogeneous nonlinear wave equations like

- $u_{tt} - u_{xx} - V'(u;x) = 0$ (e.g. $V'(u,x) = D(x)\sin u$, sine-Gordon);
- $i\psi_t + \psi_{xx} - f'(|\psi|^2;x)\psi = 0$ (e.g. $f'(|\psi|^2) = a(x) + b(x)|\psi|^2$, NLS);

often have (families of) travelling/stationary solitary waves or kinks.

What happens with these coherent structures if there are inhomogeneities (e.g. related to periodic media, external potentials, or interfaces)?

- Do the fronts/solitons persist?
- What about their stability?

In this talk, we consider an inhomogeneous sine-Gordon-like equation for

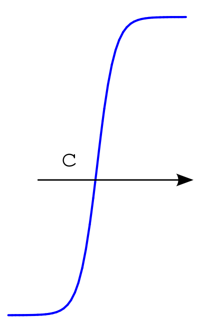
- superconductors: long Josephson junction with impurity;
- DNA-RNAP interaction.

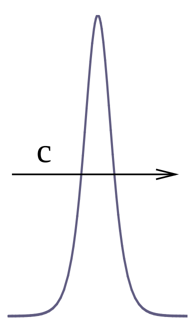
A nonlinear (semi-linear) wave equation on the real line:

$$u_{tt} = u_{xx} + V'(u).$$

A travelling wave front or solitary wave solution is of the form

$$u(x,t) = \widehat{u}(x - ct) \quad \text{and} \quad \lim_{|\xi|\to\infty} \widehat{u}_\xi(\xi) = 0.$$



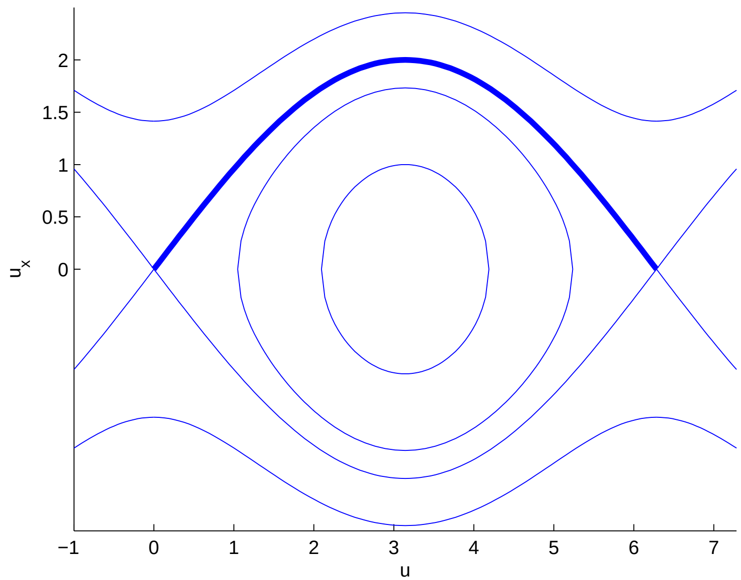


Existence and stability of such solutions?

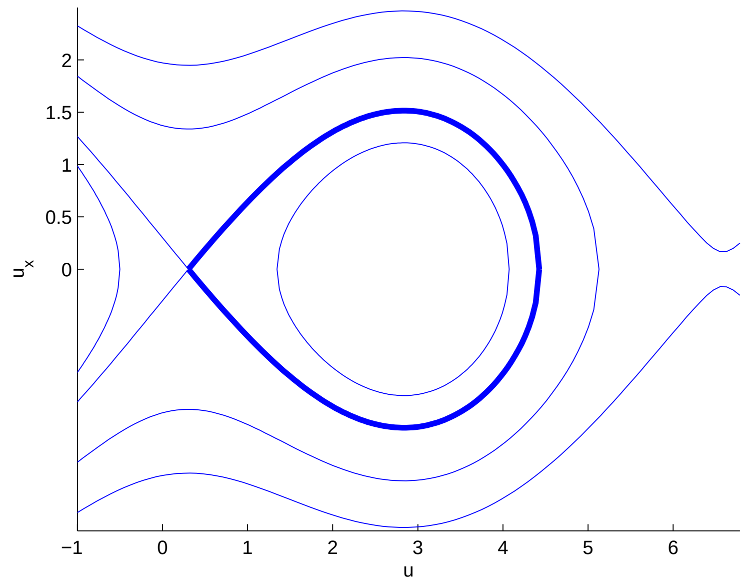
Existence of fronts/solitary waves

Write $\xi = \frac{x-ct}{\sqrt{1-c^2}}$, then $\widehat{u}(\xi)$ satisfies $\widehat{u}_{\xi\xi} + V'(\widehat{u}) = 0$, $\lim\limits_{|\xi|\to\infty} \widehat{u}_\xi(\xi) = 0$.

This is a Hamiltonian ODE, we can use phase plane analysis.



$$V(u) = \cos(u)$$



$$V(u) = \cos(u) + u/3$$

- Fronts correspond to heteroclinic connections;
- Solitary waves correspond to homoclinic connections.

Linearise about the front/solitary wave: $u(x,t) = \widehat{u}(\xi) + v(\xi,t)$ with $\xi = \frac{x-ct}{\sqrt{1-c^2}}$ and $v(\xi,t) = e^{\lambda t} v(\xi)$ gives the eigenvalue problem

$$\lambda^2 v = \mathcal{L} v, \quad \text{with} \quad \mathcal{L} = D_{\xi\xi} + V''(\widehat{u}).$$

- This is a Sturm-Liouville problem, so the eigenfunction associated to the largest eigenvalue of $\mathcal{L}$ has no zeros.
- Differentiating the ODE for $\widehat{u}$ with respect to ξ gives $\mathcal{L}\widehat{u}_\xi = 0$. As $\widehat{u}$ is smooth, $\widehat{u}_\xi$ is an eigenfunction of $\mathcal{L}$ with the eigenvalue zero.

Lemma *The front $\widehat{u}$ is nonlinearly stable iff $\widehat{u}_\xi$ has no zeros. The solitary wave $\widehat{u}$ is non-monotonic, hence unstable.*

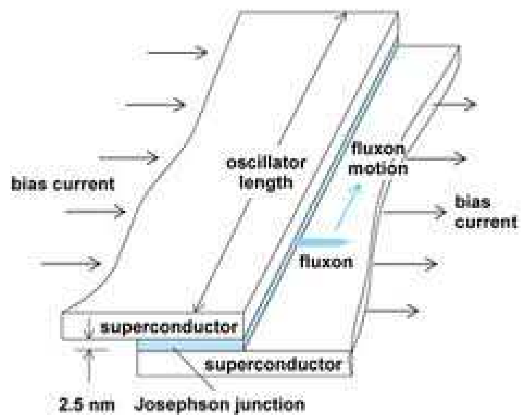
To prove nonlinear stability, use the Hamiltonian

$$H(u,P) = \frac{1}{2} \int_{-\infty}^{\infty} \left[P^2 + u_\xi^2 - V(u)\right] d\xi.$$

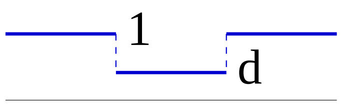
Josephson junctions & sine-Gordon models

Propagation of the magnetic flux $\phi(x,t)$ in a long Josephson junction (LJJ) is described by an inhomogeneous perturbed sine-Gordon equation:

$$\phi_{tt} = \phi_{xx} - D(x)\sin\phi + \gamma - \alpha\phi_t, \quad x \in \mathbb{R},\ t > 0.$$



The meaning of the various terms:

- γ: induced current;
- $\alpha \geq 0$: dissipation;
- $D(x)$: magnetic variations/impurities:
  - $D = 1$: long Josephson junction without variations/impurities;
  - $D(x) = 1$, for $x < 0$ and $D(x) = -1$ for $x > 0$: 0-π Josephson junction. Magnetic variation induces phase shift of π;
  - $D(x) = 1$ for $|x| > L$ and $D(x) = d > 0$ for $|x| < L$:  magnetic impurities;
  - Combinations of above.

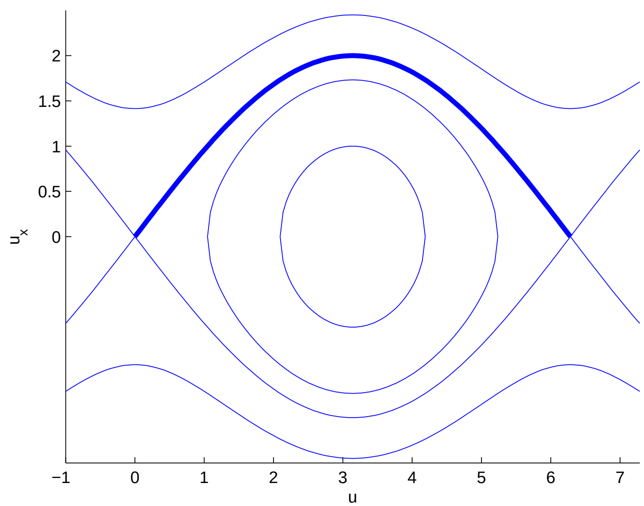
The perturbed sine-Gordon model

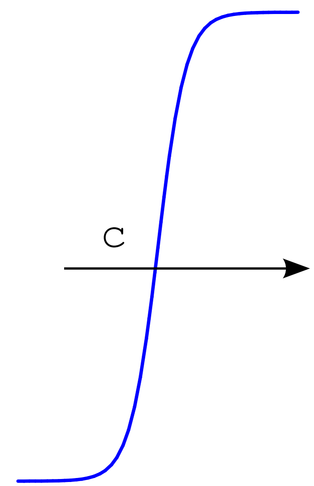
No magnetic variations/impurities ($D = 1$): perturbed sine-Gordon equation

$$\phi_{tt} = \phi_{xx} - \sin\phi + \gamma - \alpha\phi_t, \quad x \in \mathbb{R},\ t > 0.$$

Solutions:

- Fixed points are solutions of $\sin\phi - \gamma = 0$.
- If $\gamma = 0 = \alpha$, then it is the sine-Gordon equation. There is a family of stable travelling fronts ($|c| < 1$), called fluxons.





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No magnetic variations/impurities ($D = 1$): perturbed sine-Gordon equation

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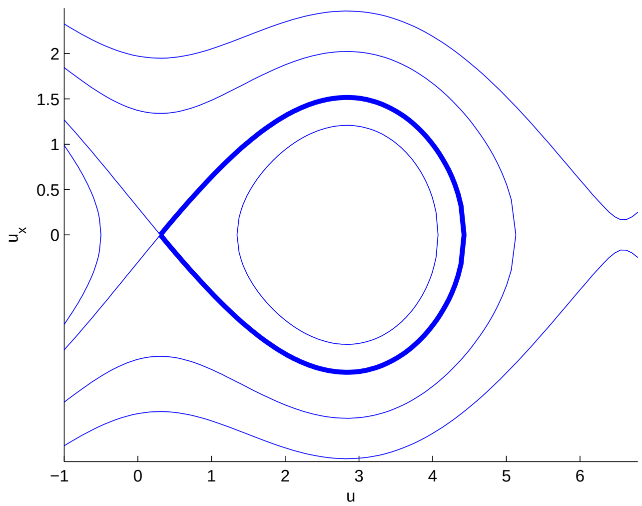
- Fixed points are solutions of $\sin\phi - \gamma = 0$.
- If $\gamma = 0 = \alpha$, then it is the sine-Gordon equation. There is a family of stable travelling fronts ($|c| < 1$), called fluxons.
- If $\alpha = 0$, $\gamma \neq 0$, then there are unstable travelling solitary waves.



The perturbed sine-Gordon model

No magnetic variations/impurities ($D = 1$): perturbed sine-Gordon equation

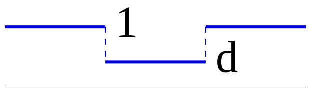
$$\phi_{tt} = \phi_{xx} - \sin\phi + \gamma - \alpha\phi_t, \quad x \in \mathbb{R},\ t > 0.$$

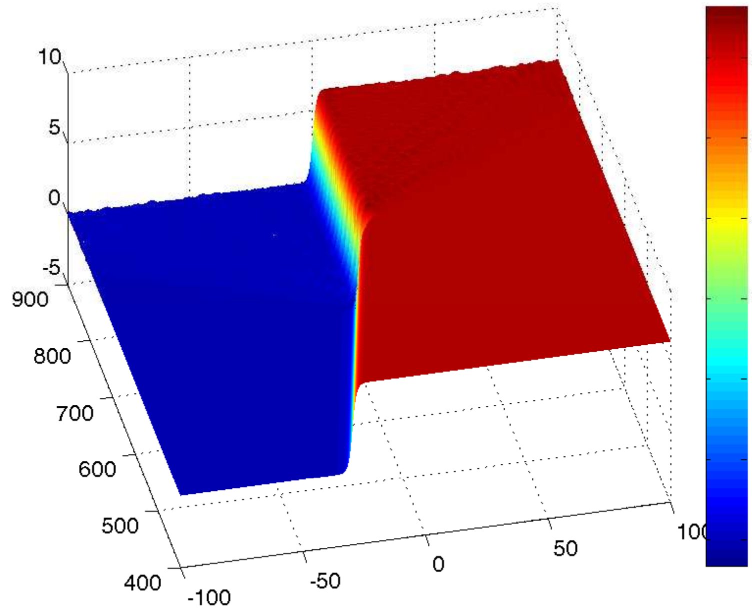
Solutions:

- Fixed points are solutions of $\sin\phi - \gamma = 0$.
- If $\gamma = 0 = \alpha$, then it is the sine-Gordon equation. There is a family of stable travelling fronts ($|c| < 1$), called fluxons.
- If $\alpha = 0$, $\gamma \neq 0$, then there are unstable travelling solitary waves.
- If $\gamma \neq 0$, $\alpha \neq 0$: there is one travelling fluxon with speed $c(\gamma, \alpha)$. This fluxon connects $\arcsin\gamma$ with $2\pi + \arcsin\gamma$ and is stable.

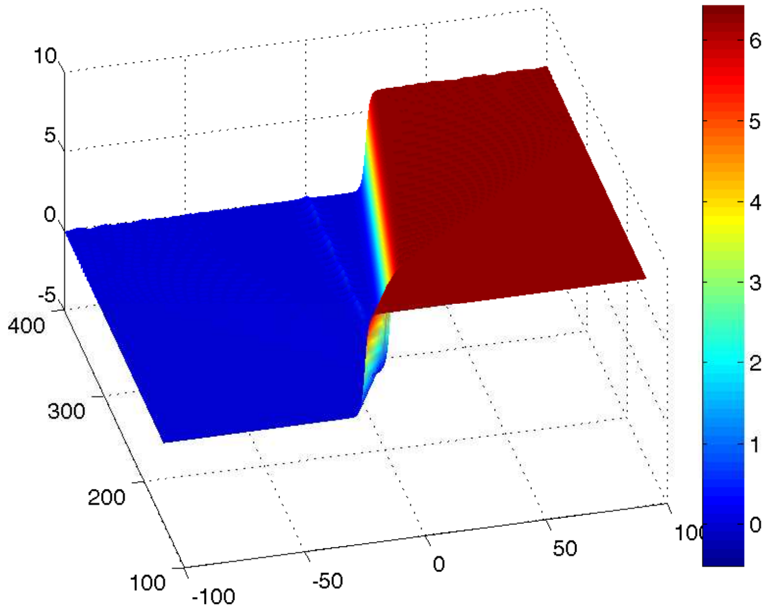
The effect of impurities

What happens with the travelling fluxon if an impurity is present?





A slow-ish fluxon gets trapped.

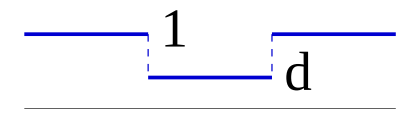


A faster fluxon gets slowed down.

Questions:

- are there pinned fluxons?
- if so, which ones are stable?

Sine-Gordon with impurity



The sine-Gordon equation with an impurity:

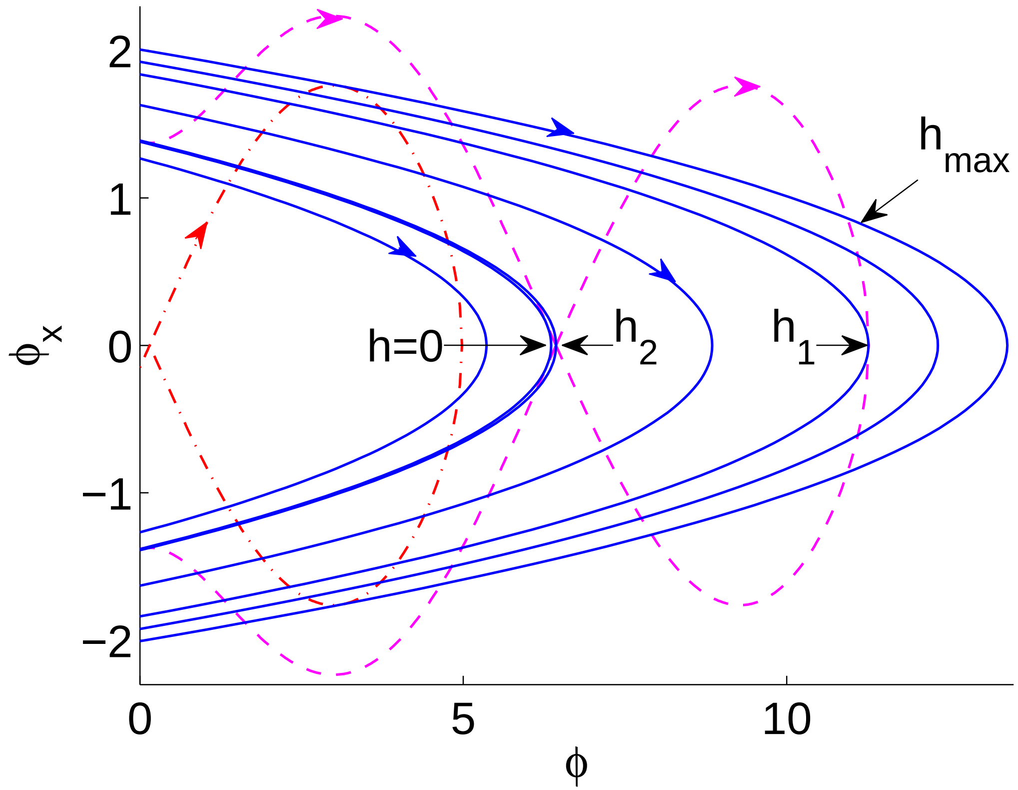
$$\phi_{tt} = \phi_{xx} - D(x)\sin\phi + \gamma - \alpha\phi_t, \quad \text{with} \quad D(x) = \begin{cases} 1, & |x| > L \\ d > 0, & |x| < L \end{cases}$$

We focus on the existence and stability of stationary (pinned) fluxons.

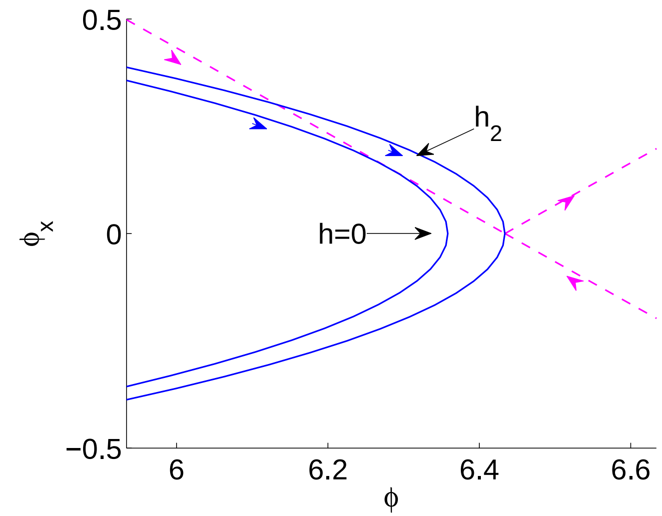
- A pinned fluxon connects $(\arcsin\gamma)$ with $(2\pi + \arcsin\gamma)$.
- Stationary solutions satisfy $0 = \phi_{xx} - D(x)\sin\phi + \gamma$.
  This is a spatially Hamiltonian system with Hamiltonian
  $$H = \tfrac{1}{2}(\phi_x)^2 - D(x)(1-\cos\phi) + \gamma\phi.$$
- Hamiltonian H is conserved on $(-\infty, -L)$, $(-L, L)$, (L, ∞):
  - On $(-\infty, -L)$ and (L, ∞), H is determined by the fixed points;
  - On $(-L, L)$, the H-value depends on the length L, denote by $h(L)$.

Existence of pinned fluxons

$\gamma = 0.15,\ d = 0$



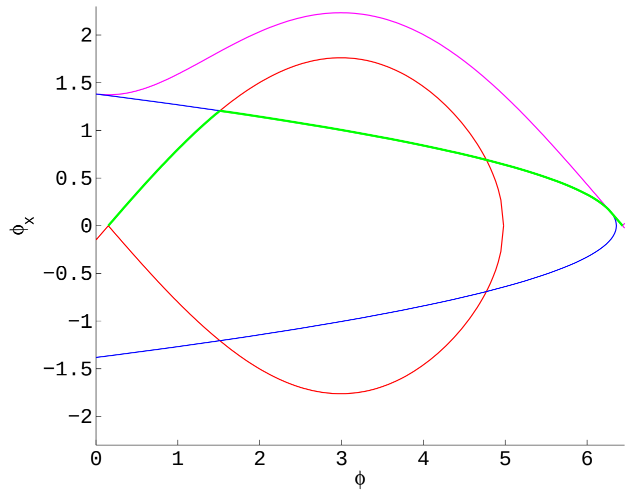
Zoom near $(2\pi, 0)$

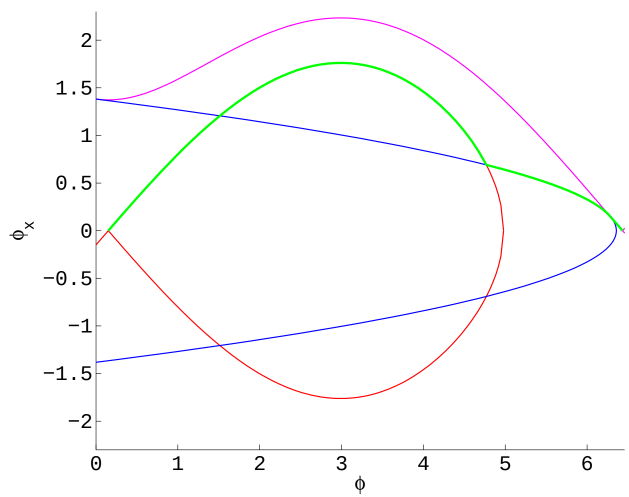


Red: leaving $\arcsin\gamma$ (unstable manifold);
Magenta: returning to $2\pi + \arcsin\gamma$ (stable manifold);
Blue: dynamics for $d = 0$.

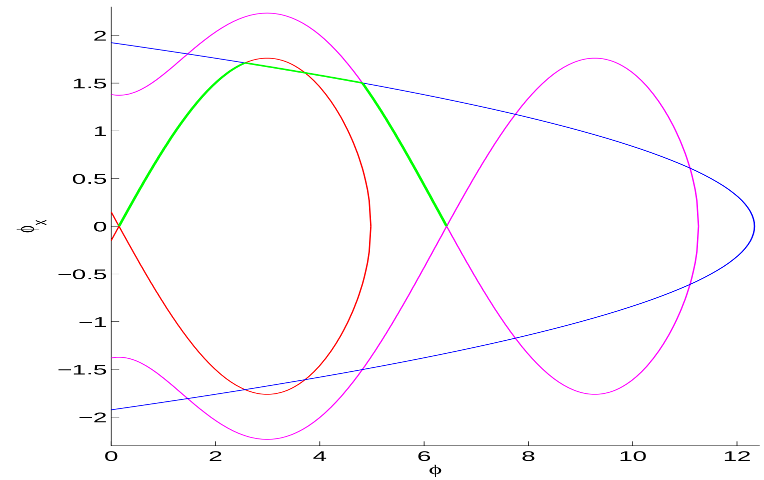
Monotonic pinned fluxons

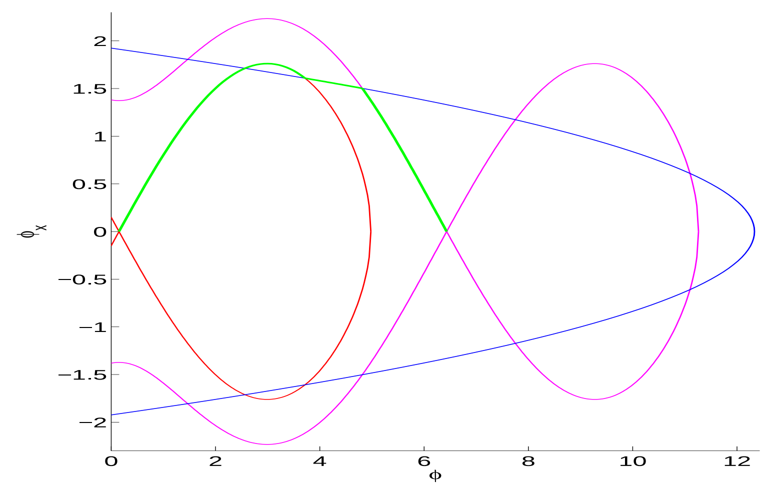
Two monotonic fluxons for $h = 0$:





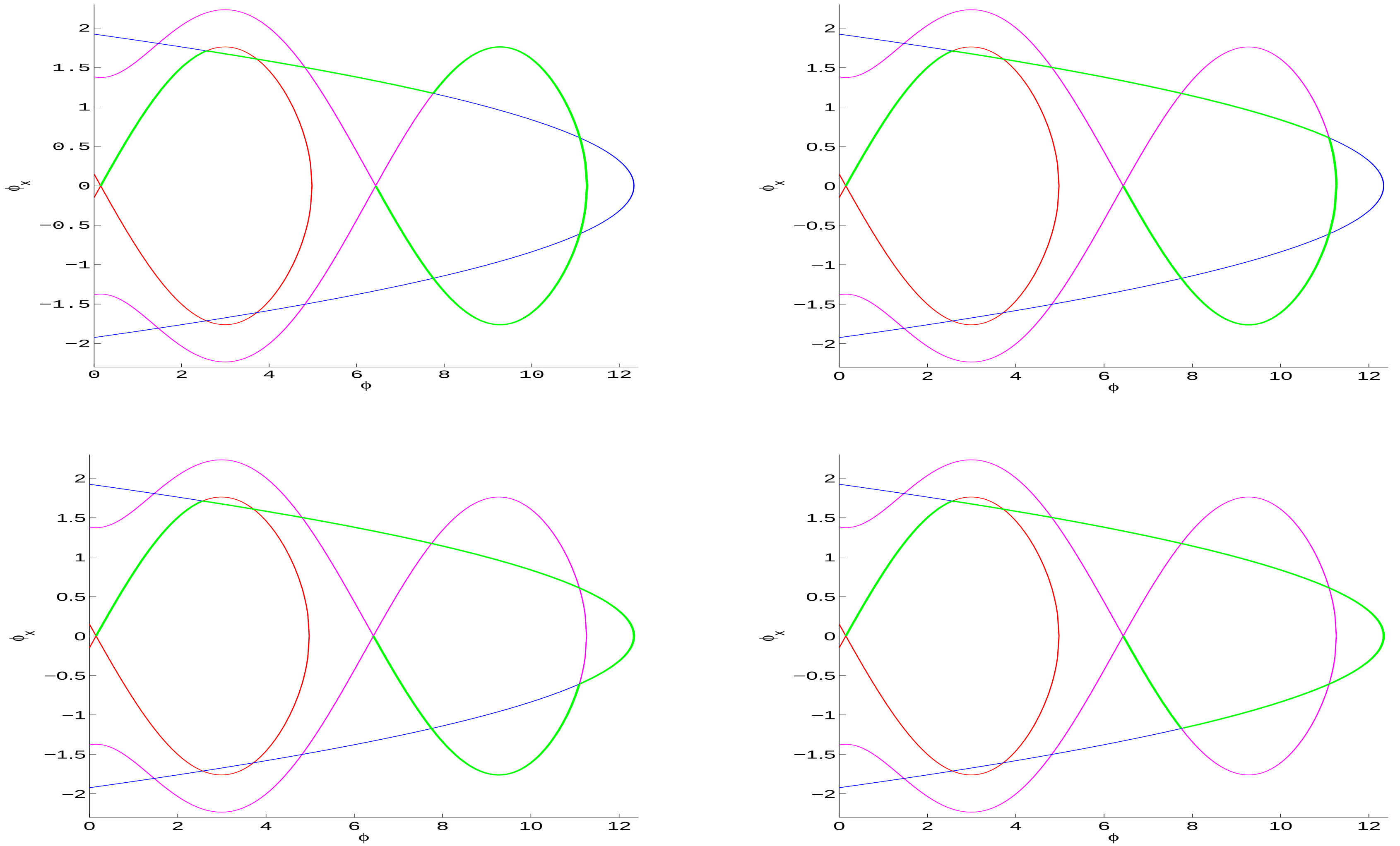
Two monotonic fluxons for $h = \frac{h_1 + h_{\max}}{2}$:



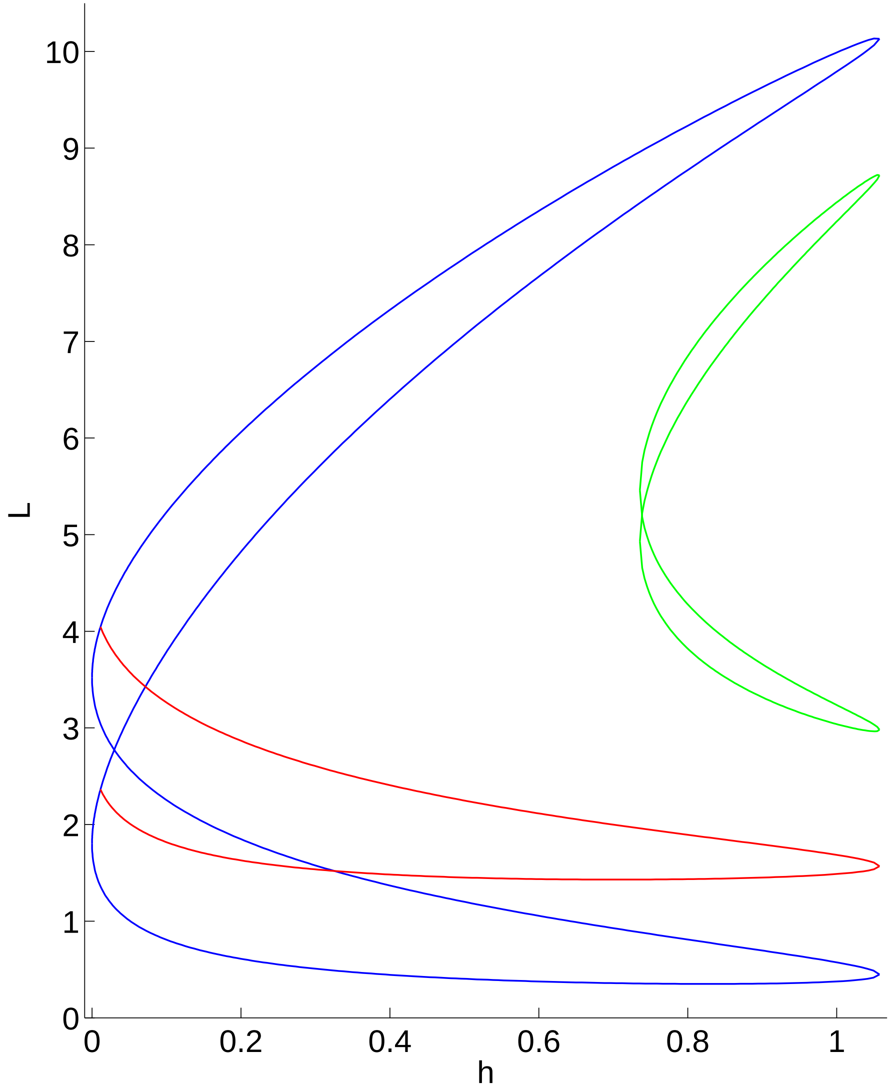


Other pinned fluxons

Eight non-monotonic fluxons for $h = (h_1 + h_{\max})/2$ (four shown):



Impurity lengths, $\gamma = 0.15$, $d = 0$



[Derks, Doelman, Knight, Susanto (2012)]

Observations:

- There is strictly positive minimum length and maximum length;
- For a fixed length, there are up to five pinned fluxons.
- Most fluxons are non-monotonic

How about stability??

Wave equation with one inhomogeneity

Consider a wave equation with one finite length inhomogeneity

$$u_{tt} = u_{xx} + \frac{\partial V}{\partial u}(u,x;L), \text{ where } V(u,x;L) = \begin{cases} V_l(u), & x < -L; \\ V_m(u), & -L < x < L; \\ V_r(u), & x > L. \end{cases}$$

Assumptions:

- Hamiltonian equations with outer potentials $V_l(u)$ and $V_r(u)$ have fixed points, called $u_{-\infty}$ resp. u_∞, which are saddles in the spatial dynamics (stable in temporal dynamics).
- There is an interval of L-values for which there exist stationary fronts connecting $u_{-\infty}$ and u_∞. The L-values and the fronts can be parametrised by

  $$h = \tfrac{1}{2}\widehat{u}_x^2 + V_m(\widehat{u}),\ 0 < x < L;$$

  where $\widehat{u}(x;h)$ is the front.

Stability and the L-h curve

Linearisation about a front $\widehat{u}(x;h)$ gives (using $u(x,t) = \widehat{u}(x;h) + e^{\lambda t}v(x)$):

$$\lambda(\lambda - \alpha)v = \mathcal{L}(h)v, \text{ with } \mathcal{L}(h) = D_{xx} + \frac{\partial^2 V}{\partial u^2}(\widehat{u}(x;h), x; L(h)).$$

Theorem [Knight, Derks, Doelman, Susanto (2013)] *If the solution $\widehat{u}(x;h)$ is such that $\widehat{u}_x \not\equiv 0$ on the middle interval, then its linearisation operator $\mathcal{L}(h)$ has an eigenvalue zero in $H^2(\mathbb{R})$ if and only if*

$$[V_m'(\widehat{u}_l(h)) - V_l'(\widehat{u}_l(h))]\, p_l(h)\ [V_m'(\widehat{u}_r(h)) - V_r'(\widehat{u}_r(h))]\, p_r(h)\ \mathbf{L}'(\mathbf{h}) = 0.$$

where
$$\begin{aligned} \widehat{u}_l(h) &= \widehat{u}(0;h), & p_l(h) &= \widehat{u}_x(0;h), \\ \widehat{u}_r(h) &= \widehat{u}(L(h);h), & p_r(h) &= \widehat{u}_x(L(h);h). \end{aligned}$$

- Bifurcation points if $[V_m'(\widehat{u}_l(h)) - V_l'(\widehat{u}_l(h))] \to 0$, or $p_l(h) \to 0$, etc. and then $L'(h) \to \infty$.
- For "most" fronts, the stability criterion is $\mathbf{L}'(\mathbf{h}) = 0$.

Ideas behind the proof

- Get a compatibility condition for the existence of an eigenvalue zero by using that the eigenfunction will consist of linear combinations of

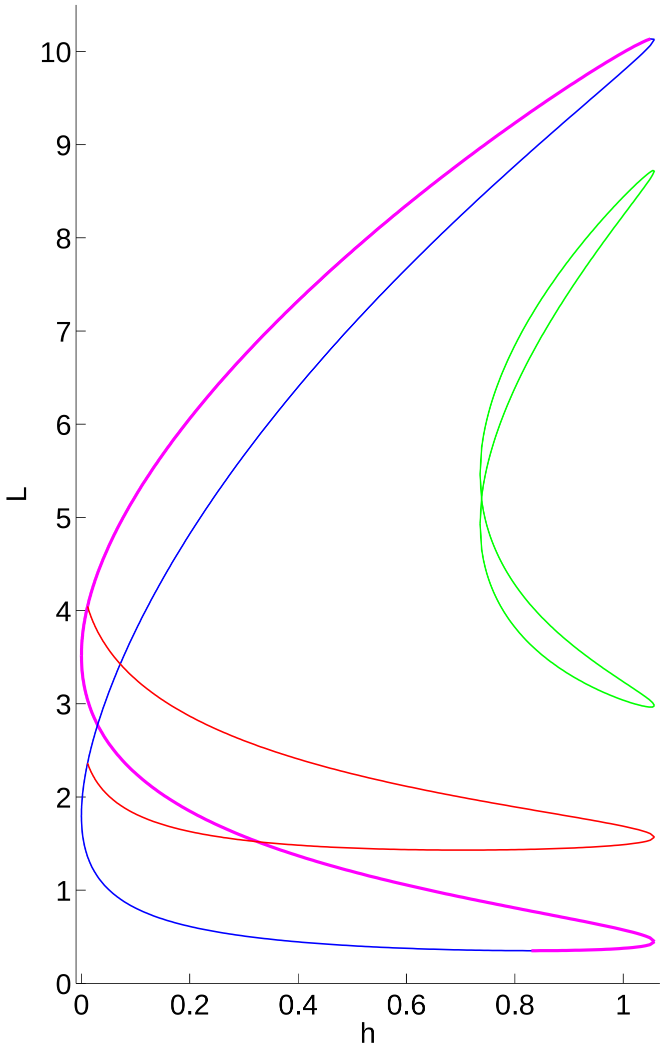
$$\widehat{u}_x \quad \text{and} \quad \widehat{u}_x \int \frac{dx}{(\widehat{u}_x(x))^2},$$

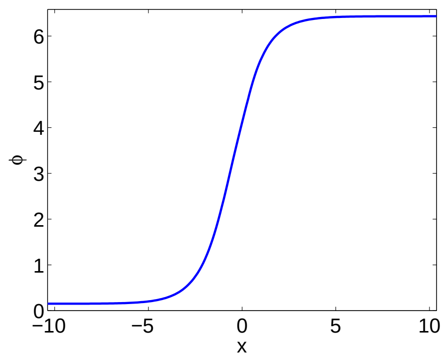
  which have to be patched together smoothly (C^1) at the points $x = -L$ and $x = L$.
- In the simplest case, $u_x(u, h) = \sqrt{2[h - V_m(u)]}$ and

$$L(h) = \int_{\widehat{u}_l(h)}^{\widehat{u}_r(h)} \frac{du}{\sqrt{2[h - V_m(u)]}}.$$

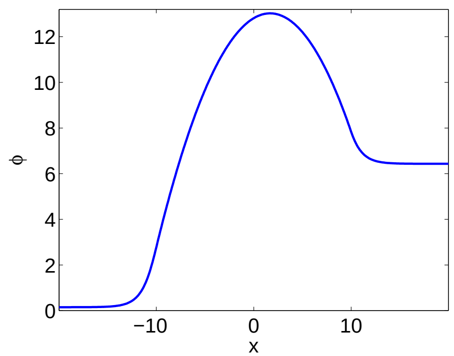
  For more complex functions $\widehat{u}$, the length consists of sums of similar integrals.
- Relate the derivative of the length curve $L(h)$ to compatibility condition.

Stability in long JJ with impurity





Stable monotonic fluxon ($L = 0.4$)



Stable non-monotonic fluxon ($L = 10$)

For each length, there is exactly one stable pinned fluxon: they are on the magenta curve.

- Full proof uses continuity via $(\gamma, d) = (0, 0)$;
- Nonlinear stability can be shown via the Hamiltonian.

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Wave equation with N inhomogeneities

Consider a wave equation with N inhomogeneities

$$u_{tt} = u_{xx} + \frac{\partial V}{\partial u}(u,x), \text{ where } V(u,x) = \begin{cases} V_l(u), & x < \chi_0; \\ V_1(u), & \chi_0 < x < \chi_1; \\ \vdots & \vdots \\ V_N(u), & \chi_{N-1} < x < \chi_N; \\ V_r(u), & x > \chi_N. \end{cases}$$

Assumptions:

- Hamiltonian equations with outer potentials $V_l(u)$ and $V_r(u)$ have fixed points, called $u_{-\infty}$ resp. u_∞, which are saddles in the spatial dynamics.
- There is region of $L_i = \chi_i - \chi_{i-1}$-values for which there exist stationary fronts connecting $u_{-\infty}$ and u_∞. The L_i-values and the fronts can be parametrised by

$$h_i = \tfrac{1}{2}\widehat{u}_x^2 + V_i(\widehat{u}), \ \chi_{i-1} < x < \chi_i, \quad i = 1, \ldots, N,$$

where $\widehat{u}(x; h_1, \ldots, h_N)$ denote the fronts.

Stability and the L_i-h_j surfaces

Linearisation about a front $\widehat{u}(x;\mathbf{h})$ gives (with $\mathbf{h} = (h_1,\dots,h_N)$)

$$\lambda(\lambda-\alpha)v = \mathcal{L}(\mathbf{h})v, \text{ with } \mathcal{L}(\mathbf{h}) = D_{xx} + \frac{\partial^2 V}{\partial u^2}(u(x;\mathbf{h}); L(\mathbf{h})).$$

Theorem [Knight, Derks, Doelman, Susanto (2013)] *If the solution $\widehat{u}(x;\mathbf{h})$ is such that $\widehat{u}_x \not\equiv 0$ on any interval, then its linearisation operator $\mathcal{L}(\mathbf{h})$ has an eigenvalue zero if and only if*

$$\det(\Gamma_N)\prod_{i=0}^{N}\mathscr{B}_i = 0,$$

where

$$\begin{aligned}
\mathscr{B}_i(\mathbf{h}) &= [V'_{i+1}(\widehat{u}(\chi_i)) - V'_i(\widehat{u}_i(\chi_i))]\,\widehat{u}_x(\chi_i), \quad i = i,\dots,N-1\\
\mathscr{B}_0(\mathbf{h}) &= [V'_1(\widehat{u}_m(\chi_0)) - V'_l(\widehat{u}_m(\chi_0))]\,\widehat{u}_x(\chi_0),\\
\mathscr{B}_N(\mathbf{h}) &= [V'_r(\widehat{u}(\chi_N)) - V'_N(\widehat{u}(\chi_N))]\,\widehat{u}_x(\chi_N)
\end{aligned}$$

and

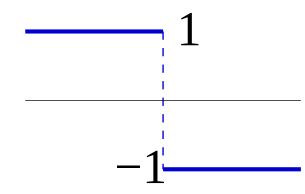
$$\Gamma_N = \frac{\partial(L_1,\dots,L_N)}{\partial(h_1,\dots,h_N)} = \begin{pmatrix} \frac{\partial L_1}{\partial h_1} & \mathscr{B}_1^{-1} & 0 & \cdots & 0\\ \mathscr{B}_1^{-1} & \frac{\partial L_2}{\partial h_2} & \mathscr{B}_2^{-1} & \ddots & 0\\ 0 & \mathscr{B}_2^{-1} & \ddots & \ddots & \\ \vdots & & \ddots & & \mathscr{B}_{N-1}^{-1}\\ 0 & 0 & \dots & \mathscr{B}_{N-1}^{-1} & \frac{\partial L_N}{\partial h_N}\end{pmatrix},$$

as $\mathscr{B}_i^{-1} = \frac{\partial L_{i+1}}{\partial h_i} = \frac{\partial L_i}{\partial h_{i+1}}$.

A 0-π junction with impurity

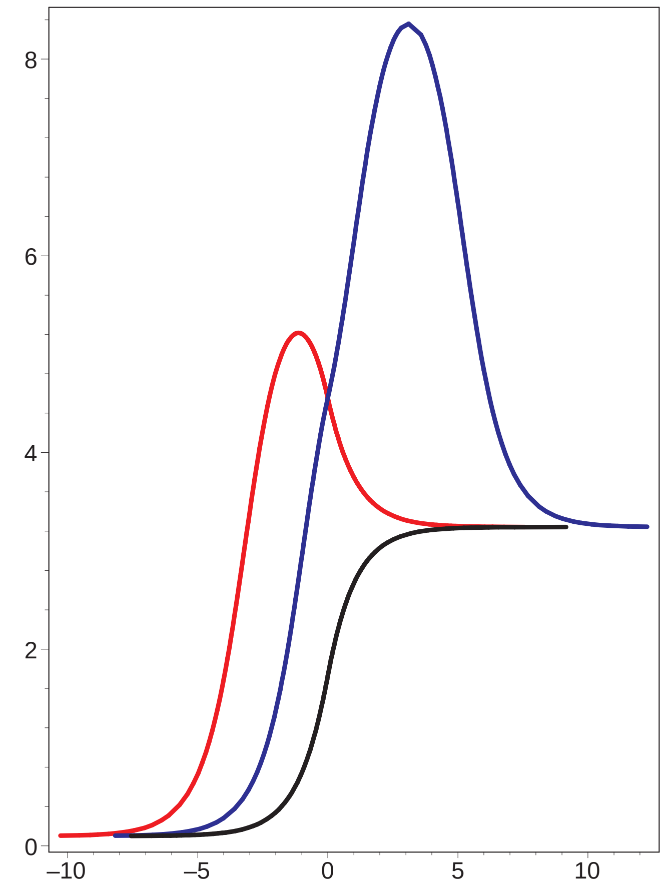
A 0-π junction is given by

$$\phi_{tt} = \phi_{xx} - D(x)\,\sin\phi + \gamma - \alpha\phi_t, \text{ with } D(x) = \begin{cases} 1, & x < L_2 \\ -1, & x > L_2 \end{cases}$$

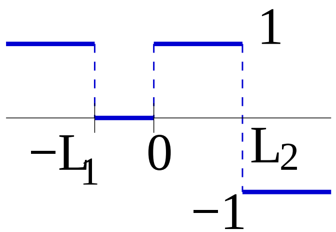


It can be shown that there are three types of stationary fluxons (see right, used $L_2 = 0$).

The black π-fluxon is unstable and the red and blue ones are unstable (red one marginally unstable if $\gamma \approx 0$).



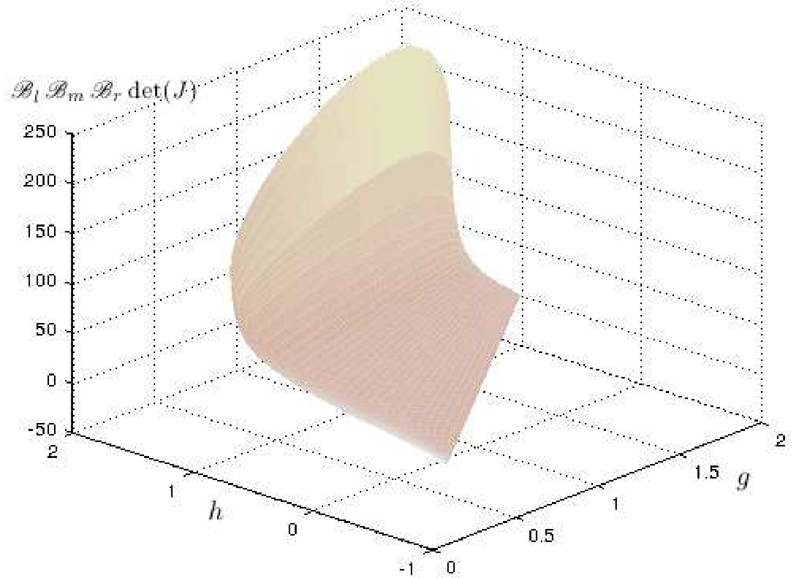
Stabilitization of π-fluxon by impurity



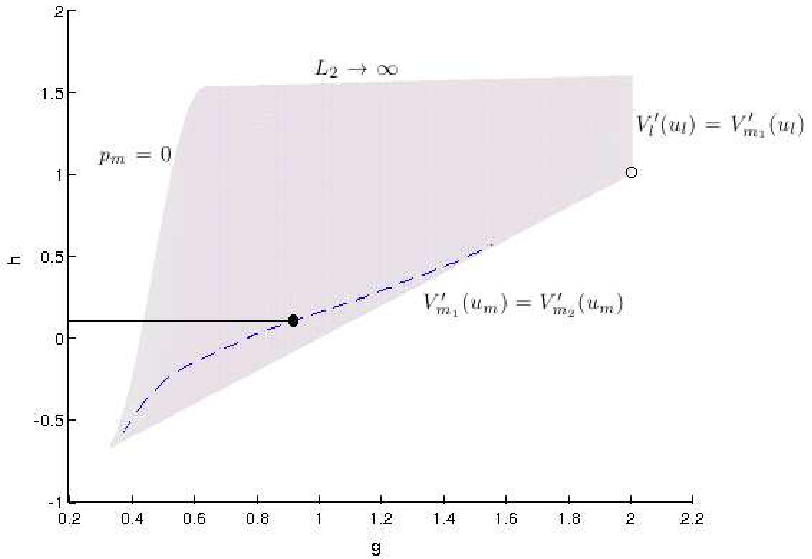
Now add an impurity with $d = 0$ on the left of the junction, does this stabilise the red π-fluxon?

We have a wave equation with 2 inhomogeneties (N=2):

Sketch of $\Gamma_2 = \det\left[\frac{\partial(L_1,L_2)}{\partial(h_1,h_2)}\right]$ for $\gamma = 0.1$:



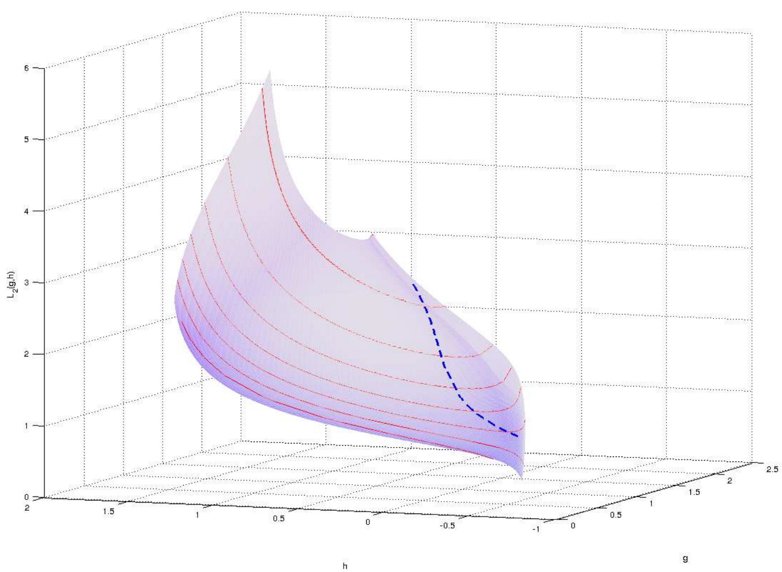
The curve $\det\left[\frac{\partial(L_1,L_2)}{\partial(h_1,h_2)}\right] = 0$:

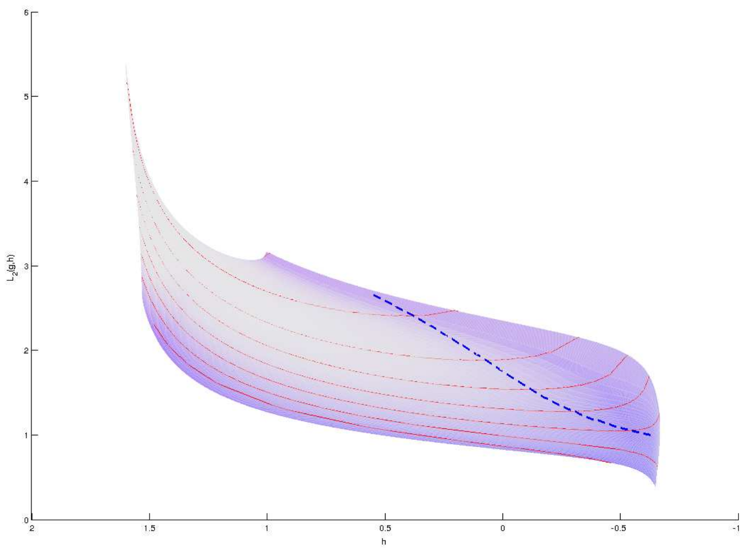


Eigenfunctions without zeros on the zero curve: red π-fluxon is stabilised by impur

Link with one inhomogeneity theorem

- If one of the two lengths are fixed, then we are in the situation of one inhomogeneity. Fixing one length (say L_1), leads to a curve $h_1(h_2)$.
- The theorem for one inhomogeneity checks for a extremal point of $L_2(h_1(h_2), h_2)$ with $h_1(h_2)$ given by $L_1(h_1, h_2) = \text{const}$. It turns out that this is the condition that the determinant vanishes at a point on the curve $h_1(h_2)$.

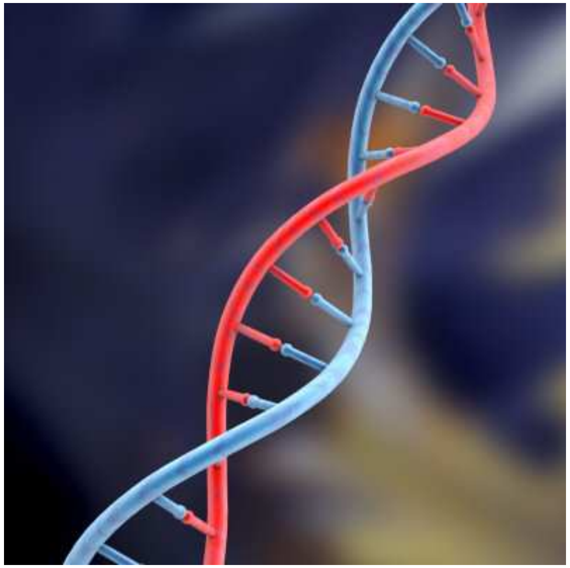




Yakushevich model for DNA dynamics

Assumptions and notation:

- The DNA is homogeneous, the sugar-phosphate backbone (SPB) doesn't move, the only dynamics are the nitrogen base rotations. The rotation angle of the base n on the "$\pm$" (red/blue) chain, away from the equilibrium, is denoted $\phi_n^\pm$.



From: http://www.csb.yale.edu/

- Interactions are modelled by potentials:
  - a *stacking* potential $\widehat{V}_s(\phi_{n+1}^\pm, \phi_n^\pm)$:
    $\widehat{V}_s(\phi_{n+1}^\pm, \phi_n^\pm) = \frac{K_s}{2} \left(\phi_{n+1}^\pm - \phi_n^\pm\right)^2$;
  - a *pairing* potential $\widehat{V}_p(\phi_n^+, \phi_n^-)$:
    $\widehat{V}_p(\phi_n^+, \phi_n^-) = \frac{K_p}{2} r^2 \left[(2 - \cos\phi_n^+ - \cos\phi_n^-)^2 + (\sin\phi_n^+ + \sin\phi_n^-)^2\right]$.
- The kinetic energy of a base is $\widehat{T} = \frac{I}{2}\left(\dot{\phi}_n^\pm\right)^2$.

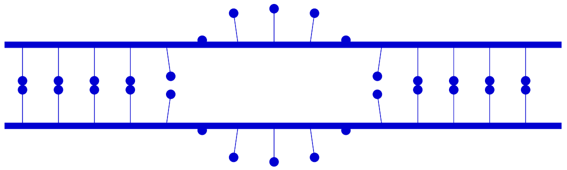
The continuum Y-model

- Change of coordinates: $\psi_n = (\phi_n^+ + \phi_n^-)/2$ and $\chi_n = (\phi_n^+ - \phi_n^-)/2$.
- The distance δ between the base sites is small: assume interpolating fields $\psi(x,t)$ and $\chi(x,t)$ with $\psi(n\delta,t) = \psi_n(t)$ and $\chi(n\delta,t) = \chi_n(t)$.
- The equations of motion are

$$\begin{aligned} \psi_{tt} &= \kappa_s \psi_{xx} - \kappa_p \sin\psi \cos\chi - \mu\psi_t; \\ \chi_{tt} &= \kappa_s \chi_{xx} - \kappa_p \sin\chi(\cos\psi - \cos\chi) - \mu\chi_t. \end{aligned}$$

where $\kappa_s = K_s\delta^2/I$, $\kappa_p = K_p r^2/I$, and μ is the dissipation coefficient.

- The symmetric configuration $\chi = 0$ is invariant and gives the (damped) sine-Gordon equation



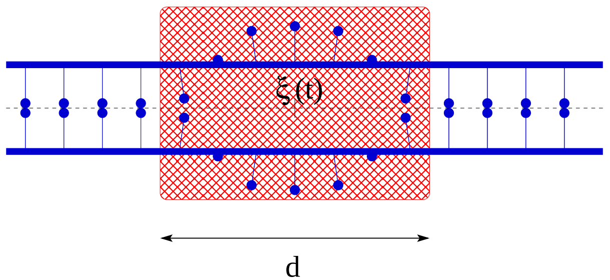
$$\psi_{tt} = \kappa_s \psi_{xx} - \kappa_p \sin\psi - \mu\psi_t$$

and has a family of travelling soliton solutions ($\mu = 0$).

- Similar for the anti-symmetric configuration with $\psi = 0$ (double sG).

Including the interaction with RNAP

- Assumptions:
  - RNAP is present on DNA chain;
  - RNAP acts locally on DNA;
  - RNAP moves by pulling on DNA chain;
  - There is sufficient ATP to feed RNAP.



- Write $\xi(t)$ for the centre of the RNAP and d for the radius of the region of binding with the DNA, i.e., the fields feel the RNA for $\xi - d < x < \xi + d$. The interaction potential is

$$W(\phi, x, \xi) = W_0(\phi)\, R(\xi, x), \quad \text{with} \quad R(\xi, x) = \begin{cases} 1, & |x-\xi| < d \\ 0, & |x-\xi| > d \end{cases}$$

- The interaction potential $W_0(\phi)$ should keep the DNA open, thus $\phi = \pi$ should be a stable equilibrium. We will use $W_0(\phi) = K_r \cos\phi$.
- Constant RNAP pulling force P along the axis: $\xi_{tt} = P - \nu \xi_t$.

DNA-RNAP interaction dynamics

The DNA-RNAP interaction gives for the equations of motion

$$\begin{aligned}\psi_{tt} &= \kappa_s\psi_{xx} - [\kappa_p - \kappa_r R(\xi,x)]\sin\psi\cos\chi - \mu\psi_t;\\ \chi_{tt} &= \kappa_s\chi_{xx} - \kappa_p\sin\chi(\cos\psi-\cos\chi) + \kappa_r R(\xi,x)\sin\chi\cos\psi - \mu\chi_t;\\ \xi_{tt} &= P - \nu\xi_t.\end{aligned}$$

Asymptotically, the RNAP moves with speed $c = P/\nu$, thus $\xi(t) = ct + \xi_0$.

Going to a moving frame $z = \dfrac{x-\xi(t)}{\sqrt{\kappa_s - c^2}}$ gives

$$\begin{aligned}\psi_{tt} - \tfrac{2\sigma}{\mu}\psi_{zt} + \mu\psi_t &= \psi_{zz} - [\kappa_p - \kappa_r\rho(z)]\sin\psi\cos\chi - \sigma\psi_z;\\ \chi_{tt} - \tfrac{2\sigma}{\mu}\chi_{zt} + \mu\chi_t &= \chi_{zz} - \kappa_p\sin\chi(\cos\psi-\cos\chi) + \kappa_r\rho(z)\sin\chi\cos\psi - \sigma\chi_z;\end{aligned}$$

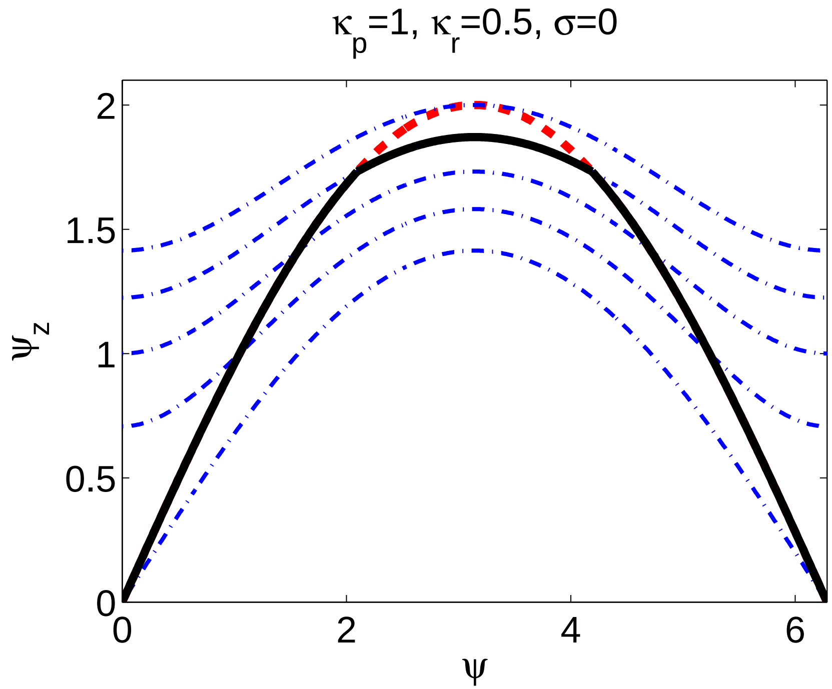
with $\rho(z) = \begin{cases}1, & |z| < \widehat{d}\\ 0, & |z| > \widehat{d}\end{cases}$, $\widehat{d} = \dfrac{d}{\sqrt{\kappa_s - c^2}}$ and $\sigma = \dfrac{\mu c}{\sqrt{\kappa_s - c^2}}$.

Undamped symmetric travelling kinks

Kink equations for the symmetric solutions:

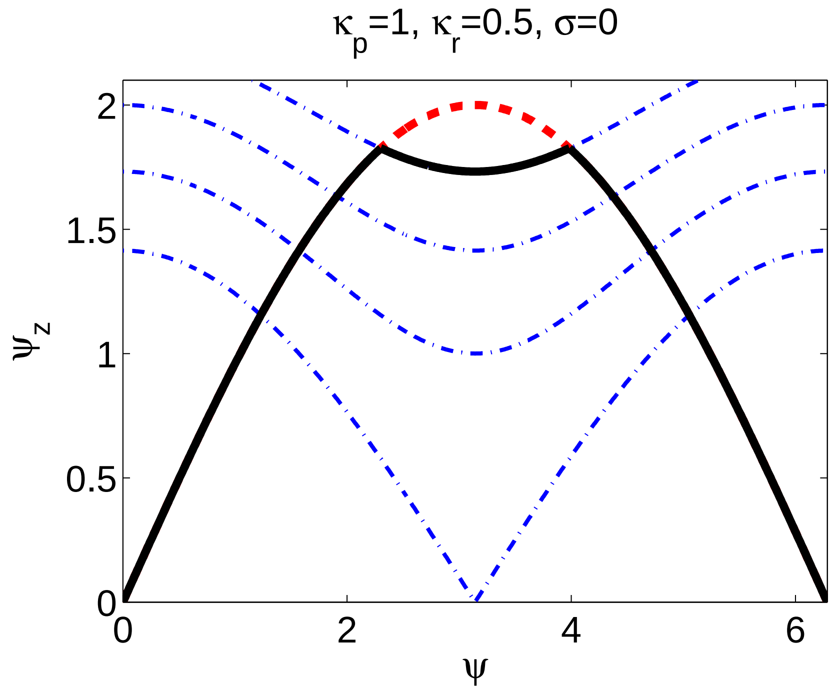
$$\begin{aligned} \psi_{zz} &= \kappa_p \sin\psi - \sigma\psi_z, & |z| > \widehat{d}; \\ \psi_{zz} &= [\kappa_p - \kappa_r] \sin\psi - \sigma\psi_z, & |z| < \widehat{d}. \end{aligned}$$

No damping ($\sigma = 0$), $\kappa_r < \kappa_p$



Exist for any $\widehat{d} > 0$

No damping ($\sigma = 0$), $\kappa_r > \kappa_p$



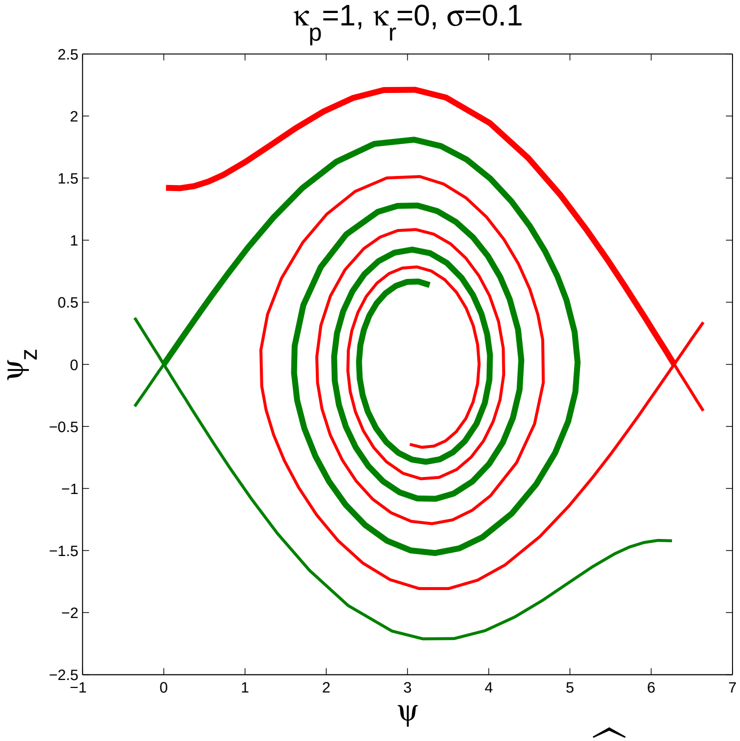
Exist for any $\widehat{d} > 0$

Damped symmetric travelling kinks

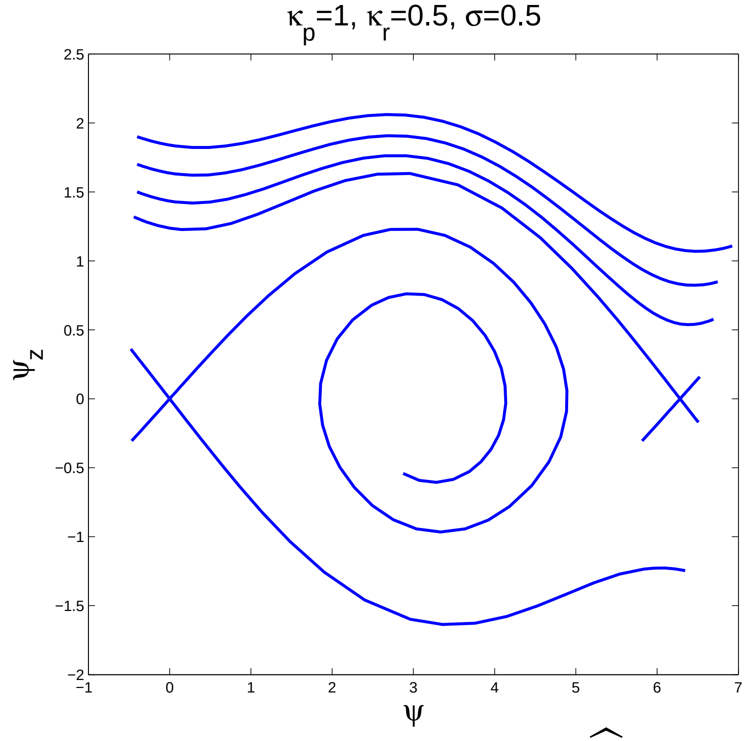
Kink equations for the symmetric solutions:

$$
\begin{aligned}
\psi_{zz} &= \kappa_p \sin\psi - \sigma\psi_z, && |z| > \widehat{d}; \\
\psi_{zz} &= [\kappa_p - \kappa_r] \sin\psi - \sigma\psi_z, && |z| < \widehat{d}.
\end{aligned}
$$

Damped $(\sigma > 0)$, $\kappa_r < \kappa_p$:



Outer: $|z| > \widehat{d}$



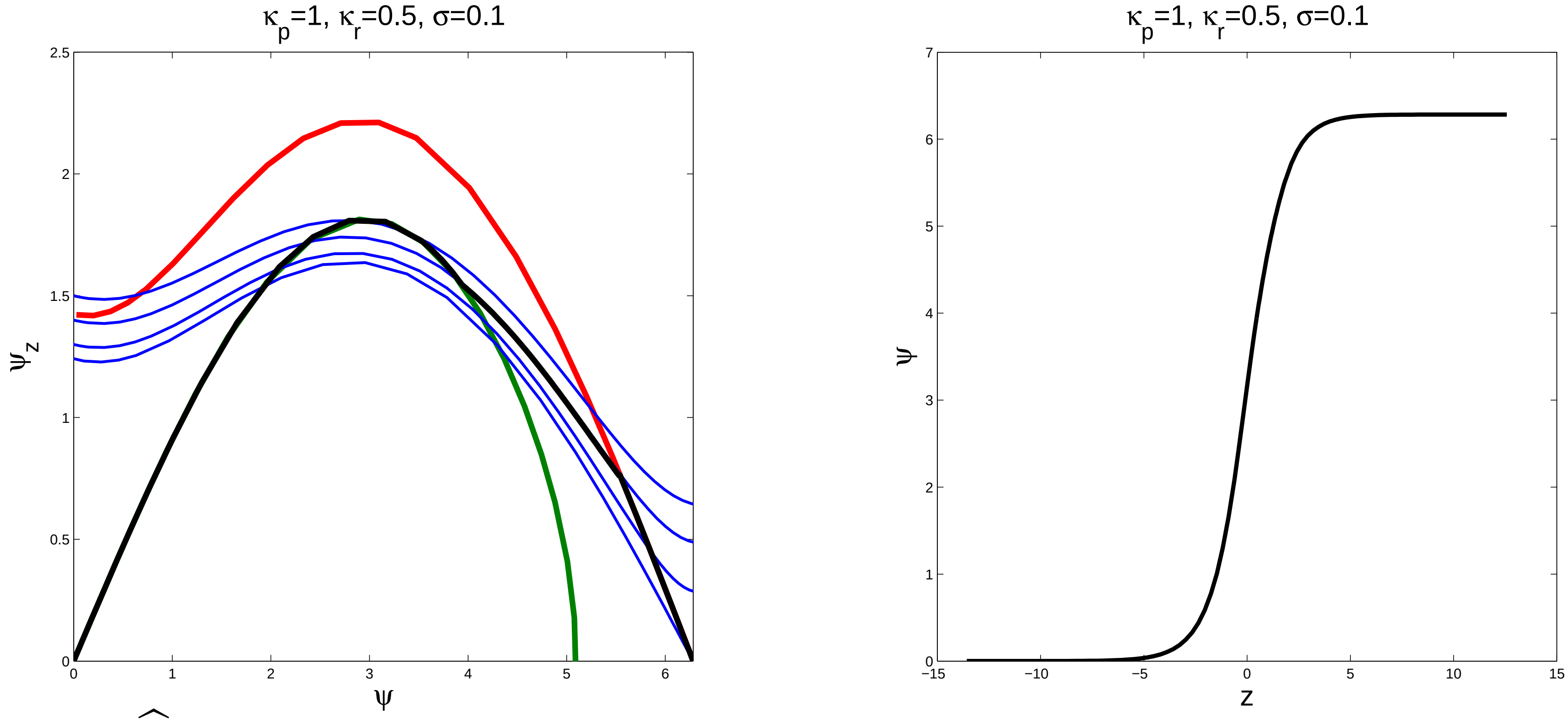
Inner: $|z| < \widehat{d}$

Damped symmetric travelling kinks

Kink equations for the symmetric solutions:

$$\begin{aligned} \psi_{zz} &= \kappa_p \sin\psi - \sigma\psi_z, & |z| > \widehat{d}; \\ \psi_{zz} &= [\kappa_p - \kappa_r] \sin\psi - \sigma\psi_z, & |z| < \widehat{d}. \end{aligned}$$

Damped $(\sigma > 0)$, $\kappa_r < \kappa_p$:

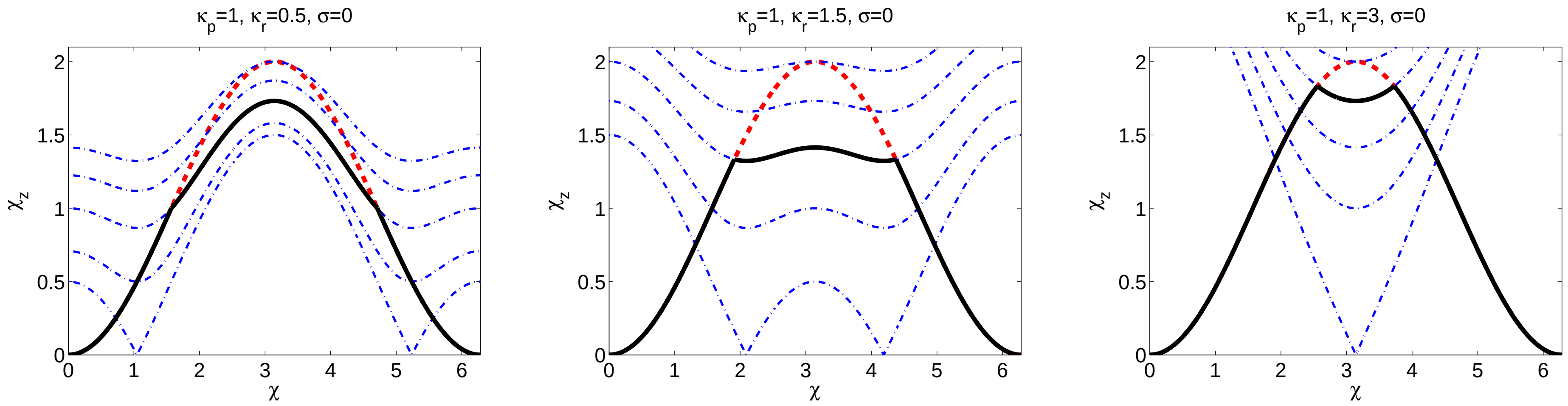


Exist for $\widehat{d}$ not too small and similar for $\kappa_r > \kappa_p$.

Anti-symmetric travelling kinks

Equations for the anti-symmetric solutions:

$$\begin{aligned} \chi_{zz} &= \kappa_p \sin\chi(1-\cos\chi) - \sigma\psi_z, & |z| > \widehat{d}; \\ \chi_{zz} &= \sin\chi\,[(\kappa_p - \kappa_r) - \kappa_p\cos\chi] - \sigma\psi_z, & |z| < \widehat{d}. \end{aligned}$$

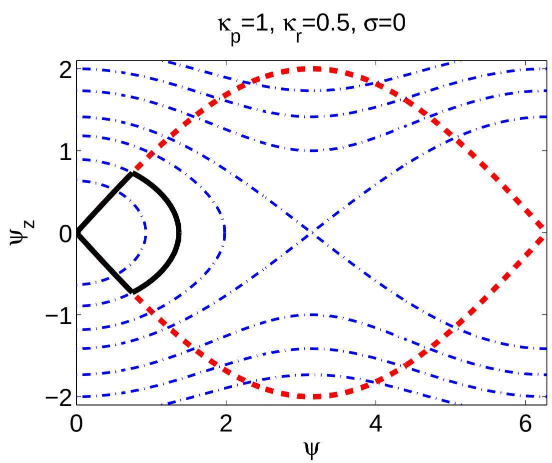


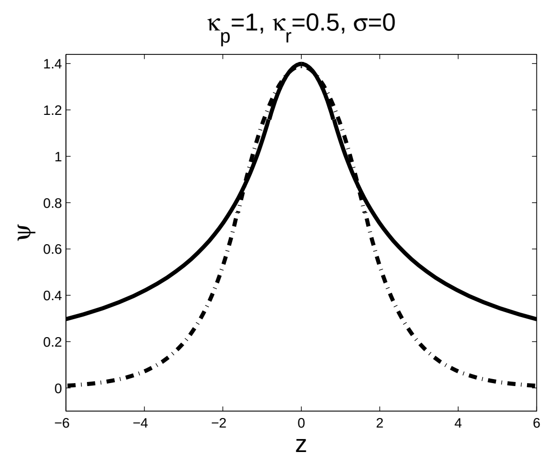
- Undamped kinks exist for any $\widehat{d} > 0$;
- Damped kinks exist for $\widehat{d}$ not too small

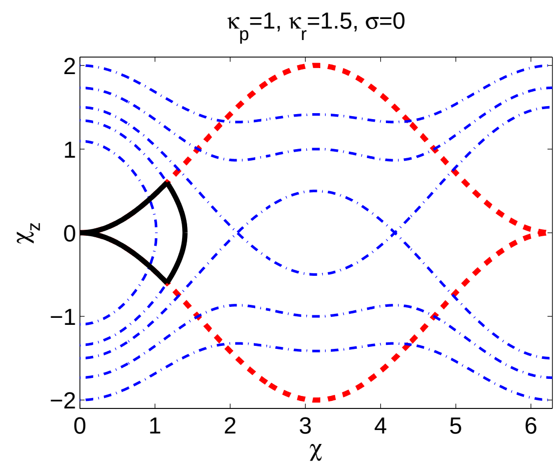
New soliton solutions:

Symmetric

Anti-symmetric



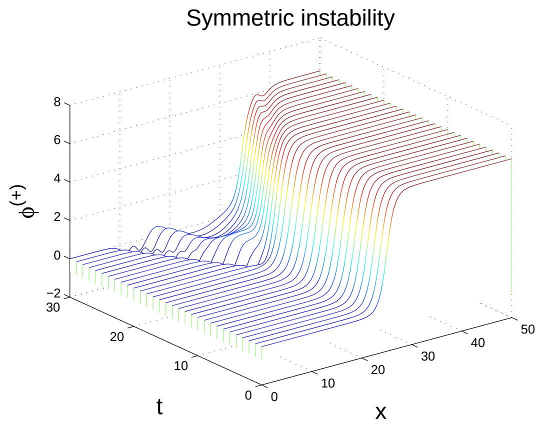


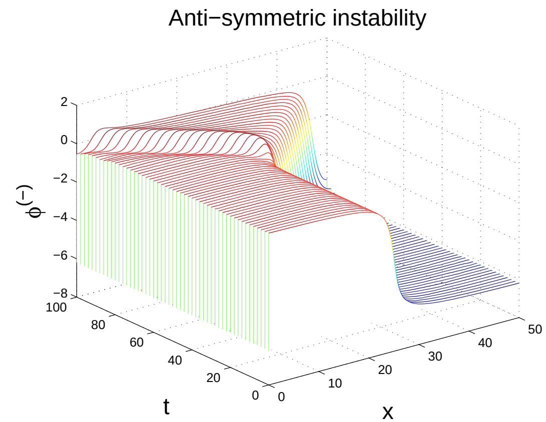


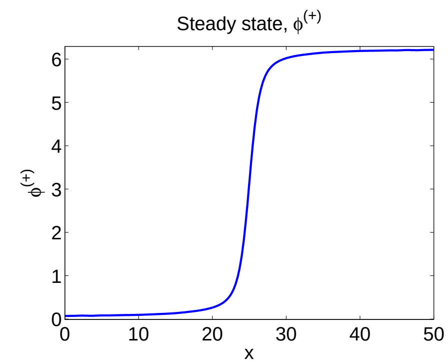
- In the symmetric section (left), solitons exist for $\kappa_r > \kappa_p$ and any $\widehat{d} > 0$;
- In the anti-symmetric section (right), these solitons exist always.
- Middle plot show the soliton shapes, the dashed curve is the symmetric one, the solid curve the anti-symmetric one.

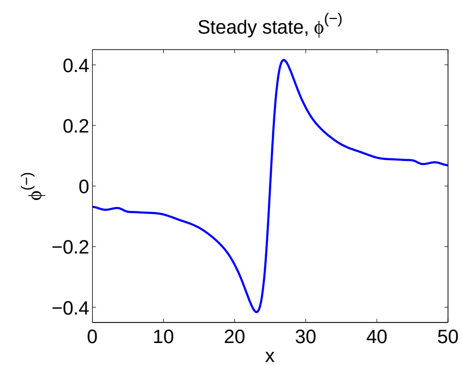
Kink stability

- Without the RNAP the symmetric and anti-symmetric kinks are stable in their own invariant sub space, but unstable under symmetry breaking perturbations.



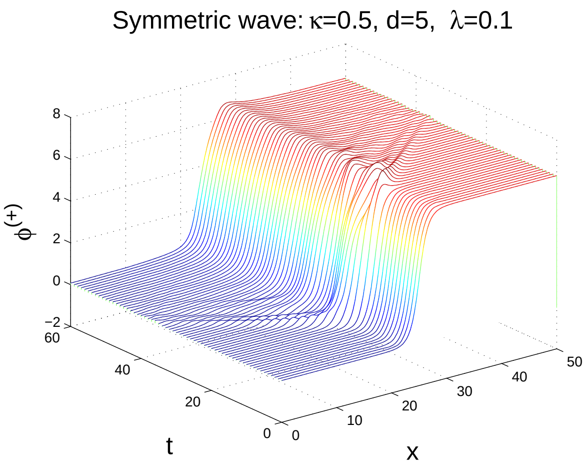


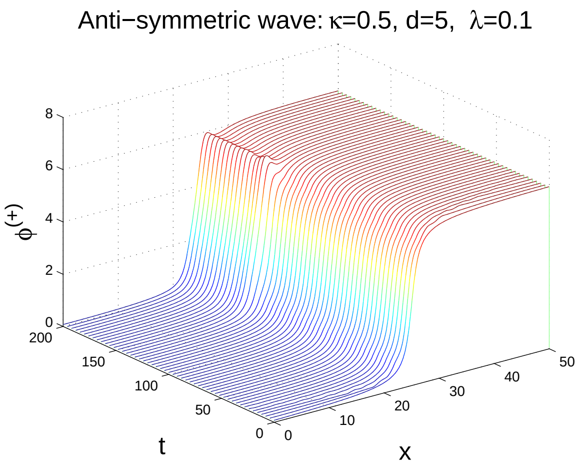




Kink stability

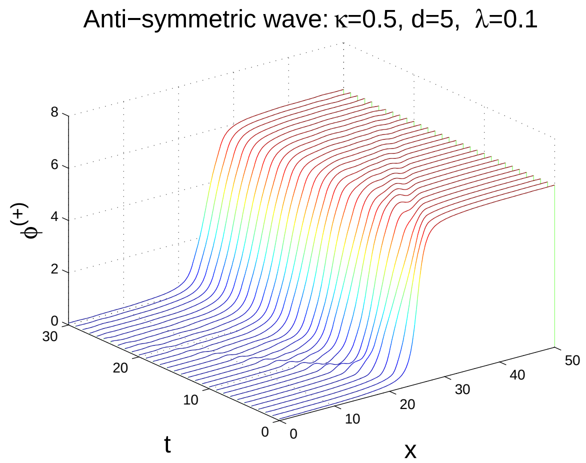
- Without the RNAP the symmetric and anti-symmetric kinks are stable in their own invariant sub space, but unstable under symmetry breaking perturbations.
- The RNAP and dissipation have a stabilising effect on the symmetric and anti-symmetric kink.





Kink stability

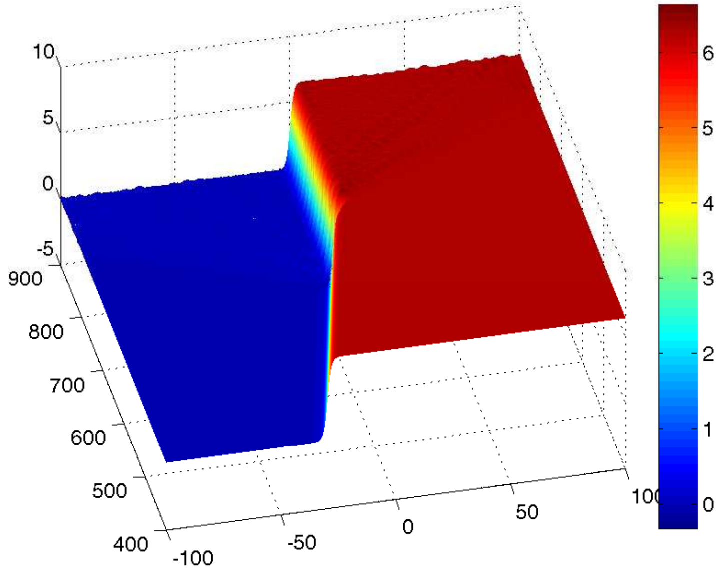
- Without the RNAP the symmetric and anti-symmetric kinks are stable in their own invariant sub space, but unstable under symmetry breaking perturbations.
- The RNAP and dissipation have a stabilising effect on the symmetric and anti-symmetric kink.
- The a-symmetric stable kink undergoes a tiny modification due to the RNAP and stays stable.



[Derks, Gaeta (2011)]

Conclusion and discussion

- For the Josephson junctions we showed existence and stability of stationary fluxons for one and more inhomogeneities. The stability results are valid for general inhomogeneous wave equations.
- In the Josephson junctions, how do travelling fluxons interact with the stationary ones or an inhomogeneity?



Projecting onto families of sine-Gordon-type fluxons could be a good avenue.

Conclusion and discussion

- For the Josephson junctions we showed existence and stability of stationary fluxons for one and more inhomogeneities. The stability results are valid for general inhomogeneous wave equations.
- In the Josephson junctions, how do travelling fluxons interact with the stationary ones or an inhomogeneity? Some results can be derived by projecting onto families of sine-Gordon-type fluxons.
- Can we extend the theory to the existence and stability of pinned fronts in coupled inhomogeneous wave equations? This is needed in the DNA-RNAP models.

Some other researchers in the UK

Some other researchers at UK universities working in related areas:

Aston: Sergei Turitsyn

Bath: Karsten Matthies, Johannes Zimmer

Heriott Watt: Margot Beck, Simon Malham, Noel Smyth

Loughborough: Gennady El, Roger Grimshaw, Karima Khusnutdinova

Nottingham: Hadi Susanto

Oxford: Mason Porter

Warwick: Claude Baesens, Robert MacKay

And many others ....

Thank you!