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Iterated Function Systems with Overlaps in Higher Dimensions

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One-dimensional beta-expansions

Let $\beta \in (1,2)$ be our parameter. Then each $x \in [0, 1/(\beta - 1)]$ has an expansion of the form

$$x = \sum_{n=1}^{\infty} \varepsilon_n \beta^{-n},$$

where $\varepsilon_n \in \{0, 1\}$.

The most common choice of ε_n is via the greedy algorithm, but we are interested here in **all** β-expansions of a given x.

The questions that arise naturally for this model are as follows:

1. Given $x \in (0,1)$, "how many" distinct β-expansions does it have? (In terms of cardinality, dimension, etc.)

2. Are there any $x \in (0,1)$ that have a unique β-expansion? If so, "how many"?

Here is the list of one-dimensional results:

Theorem 1 (P. Erdős, I. Joó and V. Komornik, 1990) *If* $\beta < \frac{1+\sqrt{5}}{2} = 1.618\ldots$, *then every* $x \in (0, 1/(\beta - 1))$ *has a continuum of* β*-expansions.*

Theorem 2 (S, 2003) *For any* $\beta \in \left[\frac{1+\sqrt{5}}{2}, 1\right)$ *the same result is true for almost every* $x \in (0, 1/(\beta - 1))$.

Put

$$\mathcal{U}_\beta = \left\{ x \in (0,1) \mid ! \ (\varepsilon_n)_1^\infty : x = \sum_{n=1}^{\infty} \varepsilon_n \beta^{-n} \right\}$$

(the *set of uniqueness*).

Corollary 3 *The set* $\mathcal{U}_\beta$ *has Lebesgue measure zero for any* $\beta \in (1, 2)$.

The question is, what can one say about the cardinality and – in case it is the continuum – about the Hausdorff dimension of this set. The answer to this question is given by P. Glendinning and myself.

Theorem 4 *(Glendinning-S, 2001) The set $\mathcal{U}_\beta$ is:*

- *countable for $\beta \in \left(\frac{1+\sqrt{5}}{2}, \beta_*\right)$;*

- *an uncountable set of zero Hausdorff dimension if $\beta = \beta_*$; and*

- *a set of positive Hausdorff dimension for $\beta \in (\beta_*, 2)$.*

Here $\beta_* = 1.787231650\ldots$ denote the (transcendental) Komornik-Loreti constant (defined via the Thue-Morse sequence).

Higher dimensions: self-similar sets

Let $\boldsymbol{p}_0, \ldots, \boldsymbol{p}_{m-1}$ be points in $\mathbb{R}^2$ and let $\{f_j\}_{j=0}^{m-1}$ be a finite collection of similitudes of $\mathbb{R}^2$:

$f_j(\boldsymbol{x}) = \beta^{-1}\boldsymbol{x} + (1-\beta^{-1})\boldsymbol{p}_j,\ j = 0, \ldots, m-1,$

where $\beta \in (1, \infty)$ is our parameter.

Then, as is well known, there exists a unique self-similar *attractor* S_β satisfying

$$S_\beta = \bigcup_j f_j(S_\beta).$$

Every $\boldsymbol{x} \in S_\beta$ has at least one *address*, i.e., $(i_1, i_2, \ldots) \in \{0, \ldots, m-1\}$ such that

$$\begin{aligned} \boldsymbol{x} &= \lim_{n \to +\infty} f_{i_1} \ldots f_{i_n}(\boldsymbol{x}_0) \\ &= (\beta - 1) \sum_{n=1}^{\infty} \beta^{-n} \boldsymbol{a}_n, \end{aligned}$$

where $\boldsymbol{x}_0 \in \mathbb{R}^2$ is arbitrary, and $\boldsymbol{a}_n \in \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_m\}$.

Recall that an IFS is said to satisfy the Open Set Condition (OSC) if there exists an open set $O \subset \mathbb{R}^2$ such that

$$O = \bigcup_j f_j(O),$$

and the union is **disjoint**. Loosely speaking, the OSC means that the images $f_j(\Delta)$ do not intersect "by much". Virtually all famous fractals (Sierpiński gasket, Sierpiński carpet, the von Koch curve, etc.) originate from IFSs that do satisfy the OSC.

We will be interested in IFSs which usually do **not** satisfy the OSC.

Main assumption: there exist i, j such that

$$f_i(\Omega) \cap f_j(\Omega)$$

has an interior point, where

$$\Omega = \mathrm{conv}(\boldsymbol{p}_0, \ldots, \boldsymbol{p}_{m-1}).$$

Analogue of Erdős-Joó-Komornik Theorem.

Theorem 5 *(S, 2006) For each* $\boldsymbol{p}_0, \ldots, \boldsymbol{p}_{m-1}$ *there exists* $\beta_0 > 1$ *such that*

1. *for any* $\beta \in (1, \beta_0)$ *each point* $x \in S_\beta$ *has* $2^{\aleph_0}$ *distinct addresses;*

2. *for* $\beta > \beta_0$ *the set of uniqueness is nonempty.*

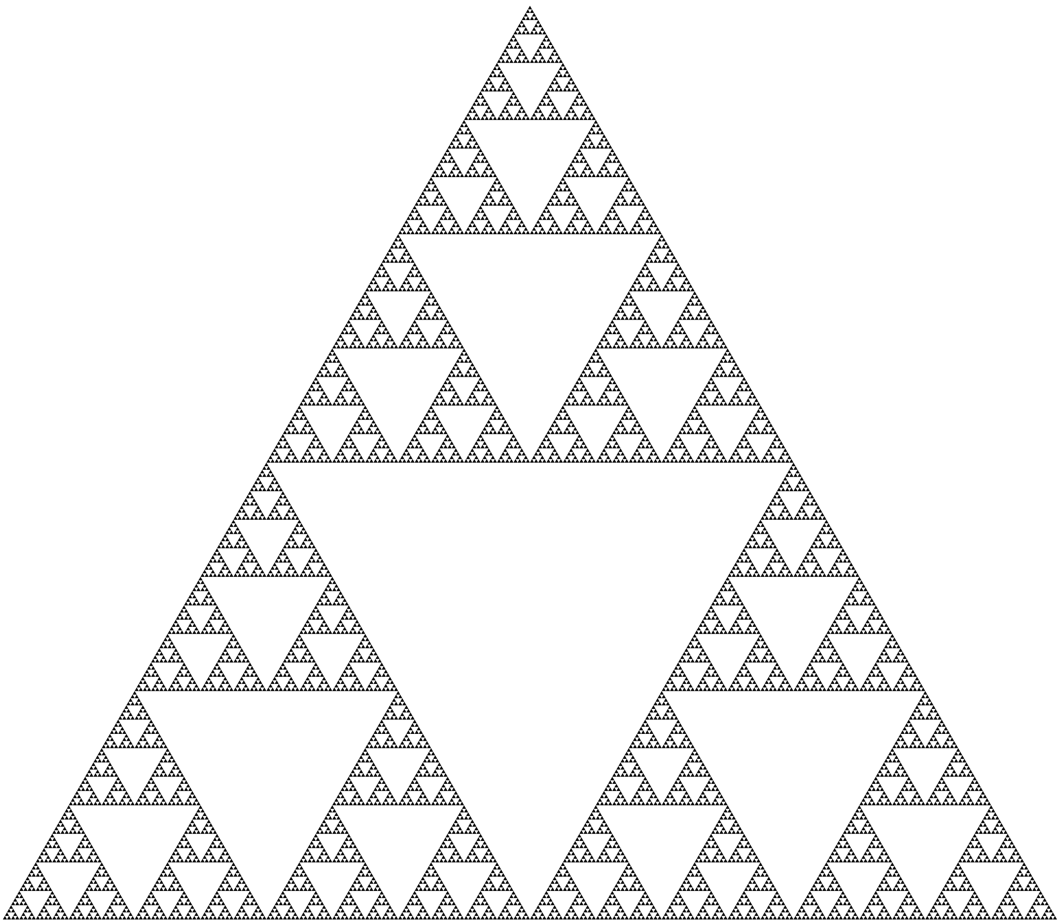
For the triangular case $\beta_0 = 1.464\ldots$ is the root of $x^3 = x^2 + 1$.

Assume $\beta > \beta_0$. What can we say about the Lebesgue measure-a.e. point?

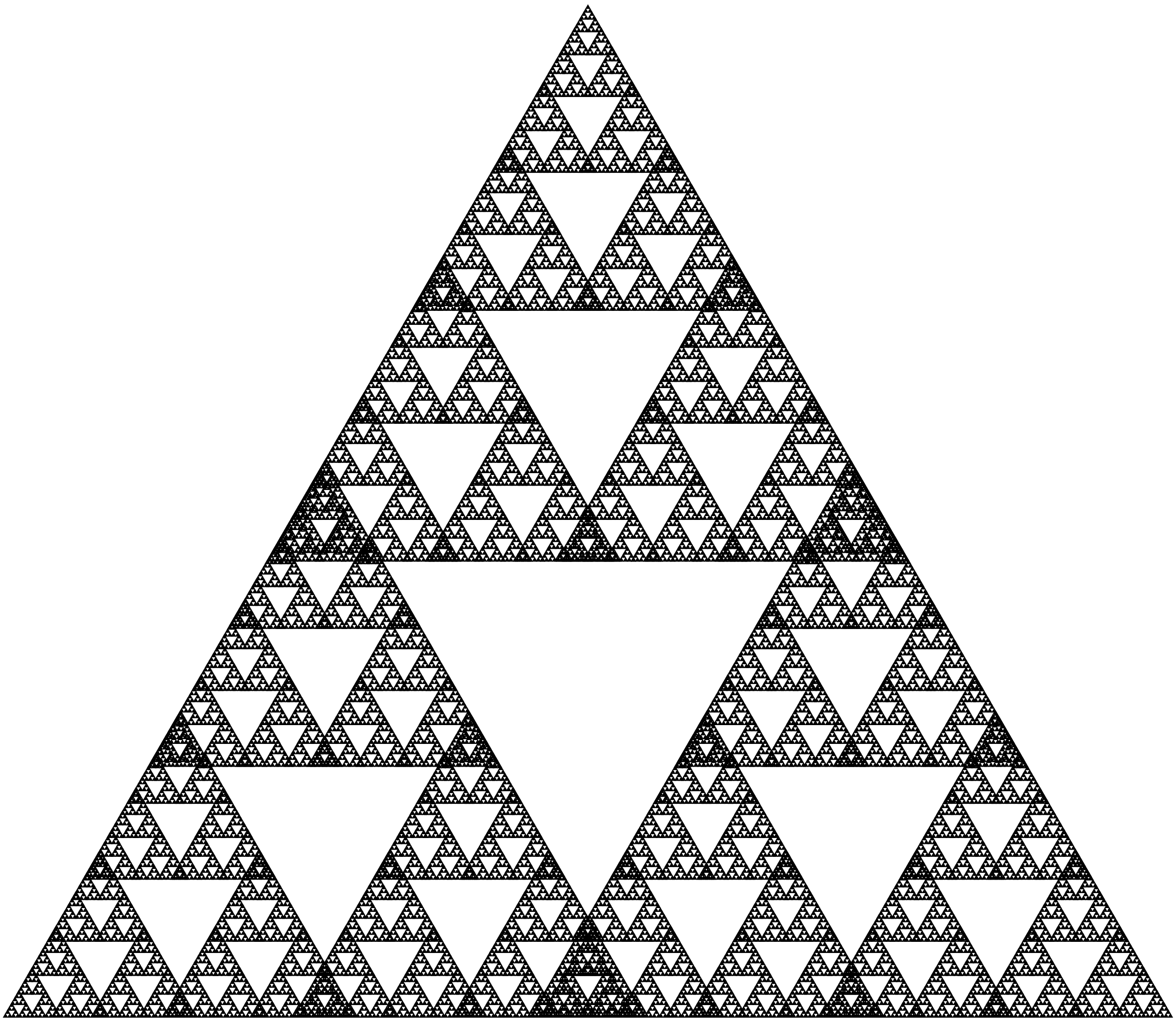
The 1D approach involves *greedy expansions* - hard to apply in 2D!

Besides, we do not always know when S_β has a positive Lebesgue measure – even in the triangular case!

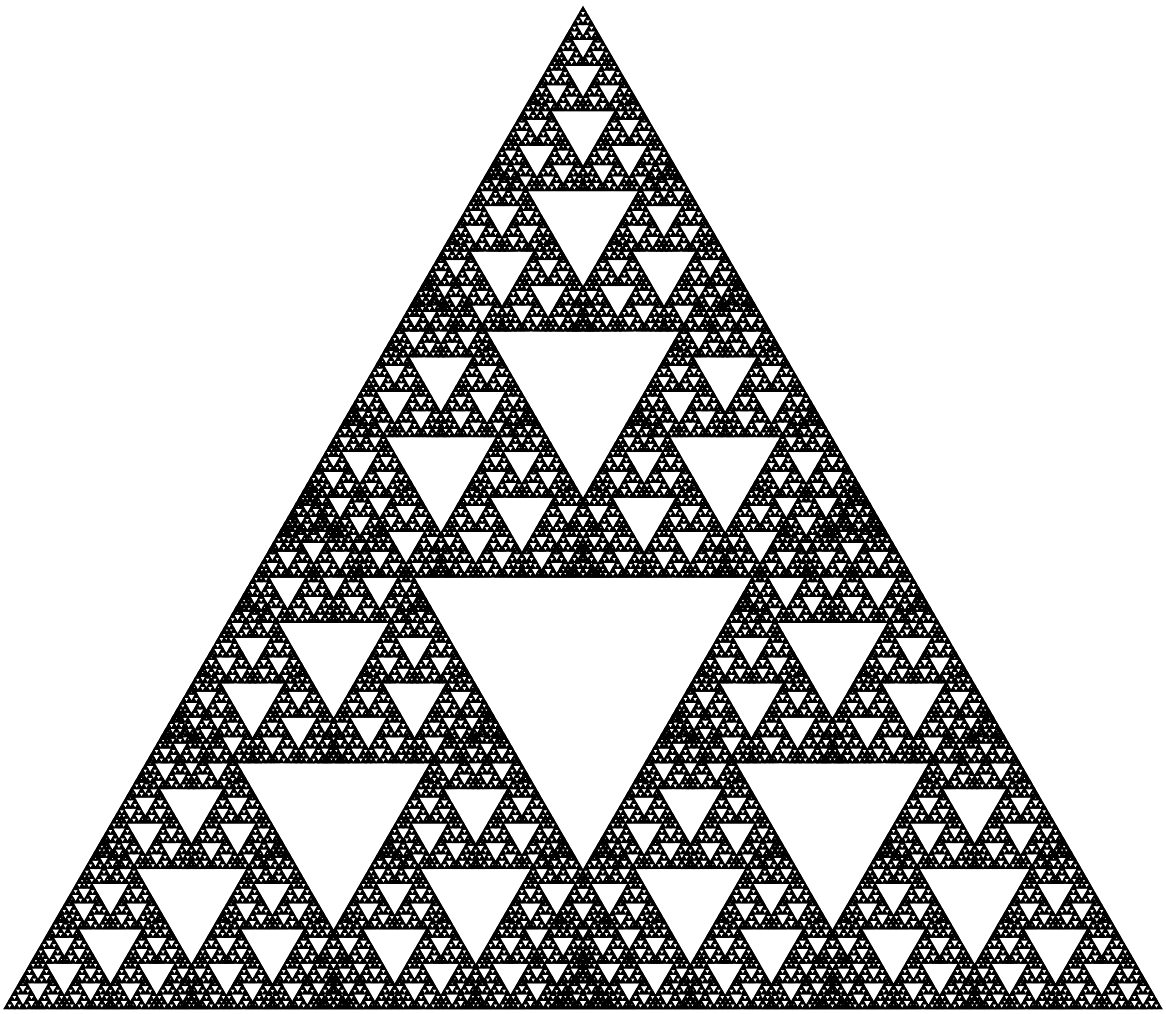
The most famous case is $\beta = 2$:



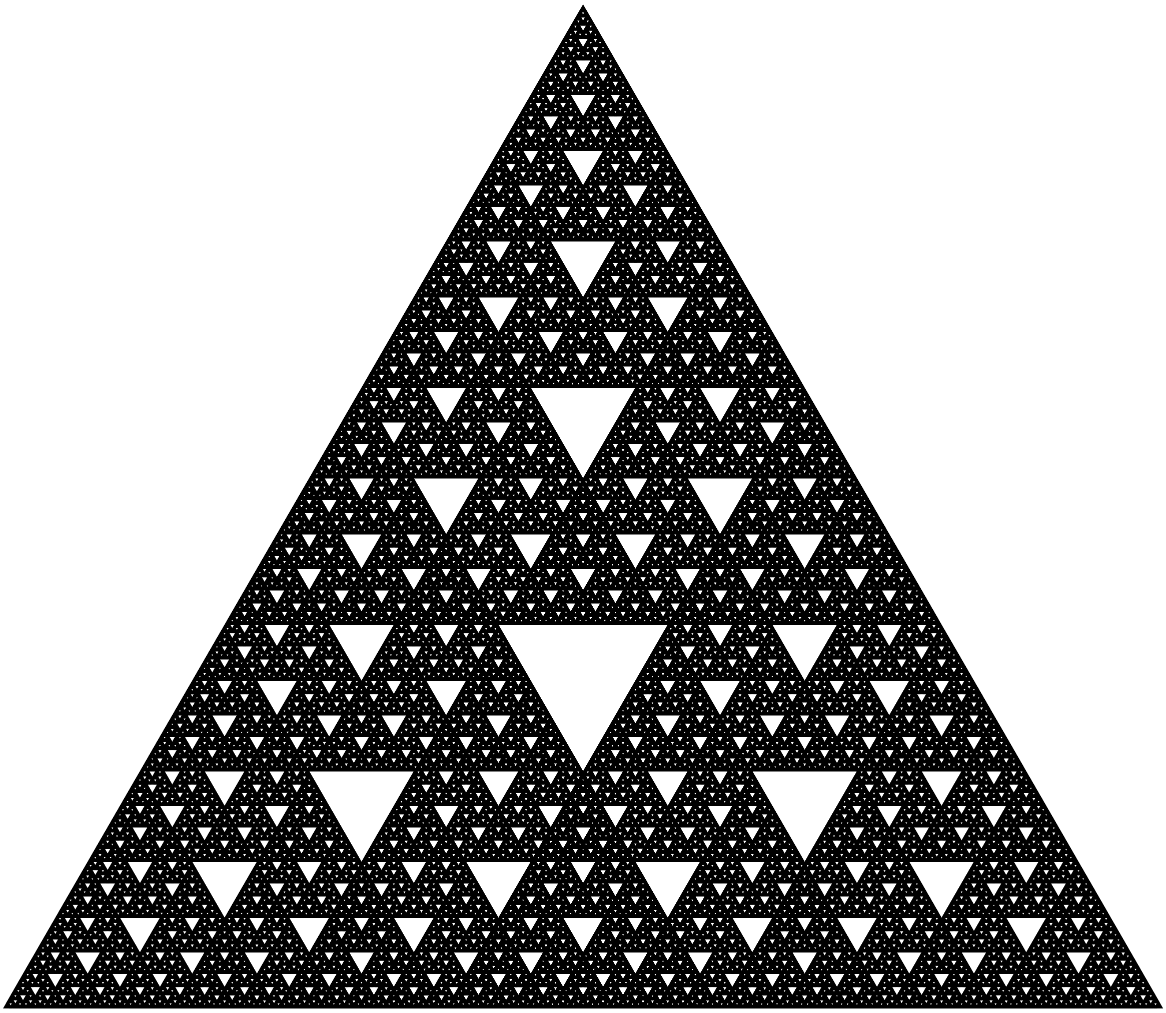
The Sierpiński Gasket



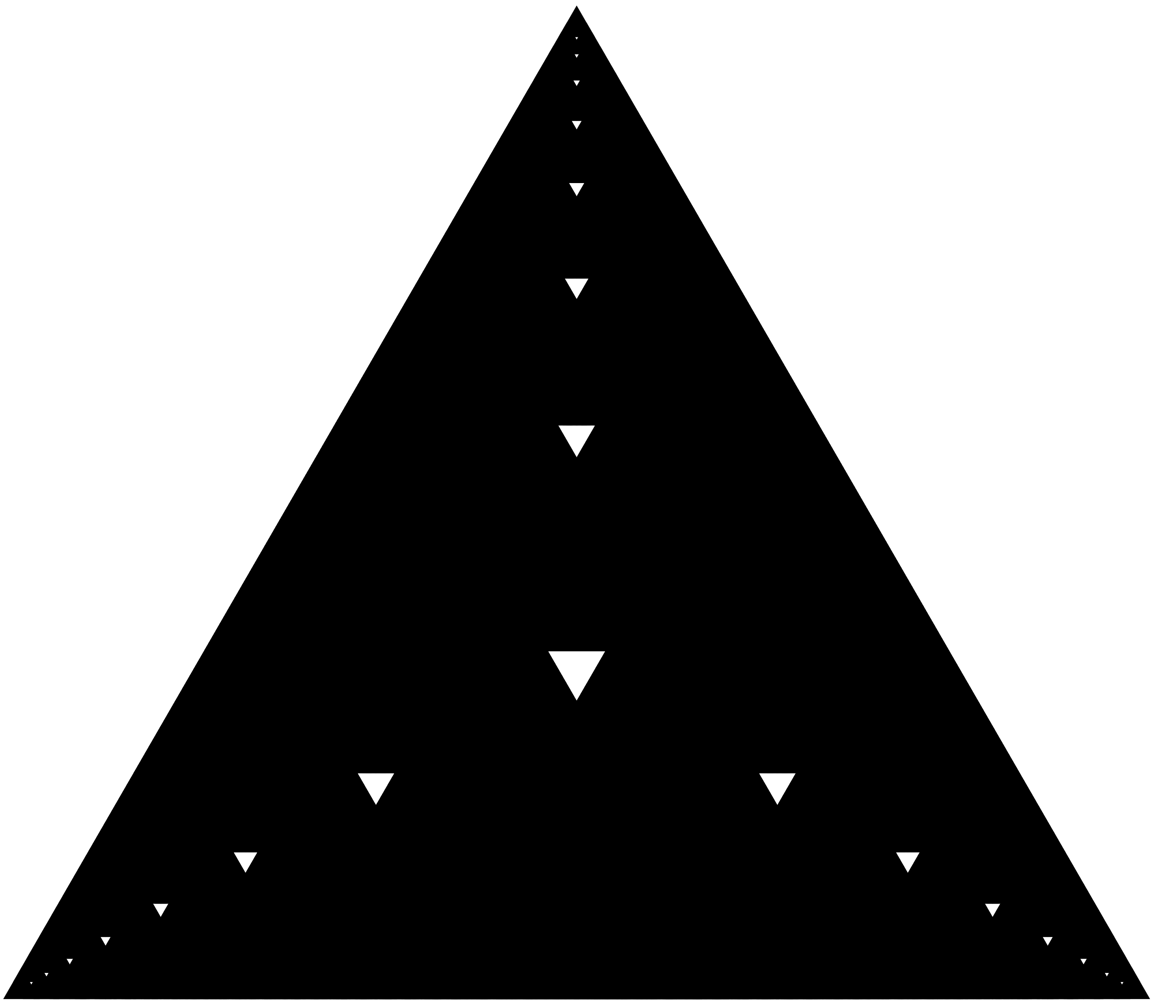
The fat Sierpiński Gasket for $\beta = 1.8$
(zero Lebesgue measure)



The fat Sierpiński Gasket for $\beta = 1.72$



The Golden Gasket, $\beta = \frac{1+\sqrt{5}}{2} \approx 1.618$.

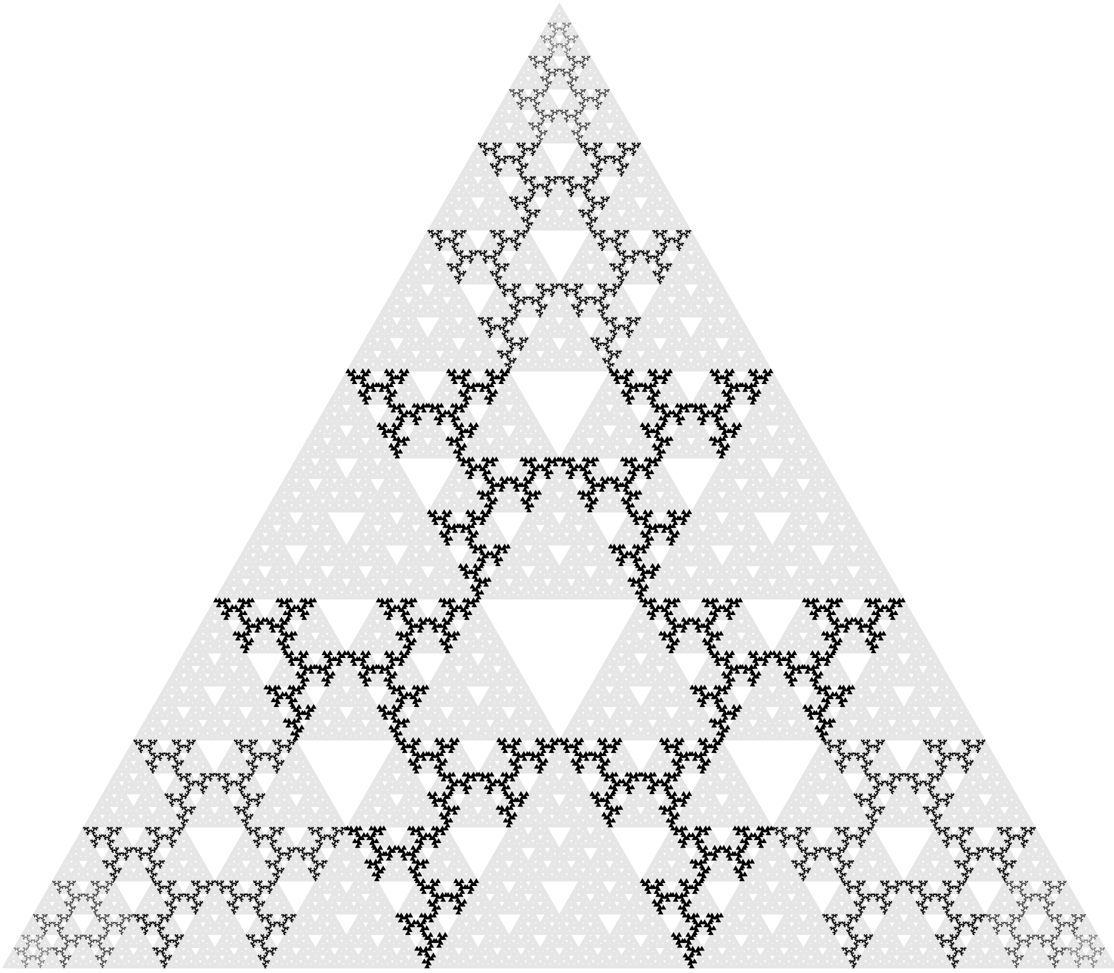


The fat Sierpiński Gasket for $\beta = 1.54$
(has a nonempty interior)

Theorem 6 *Under the main assumption (existing overlap), the set of points $x \in S_\beta$ having a continuum of addresses has the same dimension (Hausdorff as well as box-counting) as S_β itself.*

In particular, if $\mathrm{mes}(S_\beta) > 0$, *then Lebesgue-a.e. $x \in S_\beta$ has a continuum of addresses.*

Corollary 7 *The set of uniqueness has dimension less than* $\dim_H(S_\beta)$.



The set of uniqueness
superimposed on the golden gasket

Here

$$\dim_H(\mathcal{S}_\varphi) = 1.93\ldots,$$
$$\dim_H(U_\varphi) = 1.44\ldots$$

(Broomhead-Montaldi-S, 2004)

Concluding remarks and open questions.

1. All of the mentioned results are valid for $\mathbb{R}^n$, provided we slightly modify our assumption on the overlap.

2. For the future study: dynamical properties of the shift $T : U_\beta \to U_\beta$ (starting with determining for which β the set U_β is uncountable) and the greedy shift.