Anomalous Diffusion in Intermittent Maps

Nickolay Korabel[1] Rainer Klages[2] Aleksey V. Chechkin[3]

[1]Courant Institute of Mathematical Sciences, New York

[2]School of Mathematical Sciences, Queen Mary, University of London

[3]Institute for Theoretical Physics, Kharkov, Ukraine

LMS Durham Symposium, 11/07/2006

Outline

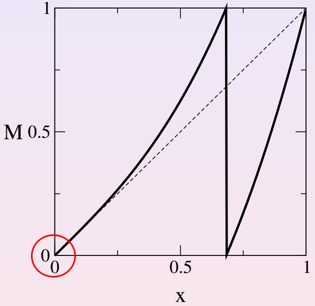
1. **motivation:**
   - a simple intermittent map modeling anomalous diffusion
   - simulations: fractal anomalous diffusion coefficient

Outline

1. **motivation:**
   - a simple intermittent map modeling anomalous diffusion
   - simulations: fractal anomalous diffusion coefficient
2. **analysis:**
   - stochastic theory of anomalous diffusion
   - fractional diffusion equation for this map
   - (towards a deterministic theory of anomalous diffusion)

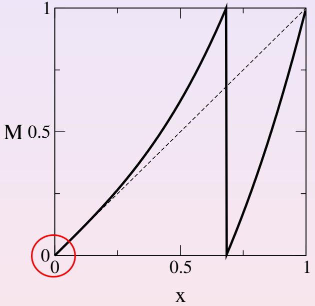
Intermittency in the Pomeau-Manneville map

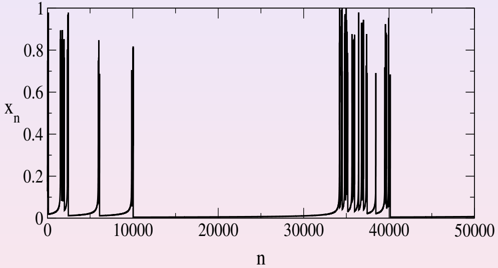
intermittent map: $x_{n+1} = M(x_n) = x_n + a x_n^z \mod 1 \; , \; z \geq 1 \; , \; a = 1$



Intermittency in the Pomeau-Manneville map

intermittent map: $x_{n+1} = M(x_n) = x_n + a x_n^z \mod 1 \; , \; z \geq 1 \; , \; a = 1$





phenomenology of intermittency (I): long periodic *laminar phases* interrupted by chaotic *bursts*; here due to a marginally stable fixed point, $M'(0) = 1$ (Pomeau, Manneville, 1980)

$\Rightarrow$ map not hyperbolic

Dynamical regimes in the Pomeau-Manneville map

three types of dynamics (Gaspard, Wang, 1988):

- $1 \leq z < 3/2$: normal dynamics (Gaussian fluctuations)
- $3/2 \leq z < 2$: transient anomalous dynamics
- $z \geq 2$: anomalous dynamics (Lévy fluctuations)

Dynamical regimes in the Pomeau-Manneville map

three types of dynamics (Gaspard, Wang, 1988):

- $1 \leq z < 3/2$: normal dynamics (Gaussian fluctuations)
- $3/2 \leq z < 2$: transient anomalous dynamics
- $z \geq 2$: anomalous dynamics (Lévy fluctuations)

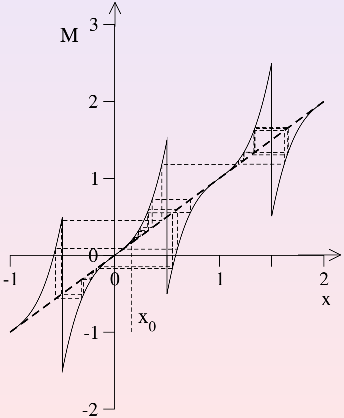
example: let $L_n := \sum_{i=0}^{n-1} \ln |M'(x_i)|$; then for $n \to \infty$

$$< L_n >_{\varrho 0} \sim \begin{cases} n, & 1 \leq z < 2 \\ n / \ln n, & z = 2 \\ n^{\alpha}, & z > 2 \quad \text{with} \quad \alpha := 1/(z-1) \end{cases}$$

note: $\exists$ *invariant probability density* $\delta(0)$ and a *non-normalizable infinite invariant measure* on $(0, 1)$ (Thaler 1983; Hu, Young, 1995)

An intermittent map with anomalous diffusion

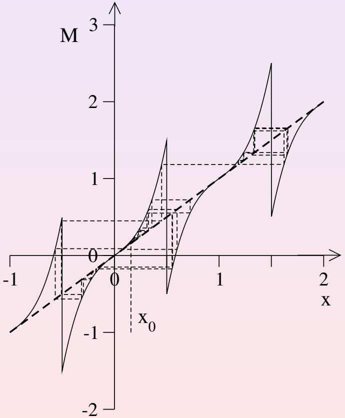
continue PM map by $M(-x) = -M(x)$ and $M(x+1) = M(x)+1$:
(Geisel, Thomae, 1984; Zumofen, Klafter, 1993)



$\Rightarrow$ *deterministic* random walk on the line

An intermittent map with anomalous diffusion

continue PM map by $M(-x) = -M(x)$ and $M(x+1) = M(x)+1$:
(Geisel, Thomae, 1984; Zumofen, Klafter, 1993)



⇒ *deterministic* random walk on the line

define anomalous diffusion coefficient (Metzler, Klafter, 2000)

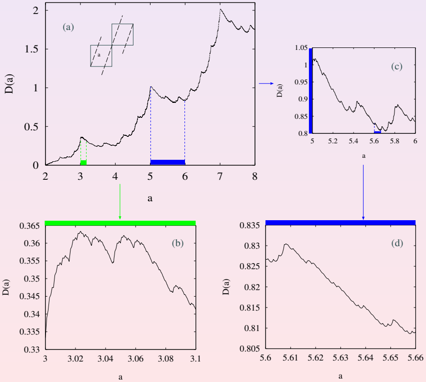
$$D(z,a) := \lim_{n\to\infty} \frac{\Gamma(1+\alpha)}{2}\frac{\langle x^2\rangle}{n^\alpha}$$

with $\alpha := \begin{cases} 1, & 1 \le z \le 2 \\ \frac{1}{z-1}, & z > 2 \end{cases}$

here: look at $K(z,a) := \frac{2D}{\Gamma(1+\alpha)}$

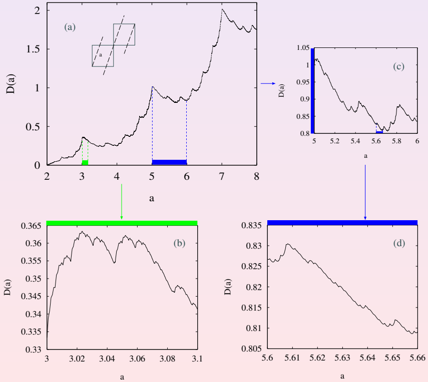
Fractal normal diffusion coefficient

$z = 1$: piecewise linear map with normal diffusion; $\exists$ exact analytical results for diffusion coefficient $D(a) = K(1, a + 1)$ (R.K., Dorfman, 1995; Groeneveld, R.K., 2002; Cristadoro, 2006)



Fractal normal diffusion coefficient

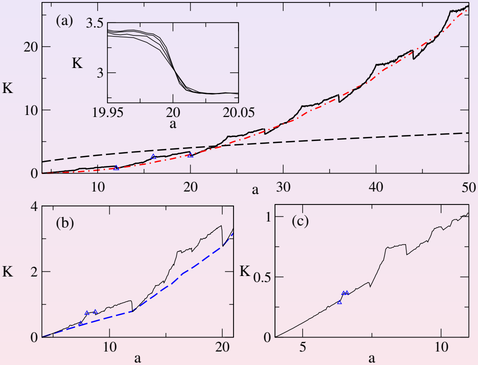
$z = 1$: piecewise linear map with normal diffusion; $\exists$ exact analytical results for diffusion coefficient $D(a) = K(1, a+1)$ (R.K., Dorfman, 1995; Groeneveld, R.K., 2002; Cristadoro, 2006)



- $D(a)$ is a fractal function of the slope a
- fractal dimension?
- structure related to (dense) Markov partitions
- $D(a)$ continuous

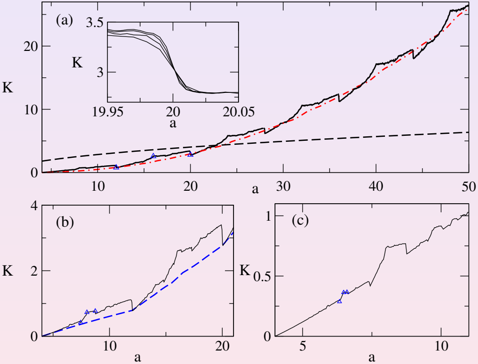
Fractal anomalous diffusion coefficient

$K(3, a)$ from computer simulations (Korabel, R.K. et al., 2005):



Fractal anomalous diffusion coefficient

$K(3, a)$ from computer simulations (Korabel, R.K. et al., 2005):



- $\exists$ fractal structure
- $K(3, a)$ conjectured to be *discontinuous* (inset) on dense set
- comparison with *stochastic theory*, see dashed lines

CTRW theory I: Montroll-Weiss equation

first applied to photocopying machines (Scher, Montroll, 1973)

CTRW theory I: Montroll-Weiss equation

first applied to photocopying machines (Scher, Montroll, 1973)

master equation for a stochastic process defined by *waiting time distribution* $w(t)$ and *distribution of jumps* $\lambda(x)$:

$$\varrho(x,t) = \int_{-\infty}^{\infty} dx' \lambda(x-x') \int_0^t dt' \, w(t-t') \, \varrho(x',t') + \\ +(1 - \int_0^t dt' w(t'))\delta(x)$$

structure: jump + no jump for particle starting at $(x,t)=(0,0)$

CTRW theory I: Montroll-Weiss equation

first applied to photocopying machines (Scher, Montroll, 1973)

master equation for a stochastic process defined by *waiting time distribution* $w(t)$ and *distribution of jumps* $\lambda(x)$:

$$\varrho(x,t) = \int_{-\infty}^{\infty} dx' \lambda(x-x') \int_0^t dt' \; w(t-t') \; \varrho(x',t') + \\ +(1-\int_0^t dt' w(t'))\delta(x)$$

structure: jump + no jump for particle starting at $(x,t)=(0,0)$

F̂ourier-L̃aplace transform yields Montroll-Weiss equation (1965)

$$\hat{\tilde{\varrho}}(k,s) = \frac{1-\tilde{w}(s)}{s} \frac{1}{1-\hat{\lambda}(k)\tilde{w}(s)}$$

with mean square displacement $\langle \tilde{x^2}(s) \rangle = -\left. \frac{\partial^2 \hat{\tilde{\varrho}}(k,s)}{\partial k^2} \right|_{k=0}$

CTRW theory II: application to maps

apply CTRW to maps (Geisel, Klafter, 1984ff): need $w(t), \lambda(x)$!

- continuous-time approximation for the PM-map

$$x_{n+1} - x_n \simeq \frac{dx}{dt} = ax^z , \quad x \ll 1$$

solve for initial condition $x(0) = x_0$, define jump as $x(t) = 1$ and compute

$$w(t) \simeq \varrho_{in}(x_0) \left| \frac{dx_0}{dt} \right|$$

by assuming uniform density of injection points $\varrho_{in}(x) \simeq 1$

CTRW theory II: application to maps

apply CTRW to maps (Geisel, Klafter, 1984ff): need $w(t), \lambda(x)$!

- continuous-time approximation for the PM-map

$$x_{n+1} - x_n \simeq \frac{dx}{dt} = ax^z \, , \quad x \ll 1$$

solve for initial condition $x(0) = x_0$, define jump as $x(t) = 1$ and compute

$$w(t) \simeq \varrho_{in}(x_0) \left| \frac{dx_0}{dt} \right|$$

by assuming uniform density of injection points $\varrho_{in}(x) \simeq 1$

- *revised* ansatz for jumps:

$$\lambda(x) = \frac{p}{2}\delta(|x| - \ell) + (1-p)\delta(x)$$

with jump length ℓ, escape probability $p := 2(\frac{1}{2} - x_c)$, $M(x_c) := 1$
CTRW machinery ... yields exactly

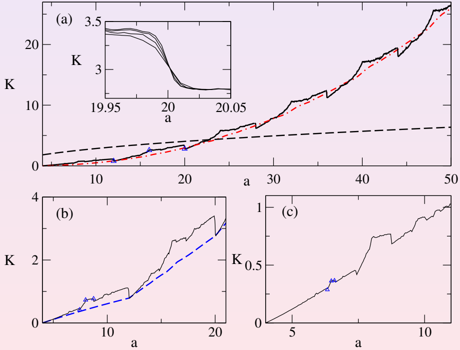
$$K = p\ell^2 \begin{cases} \frac{a^\gamma \sin(\pi\gamma)}{\pi\gamma^{1+\gamma}}, & 0 < \gamma < 1 \\ a\frac{\gamma-1}{\gamma}, & \gamma \geq 1 \end{cases} \quad , \quad \gamma := 1/(z-1)$$

Dynamical crossover in anomalous diffusion

define average jump length:

$\ell_1 := < |M(x) - x| >_{\varrho_0=1,escape} \quad \Rightarrow K \sim a^{5/2} \quad$ for $\ell_1 \gg 2$

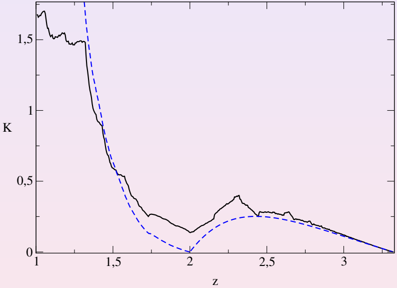
$\ell_2 := < |[M(x)]| >_{\varrho_0=1,escape} \quad \Rightarrow K \sim p(a) \quad$ for $\ell_2 \ll 2$



- $\exists$ dynamical crossover for $K(3, a)$ between small and large a
- same crossover in periodic Lorentz gas (R.K., Dellago, 2002)

Phase transition from normal to anomalous diffusion

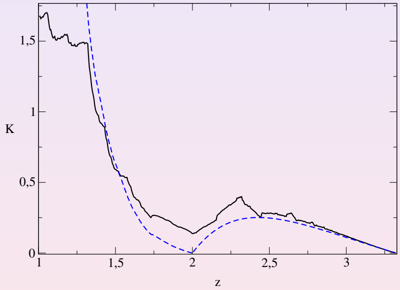
now $K(z,5)$: compare CTRW approximation (blue line, with integer jump length ℓ_2) with simulation results



$\Rightarrow$ full suppression of diffusion due to logarithmic corrections

Phase transition from normal to anomalous diffusion

now $K(z,5)$: compare CTRW approximation (blue line, with integer jump length ℓ_2) with simulation results



$\Rightarrow$ full suppression of diffusion due to logarithmic corrections

$$< x^2 > \sim \begin{cases} n/\ln n, n < n_{cr} \text{ and } \sim n, n > n_{cr}, & z < 2 \\ n/\ln n, & z = 2 \\ n^\alpha/\ln n, n < \tilde{n}_{cr} \text{ and } \sim n^\alpha, n > \tilde{n}_{cr}, & z > 2 \end{cases}$$

Time-fractional equation for subdiffusion

from Montroll-Weiss equation for PM map a time-fractional diffusion equation can be derived:

$$\frac{\partial^\gamma \varrho(x,t)}{\partial t^\gamma} = D\,\frac{\partial^2 \varrho}{\partial x^2}$$

with Caputo fractional derivative

$$\frac{\partial^\gamma \varrho}{\partial t^\gamma} := \begin{cases} \frac{\partial \varrho}{\partial t} & \gamma = 1 \\ \frac{1}{\Gamma(1-\gamma)} \int_0^t dt'\,(t-t')^{-\gamma} \frac{\partial \varrho}{\partial t'} & 0 < \gamma < 1 \end{cases}$$

Time-fractional equation for subdiffusion

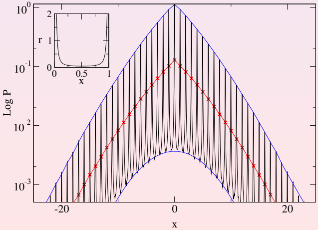
from Montroll-Weiss equation for PM map a time-fractional diffusion equation can be derived:

$$\frac{\partial^\gamma \varrho(x,t)}{\partial t^\gamma} = D\,\frac{\partial^2 \varrho}{\partial x^2}$$

with Caputo fractional derivative

$$\frac{\partial^\gamma \varrho}{\partial t^\gamma} := \begin{cases} \frac{\partial \varrho}{\partial t} & \gamma = 1 \\ \frac{1}{\Gamma(1-\gamma)} \int_0^t dt' (t-t')^{-\gamma} \frac{\partial \varrho}{\partial t'} & 0 < \gamma < 1 \end{cases}$$

can be solved exactly; compare with simulation results:



- *Gaussian and non-Gaussian envelopes* (blue) reflect intermittency
- *fine structure* due to density on the unit interval

Towards a deterministic theory of anomalous diffusion

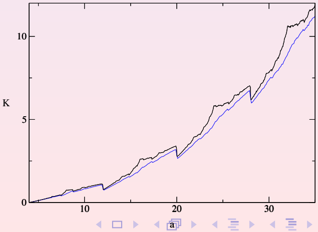
rewrite spreading in terms of velocities $v_k := x_{k+1} - x_k$:

$$< (x_n - x_0)^2 >=< \sum_{k=0}^{n-1} v_k \sum_{l=0}^{n-1} v_l > \quad (n \to \infty)$$

yields *anomalous* Taylor-Green-Kubo formula

$$K = \lim_{n\to\infty} \frac{1}{n^\alpha} \left[< \sum_{k=0}^{n-1} v_k^2 > +2 < \sum_{k=0}^{n-1} \sum_{l=1}^{n-1} v_k v_{k+l} > \right]$$

approximation by *first term only*:
some correlations included;
relation to CTRW theory?



Fractal functions for anomalous diffusion

for integer velocities

$j_k := [x_{k+1}] - [x_k]$

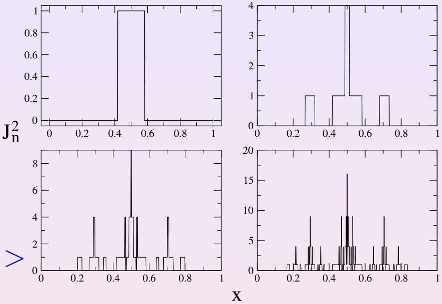
define *jump function*

$J_n(x) := \sum_{k=0}^{n} j_k$

with *recursion relation*

$J_n(x) = j_0(x) + J_{n-1}(M(x))$

yielding $K = \lim_{n\to\infty} \frac{1}{n^\alpha} < J_{n-1}^2(x) >$



Fractal functions for anomalous diffusion

for integer velocities

$j_k := [x_{k+1}] - [x_k]$

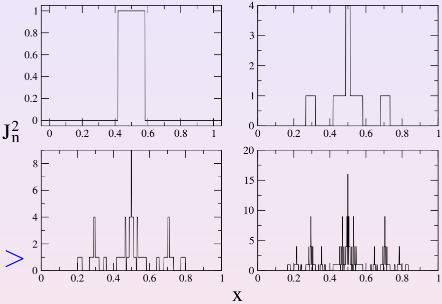
define *jump function*

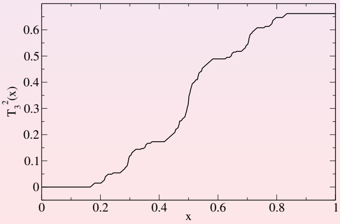
$J_n(x) := \sum_{k=0}^{n} j_k$

with *recursion relation*

$J_n(x) = j_0(x) + J_{n-1}(M(x))$

yielding $K = \lim_{n\to\infty} \frac{1}{n^\alpha} < J_{n-1}^2(x) >$





integration $T_n^2(x) := \int_0^x dy\, J_n^2(y)$

leads to $K = \lim_{n\to\infty} \frac{1}{n^\alpha} T_{n-1}^2(1)$

note: $T_\infty^2(x)$ is a fractal function

Summary

simple *intermittent map* for anomalous diffusion yields

- **fractal** anomalous diffusion coefficient
- dynamical **crossover** and dynamical **phase transition** under parameter variation, described by modified CTRW theory
- non-Gaussian probability density, described by a fractional diffusion equation, plus **fine structure**

Summary

simple *intermittent map* for anomalous diffusion yields

- **fractal** anomalous diffusion coefficient
- dynamical **crossover** and dynamical **phase transition** under parameter variation, described by modified CTRW theory
- non-Gaussian probability density, described by a fractional diffusion equation, plus **fine structure**

note: CTRW theory yields only *approximate* solutions, because *deterministic dynamical correlations* are excluded!
challenge: microscopic nonlinear theory of anomalous deterministic diffusion?

Lit.: N.Korabel et al., Europhys. Lett. **70**, 63 (2005)
http://www.maths.qmul.ac.uk/~klages

PHYSIKZENTRUM BAD HONNEF, GERMANY

373. Wilhelm und Else Heraeus-Seminar on

Anomalous Transport: Experimental Results and Theoretical Challenges

July 12 - 16, 2006

Scientific coordinators:

Rainer Klages
Queen Mary, University of London
United Kingdom

Günter Radons
TU Chemnitz
Germany

Igor M. Sokolov
Humboldt-Universität zu Berlin
Germany

Anomalous transport phenomena such as sub- and superdiffusion, non-Gaussian probability distributions, aging and dynamical localization form a rapidly growing research area within nonequilibrium statistical physics. Understanding these processes demands for establishing new cross-links between non-Markovian microscopic correlations and unusual statistical properties on macroscopic scales. This requires to combine methods from dynamical systems theory, stochastic processes and disordered systems.

The seminar will provide a unique opportunity to learn about topics ranging from mathematical foundations of anomalous dynamics to the most recent experimental results in this field. It attempts to initiate new cross-disciplinary collaborations between scientists from the above research areas and to foster fruitful interactions between theorists and experimentalists working on anomalous transport.

Scientific key topics are:

- **Applications:** Experimental results for anomalous transport under nonequilibrium conditions such as anomalous dispersion in flows, transport in porous media, aging in glassy systems, anomalous diffusion of biological cells and in cell membranes, surface diffusion, anomalous distributions in granular media and in plasma physics
- **Theoretical methods and models:** nonhyperbolic dynamics, intermittent deterministic transport, disordered dynamical systems, Lévy walks, Lévy flights, random walks in random environments, dynamical phase transitions, classical dynamical localization, thermodynamic formalism, continuous time random walks, fractional calculus

Keynote speakers:

R.Artuso (Como), R.Balescu (Brussels), E.Barkai (Bar-Ilan), C.Beck (London), D.Brockmann (Göttingen), A.V.Chechkin (Kharkov), D.Del-Castillo-Negrete (Oak Ridge), P.Dieterich (Dresden), R. Friedrich (Münster), R.Gorenflo (Berlin), R.Hilfer (Stuttgart), R. Valiullin (Leipzig), R.Kimmich (Ulm), J.Klafter (Tel Aviv), W.Kob (Montpellier), A.Kusumi (Kyoto), E.Lutz (Ulm), F. Mainardi (Bologna), R.Metzler (Copenhagen), M.J.Saxton (Davis), M.Shlesinger (Arlington), S.Tasaki (Waseda), G.Vogl (Vienna), A.Vulpiani (Rome), M.Wilkinson (Milton Keynes), S.Yuste (Badajoz)

Applications are welcome and should be made by using the application form on the conference web page, however, the number of attendees is limited. The seminar's registration fee is € 200 and will cover accommodation and meals.

Deadline for applications is April 30, 2006.

Further information is available on the conference webpage or from the organizers:

http://anotrans.physik.hu-berlin.de

r.klages@qmul.ac.uk radons@physik.tu-chemnitz.de igor.sokolov@physik.hu-berlin.de