Commutative algebras of Toeplitz operators and Berezin quantization

N. Vasilevski

Departamento de Matemáticas
CINVESTAV del I.P.N., Mexico City, México
e–mail: nvasilev@math.cinvestav.mx
http://www.math.cinvestav.mx/~nvasilev

$\mathbb{D}$ is the unit disk in $\mathbb{C}$,
$L_2(\mathbb{D})$ with the Lebesgue plane measure $d\mu(z) = dxdy$,
Bergman space $\mathcal{A}^2(\mathbb{D})$ consists of analytic functions in $\mathbb{D}$,
Bergman orthogonal projection $B_{\mathbb{D}}$ of $L_2(\mathbb{D})$ onto $\mathcal{A}^2(\mathbb{D})$:

$$(B_{\mathbb{D}}\varphi)(z) = \frac{1}{\pi}\int_{\mathbb{D}} \frac{\varphi(\zeta)\, d\mu(\zeta)}{(1 - z\overline{\zeta})^2},$$

Toeplitz operator T_a with symbol $a = a(z)$:

$$T_a : \varphi \in \mathcal{A}^2(\mathbb{D}) \longmapsto B_{\mathbb{D}}\, a\varphi \in \mathcal{A}^2(\mathbb{D}).$$

Unit Disk as a Hyperbolic Plane

Consider the unit disk $\mathbb{D}$ endowed with the hyperbolic metric

$$g = ds^2 = \frac{1}{\pi} \frac{dx^2 + dy^2}{(1 - (x^2 + y^2))^2}.$$

A *geodesic*, or a hyperbolic straight line, in $\mathbb{D}$ is (a part of) an Euclidian circle or a straight line orthogonal to the boundary $S^1 = \partial\mathbb{D}$.

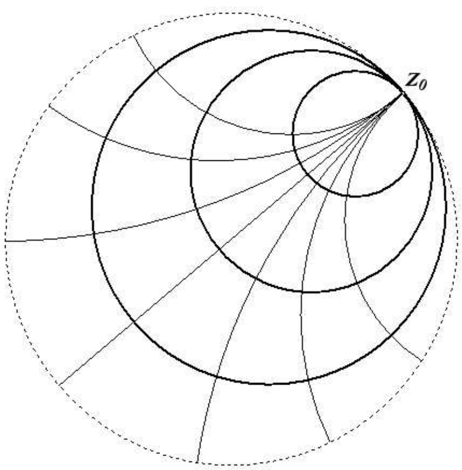
Each pair of geodesics, say L_1 and L_2, lie in a geometrically defined object, one-parameter family $\mathcal{P}$ of geodesics, which is called the *pencil* determined by L_1 and L_2. Each pencil has an associated family $\mathcal{C}$ of lines, called *cycles*, the orthogonal trajectories to geodesics forming the pencil.

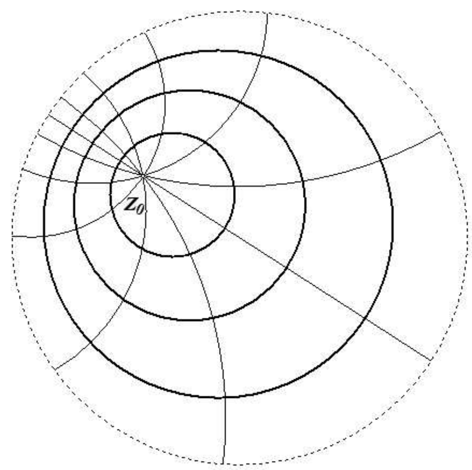
The pencil $\mathcal{P}$ determined by L_1 and L_2 is called

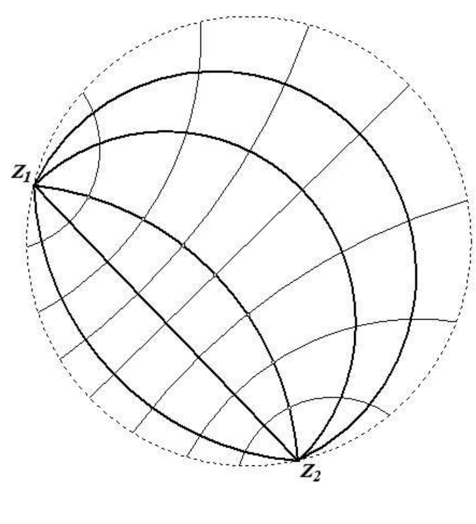
parabolic if L_1 and L_2 are parallel, in this case $\mathcal{P}$ is a set of all geodesics parallel to both L_1 and L_2, and cycles are called *horocycles*;

elliptic if L_1 and L_2 are intersecting, in this case $\mathcal{P}$ is a set of all geodesics passing through the common point of L_1 and L_2;

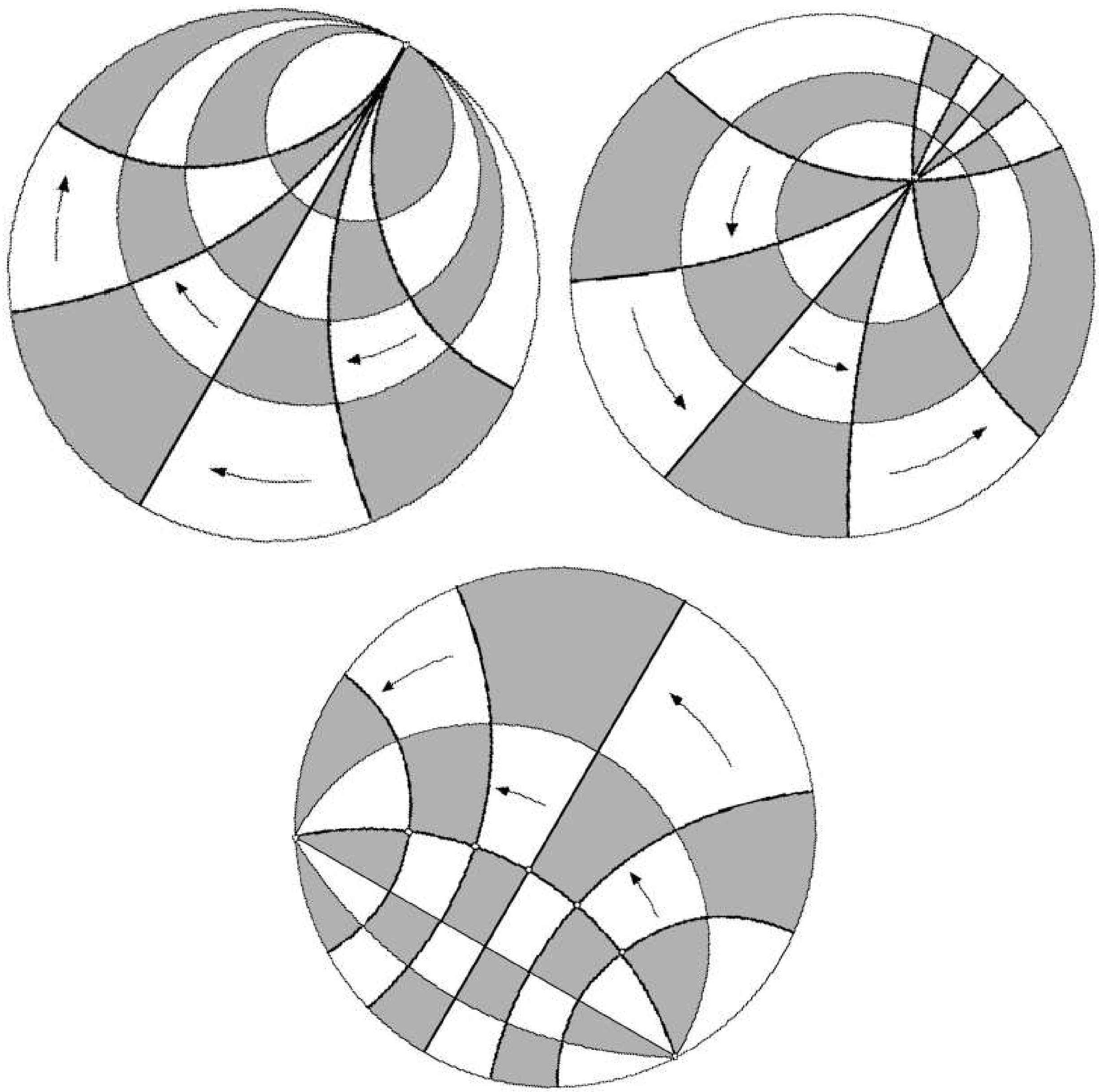
hyperbolic if L_1 and L_2 are disjoint, in this case $\mathcal{P}$ is a set of all geodesics orthogonal to the common orthogonal of L_1 and L_2, and cycles are called *hypercycles*.







Each Möbius transformation $g \in \text{Möb}(\mathbb{D})$ is a *movement of the hyperbolic plane*, determines a certain pencil of geodesics $\mathcal{P}$, and its action is as follows: *each geodesic L from the pencil $\mathcal{P}$, determined by g, moves along the cycles in $\mathcal{C}$ to the geodesic $g(L) \in \mathcal{P}$, while each cycle in $\mathcal{C}$ is invariant under the action of g.*



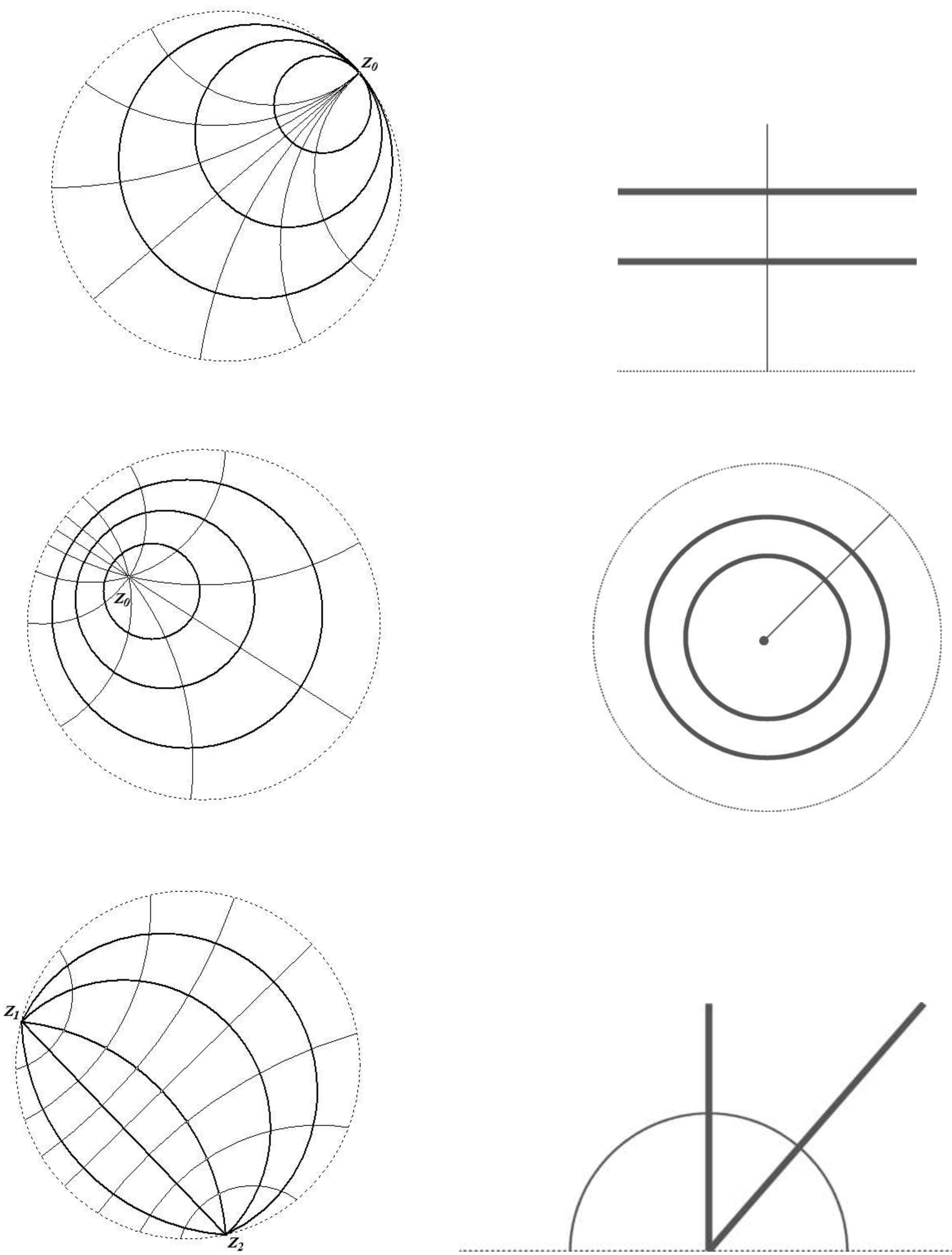
Theorem 1 *Given a pencil $\mathcal{P}$ of geodesics, consider the set of symbols which are* constant *on corresponding cycles. The C^*-algebra generated by Toeplitz operators with such symbols is* commutative.

That is, each pencil of geodesics generates a commutative C^*-algebra of Toeplitz operators.

Theorem 2 *Given a Möbius transformation $g \in \mathrm{M\ddot{o}b}(\mathbb{D})$, consider the set of symbols which are* invariant *with respect to the one-parameter group generated by g. The C^*-algebra generated by Toeplitz operators with such symbols is* commutative.

That is, each one-parameter group of Möbius transformations ($\equiv$ maximal commutative subgroup of $\mathrm{M\ddot{o}b}(\mathbb{D})$) generates a commutative C^*-algebra of Toeplitz operators.

Model cases:



Parabolic case

Consider the upper half-plane Π in $\mathbb{C}$. Introduce the unitary operators

$$U_1 = F\otimes I : L_2(\Pi) = L_2(\mathbb{R})\otimes L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R})\otimes L_2(\mathbb{R}_+),$$

where $F : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is the Fourier transform

$$(Ff)(u) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-iu\xi} f(\xi)\, d\xi,$$

and

$$U_2 : L_2(\Pi) = L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$$

which is defined by the rule

$$U_2 : \varphi(u, v) \longmapsto \frac{1}{\sqrt{2|x|}}\, \varphi(x, \frac{y}{2|x|}).$$

Letting $\ell_0(y) = e^{-y/2}$, we have $\ell_0(y) \in L_2(\mathbb{R}_+)$ and $\|\ell_0(y)\| = 1$. Denote by L_0 the one-dimensional subspace of $L_2(\mathbb{R}_+)$ generated by $\ell_0(y)$.

Theorem 3 *The unitary operator $U = U_2U_1$ is an isometric isomorphism of the space $L_2(\Pi) = L_2(\mathbb{R})\otimes L_2(\mathbb{R}_+)$ under which the Bergman space $\mathcal{A}^2(\Pi)$ is mapped onto $L_2(\mathbb{R}_+) \otimes L_0$,*

$$U : \mathcal{A}^2(\Pi) \longrightarrow L_2(\mathbb{R}_+) \otimes L_0.$$

Introduce the isometric imbedding

$$R_0 : L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$$

by the rule $\quad (R_0 f)(x, y) = \chi_+(x)\, f(x)\, \ell_0(y),$
where $\chi_+(x)$ is the characteristic function of $\mathbb{R}_+$.

Now the operator $R = R_0^* U$ maps the space $L_2(\Pi)$ onto $L_2(\mathbb{R}_+)$, and the restriction

$$R|_{\mathcal{A}^2(\Pi)} : \mathcal{A}^2(\Pi) \longrightarrow L_2(\mathbb{R}_+)$$

is an isometric isomorphism. The adjoint operator

$$R^* = U^* R_0 : L_2(\mathbb{R}_+) \longrightarrow \mathcal{A}^2(\Pi) \subset L_2(\Pi)$$

is an isometric isomorphism of $L_2(\mathbb{R}_+)$ onto the subspace $\mathcal{A}^2(\Pi)$ of the space $L_2(\Pi)$. Moreover,

$$\begin{aligned} R\, R^* = I \quad &: \quad L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R}_+), \\ R^* R = B_\Pi \quad &: \quad L_2(\Pi) \longrightarrow \mathcal{A}^2(\Pi). \end{aligned}$$

Theorem 4 *Let $a = a(v)$ be a measurable function on $\mathbb{R}_+$. Then the Toeplitz operator T_a acting on $\mathcal{A}^2(\Pi)$ is unitary equivalent to the multiplication operator $\gamma_a I = R\, T_a R^*$, acting on $L_2(\mathbb{R}_+)$. The function $\gamma_a(x)$ is given by*

$$\gamma_a(x) = \int_{\mathbb{R}_+} a(\frac{y}{2x})\, e^{-y}\, dy, \quad x \in \mathbb{R}_+.$$

Berezin quantization on the hyperbolic plane

We consider the pair $(\mathbb{D}, \omega)$, where $\mathbb{D}$ is the unit disk and

$$\omega = \frac{1}{\pi} \frac{dx \wedge dy}{(1-(x^2+y^2)^2} = \frac{1}{2\pi i} \frac{d\overline{z} \wedge dz}{(1-|z|^2)^2}.$$

Poisson brackets:

$$\begin{aligned} \{a, b\} &= \pi(1-(x^2+y^2))^2 \left(\frac{\partial a}{\partial y} \frac{\partial b}{\partial x} - \frac{\partial a}{\partial x} \frac{\partial b}{\partial y} \right) \\ &= 2\pi i (1-z\overline{z})^2 \left(\frac{\partial a}{\partial z} \frac{\partial b}{\partial \overline{z}} - \frac{\partial a}{\partial \overline{z}} \frac{\partial b}{\partial z} \right). \end{aligned}$$

Laplace-Beltrami operator:

$$\begin{aligned} \Delta &= \pi(1-(x^2+y^2))^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \\ &= 4\pi(1-z\overline{z})^2 \frac{\partial^2}{\partial z \partial \overline{z}}. \end{aligned}$$

Introduce weighted Bergman spaces $\mathcal{A}^2_h(\mathbb{D})$ with the scalar product

$$(\varphi, \psi) = \left(\frac{1}{h} - 1 \right) \int_{\mathbb{D}} \varphi(z) \overline{\psi(z)} \, (1-z\overline{z})^{\frac{1}{h}} \, \omega(z).$$

The weighted Bergman projection has the form

$$(B_{\mathbb{D},h}\varphi)(z) = \left(\frac{1}{h} - 1 \right) \int_{\mathbb{D}} \varphi(\zeta) \, \left(\frac{1-\zeta\overline{\zeta}}{1-z\overline{\zeta}} \right)^{\frac{1}{h}} \omega(\zeta).$$

Let $E = (0, \frac{1}{2\pi})$, for each $\hbar = \frac{h}{2\pi} \in E$, and consequently $h \in (0,1)$, introduce the Hilbert space $H_\hbar$ as the weighted Bergman space $\mathcal{A}_h^2(\mathbb{D})$.
For each function $a = a(z) \in C^\infty(\mathbb{D})$ consider the family of Toeplitz operators $T_a^{(h)}$ with (anti-Wick) symbol a acting on $\mathcal{A}_h^2(\mathbb{D})$, for $h \in (0,1)$, and denote by $\mathcal{T}_h$ the *-algebra generated by Toeplitz operators $T_a^{(h)}$ with symbols $a \in C^\infty(\mathbb{D})$.

The Wick symbols of the Toeplitz operator $T_a^{(h)}$ has the form

$$\widetilde{a}_h(z,\overline{z}) = (\frac{1}{h} - 1) \int_{\mathbb{D}} a(\zeta) \left(\frac{(1-|z|^2)(1-|\zeta|^2)}{(1-z\overline{\zeta})(1-\zeta\overline{z})} \right)^{\frac{1}{h}} \omega(\zeta).$$

For each $h \in (0,1)$ define the function algebra

$$\widetilde{\mathcal{A}}_h = \{\widetilde{a}_h(z,\overline{z}) : \, a \in C^\infty(\mathbb{D})\}$$

with point wise linear operations, and with the multiplication law defined by the product of Toeplitz operators:

$$\widetilde{a}_h \star \widetilde{b}_h = (\frac{1}{h} - 1) \int_{\mathbb{D}} \widetilde{a}_h(z,\overline{\zeta})\, \widetilde{b}_h(\zeta,\overline{z}) \left(\frac{(1-|z|^2)(1-|\zeta|^2)}{(1-z\overline{\zeta})(1-\zeta\overline{z})} \right)^{\frac{1}{h}} \omega.$$

The correspondence principle is given by

$$\begin{aligned}\widetilde{a}_h(z,\overline{z}) &= a(z,\overline{z}) + O(\hbar),\\ (\widetilde{a}_h \star \widetilde{b}_h - \widetilde{b}_h \star \widetilde{a}_h)(z,\overline{z}) &= i\hbar\,\{a,b\} + O(\hbar^2).\end{aligned}$$

Three term asymptotic expansion:

$$\begin{aligned}&(\widetilde{a}_h \star \widetilde{b}_h - \widetilde{b}_h \star \widetilde{a}_h)(z,\overline{z})\\ = \ & i\hbar\,\{a,b\}\\ &+i\frac{\hbar^2}{4}\left(\Delta\{a,b\} + \{a,\Delta b\} + \{\Delta a, b\} + 8\pi\{a,b\}\right)\\ &+i\frac{\hbar^3}{24}\Big[\{\Delta a, \Delta b\} + \{a, \Delta^2 b\} + \{\Delta^2 a, b\} + \Delta^2\{a,b\}\\ &+\Delta\{a,\Delta b\} + \Delta\{\Delta a, b\}\\ &+28\pi\left(\Delta\{a,b\} + \{a,\Delta b\} + \{\Delta a, b\}\right)\\ &+ 96\pi^2\{a,b\}\Big] + o(\hbar^3)\end{aligned}$$

Corollary 5 *Let $\mathcal{A}(\mathbb{D})$ be a subspace of $C^\infty(\mathbb{D})$ such that for each $h \in (0,1)$ the Toeplitz operator algebra $\mathcal{T}_h(\mathcal{A}(\mathbb{D}))$ is commutative.*
Then for all a, $b \in \mathcal{A}(\mathbb{D})$ we have

$$\begin{aligned}\{a,b\} &= 0,\\ \{a,\Delta b\} + \{\Delta a, b\} &= 0,\\ \{\Delta a, \Delta b\} + \{a, \Delta^2 b\} + \{\Delta^2 a, b\} &= 0.\end{aligned}$$

Let $\mathcal{A}(\mathbb{D})$ be a linear space of smooth functions which generates for each $h \in (0,1)$ the commutative C^*-algebra $\mathcal{T}_h(\mathcal{A}(\mathbb{D}))$ of Toeplitz operators.

First term: $\{a, b\} = 0$:

Lemma 6 *All functions in $\mathcal{A}(\mathbb{D})$ have (globally) the same set of level lines and the same set of gradient lines.*

Second term: $\{a, \Delta b\} + \{\Delta a, b\} = 0$:

Theorem 7 *The space $\mathcal{A}(\mathbb{D})$ consists of functions whose common gradient lines are geodesics in the hyperbolic geometry of the unit disk $\mathbb{D}$.*

Third term: $\{\Delta a, \Delta b\} + \{a, \Delta^2 b\} + \{\Delta^2 a, b\} = 0$:

Theorem 8 *The space $\mathcal{A}(\mathbb{D})$ consists of functions whose common level lines are cycles.*

Dynamics of spectra of Toeplitz operators

Let D be either the unit disk $\mathbb{D}$, or the upper half-plane Π in $\mathbb{C}$.

For a symbol $a = a(z)$, $z \in D$, the Toeplitz operator $T_a^{(\lambda)}$ acts on $\mathcal{A}_\lambda^2(D)$ as follows

$$T_a^{(\lambda)}\varphi = B_D^{(\lambda)} a\varphi, \qquad \varphi \in \mathcal{A}_\lambda^2(D).$$

Theorem 9 *Given any model pencil and a symbol $a \in L_\infty(D)$, constant on corresponding cycles, the Toeplitz operator $T_a^{(\lambda)}$ is unitary equivalent to the multiplication operator $\gamma_{a,\lambda} I$, where*
in the parabolic case: $a = a(y)$, $y \in \mathbb{R}_+$,
$\gamma_{a,\lambda} I : L_2(\mathbb{R}_+) \to L_2(\mathbb{R}_+)$,

$$\gamma_{a,\lambda}(x) = \frac{x^{\lambda+1}}{\Gamma(\lambda+1)} \int_0^\infty a(y/2) y^\lambda e^{-xy} dy;$$

in the elliptic case: $a = a(r)$, $r \in [0,1)$, $\quad \gamma_{a,\lambda} I : l_2 \to l_2$,

$$\gamma_{a,\lambda}(n) = \frac{1}{\mathrm{B}(n+1, \lambda+1)} \int_0^1 a(\sqrt{r}) \, (1-r)^\lambda \, r^n dr;$$

in the hyperbolic case: $a = a(\theta)$, $\theta \in (0, \pi)$,
$\gamma_{a,\lambda} I : L_2(\mathbb{R}) \to L_2(\mathbb{R})$,

$$\gamma_{a,\lambda}(\xi) = 2^\lambda \, (\lambda+1) \frac{|\Gamma(\frac{\lambda+2}{2} + i\xi)|^2}{\pi \, \Gamma(\lambda+2) \, e^{\pi\xi}} \int_0^\pi a(\theta) \, e^{-2\xi\theta} \, \sin^\lambda \theta \, d\theta.$$

Continuous symbols

Let E be a subset of $\mathbb{R}$ having $+\infty$ as a limit point, and let for each $\lambda \in E$ there is a set $M_\lambda \subset \mathbb{C}$. Define the set M_∞ as the set of all $z \in \mathbb{C}$ for which there exists a sequence of complex numbers $\{z_n\}_{n\in\mathbb{N}}$ such that

(i) for each $n \in \mathbb{N}$ there exists $\lambda_n \in E$ such that $z_n \in M_{\lambda_n}$,

(ii) $\lim_{n\to\infty} \lambda_n = +\infty$,

(iii) $z = \lim_{n\to\infty} z_n$.

We will write

$$M_\infty = \lim_{\lambda\to+\infty} M_\lambda,$$

and call M_∞ the (partial) limit set of a family $\{M_\lambda\}_{\lambda\in E}$ when $\lambda \to +\infty$.

The *a priori* spectral information for L_∞-symbols:

$$\operatorname{sp} T_a^{(\lambda)} \subset \operatorname{conv}(\text{ess-Range}\, a).$$

Given a symbol $a \in L_\infty(D)$, constant on cycles, the Toeplitz operator $T_a^{(\lambda)}$ is unitary equivalent to the multiplication operator $\gamma_{a,\lambda}I$. Thus

$$\operatorname{sp} T_a^{(\lambda)} = \overline{M_\lambda(a)}, \qquad \text{where} \quad M_\lambda(a) = \operatorname{Range} \gamma_{a,\lambda}.$$

Theorem 10 *Let a be a continuos symbol constant on cycles. Then*

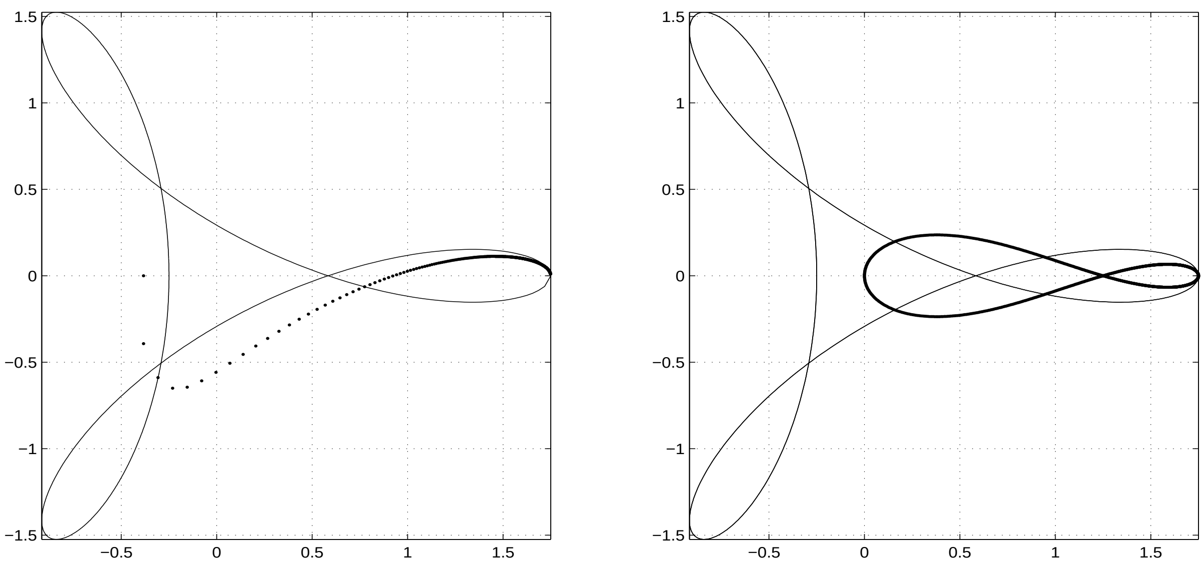
$$\lim_{\lambda \to +\infty} \operatorname{sp} T_a^{(\lambda)} = M_\infty(a) = \operatorname{Range} a.$$

The set $\operatorname{Range} a$ coincides with the spectrum $\operatorname{sp} aI$ of the operator of multiplication by $a = a(y)$, thus the another form of the above is

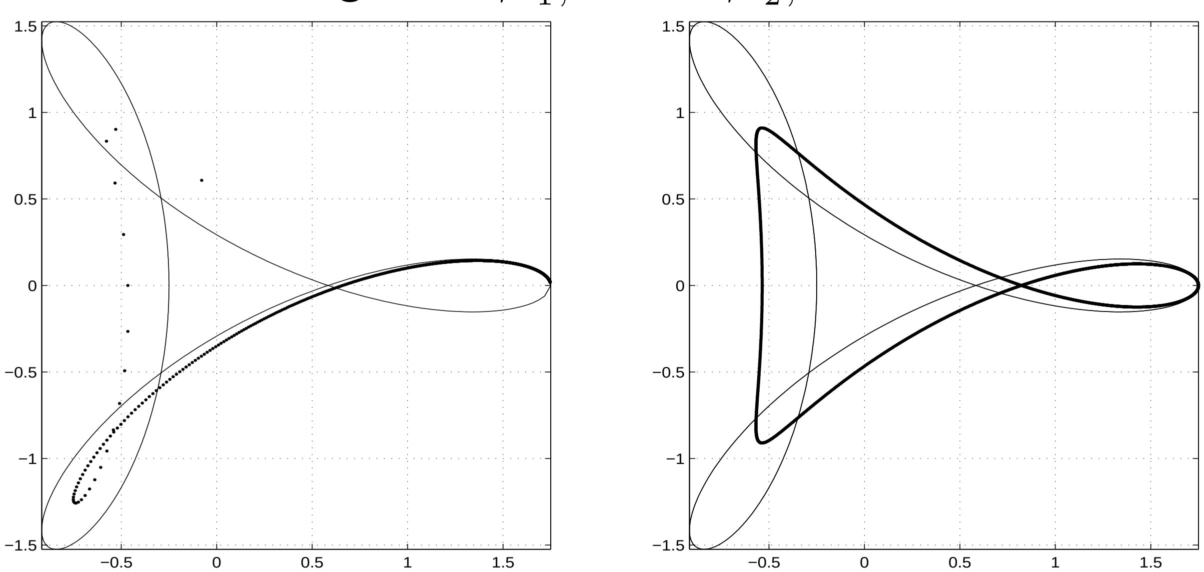
$$\lim_{\lambda \to +\infty} \operatorname{sp} T_a^{(\lambda)} = \operatorname{sp} aI.$$

Two continuous symbol (both are hypocycloids)

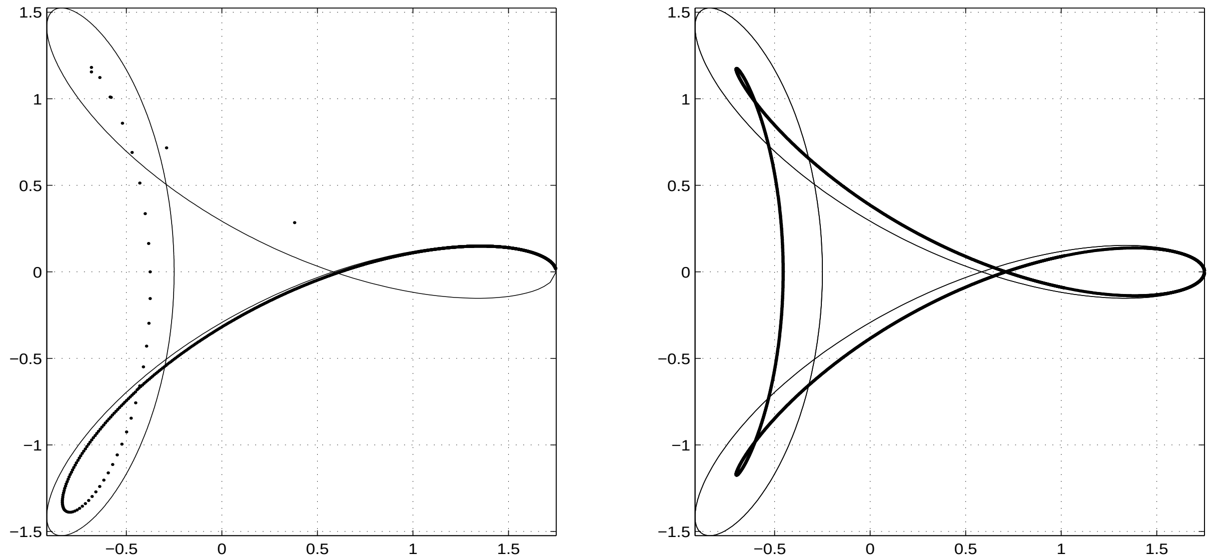
$$
\begin{aligned}
a_1(r) &= \frac{3}{4}(r + i\sqrt{1-r^2})^8 + (r - i\sqrt{1-r^2})^4 \\
a_2(\theta) &= \frac{3}{4}e^{4i\theta} + e^{-2i\theta}.
\end{aligned}
$$



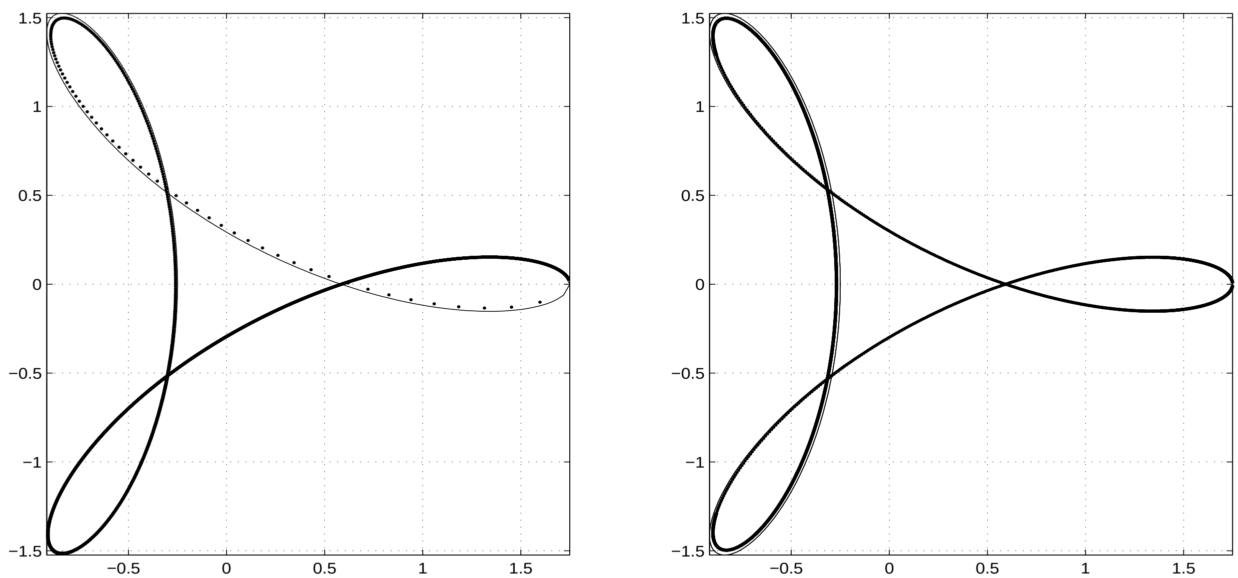
The images of $\gamma_{a_1,\lambda}$ and $\gamma_{a_2,\lambda}$ for $\lambda = 0$.



The images of $\gamma_{a_1,\lambda}$ and $\gamma_{a_2,\lambda}$ for $\lambda = 5$.



The images of $\gamma_{a_1,\lambda}$ and $\gamma_{a_2,\lambda}$ for $\lambda = 12$.



The images of $\gamma_{a_1,\lambda}$ and $\gamma_{a_2,\lambda}$ for $\lambda = 200$.

Piecewise continuous symbols

Let a be a piecewise continuous symbol constant on cycles and having a finite number m of jump points. Denote by $\bigcup_{j=1}^{m} I_j(a)$ the union of the straight line segments connecting the one-sided limit values of a at the jump points. Introduce

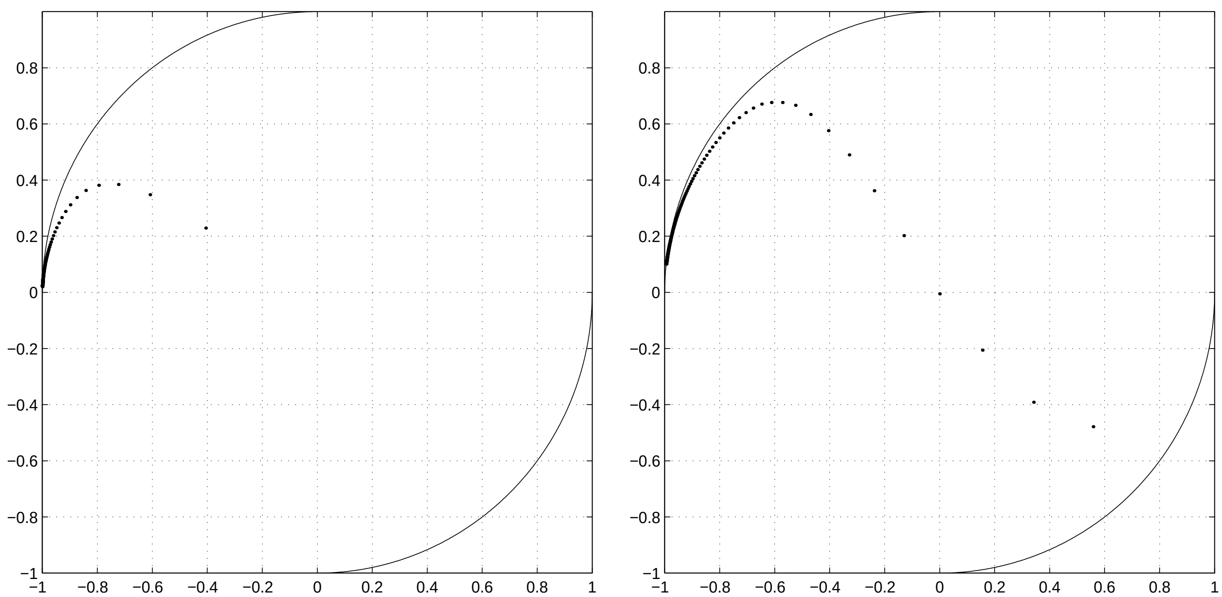
$$\widetilde{R}(a) = \mathrm{Range}\, a \cup \left(\bigcup_{j=1}^{m} I_j(a) \right).$$

Theorem 11 *Let a be a piecewise continuous symbol constant on cycles. Then*

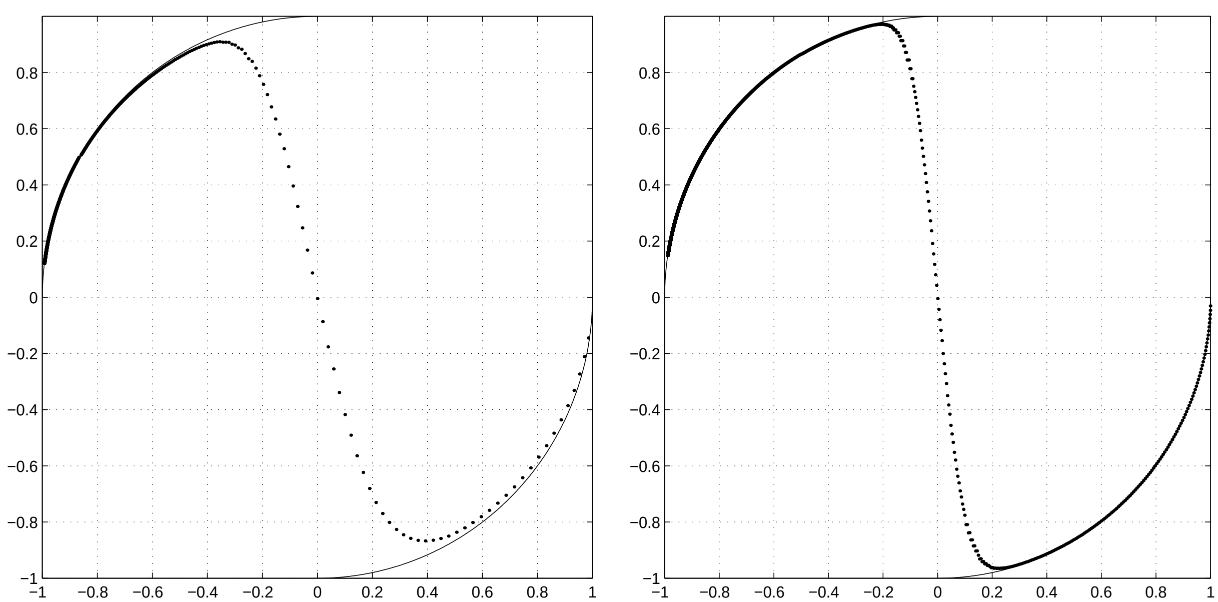
$$\lim_{\lambda \to \infty} \mathrm{sp}_\lambda\, T_a^{(\lambda)} = M_\infty(a) = \widetilde{R}(a).$$

Piecewise continuous symbol

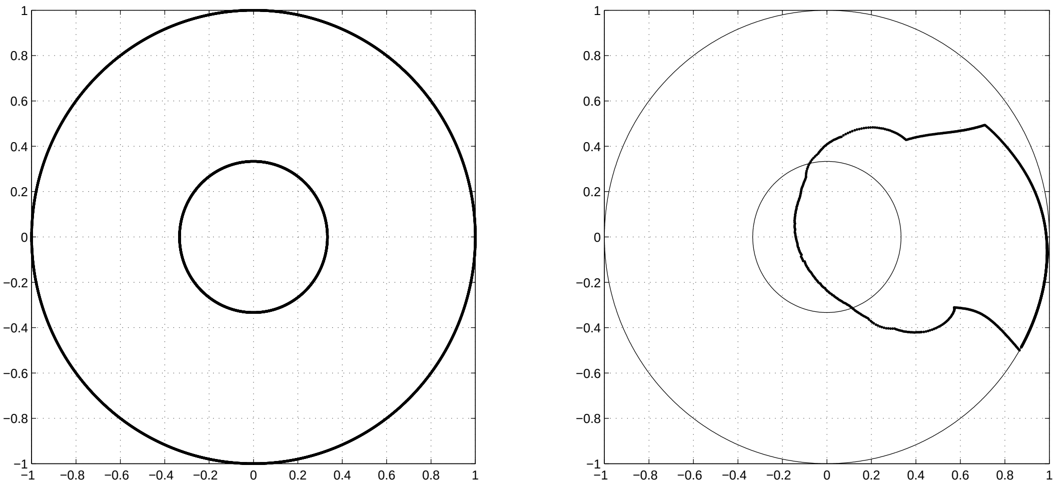
$$a(r) = \begin{cases} e^{-i\pi r^2}, & r \in [0, 1/\sqrt{2}] \\ e^{i\pi r^2}, & r \in (1/\sqrt{2}, 1] \end{cases}$$



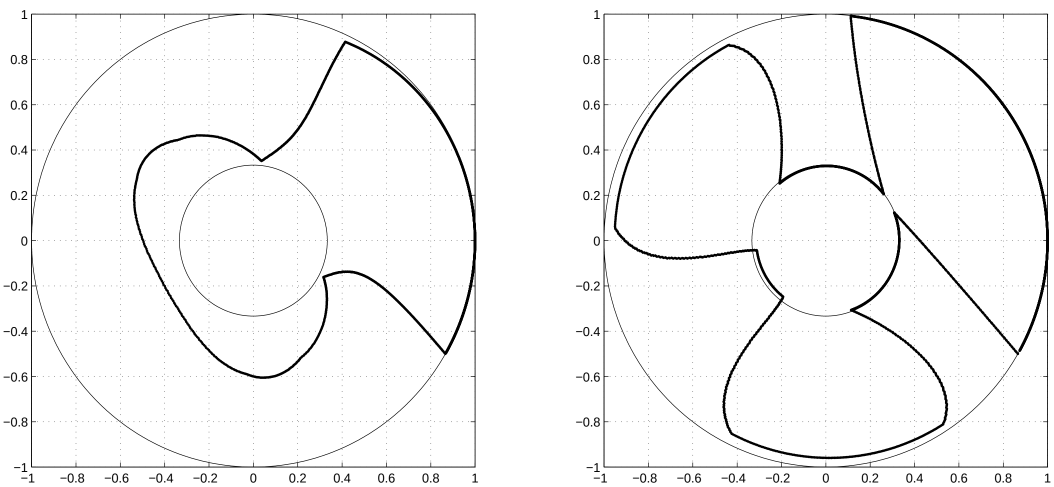
The sequence $\gamma_{a,\lambda} = \{\gamma_{a,\lambda}(n)\}$ for $\lambda = 0$ and $\lambda = 4$.



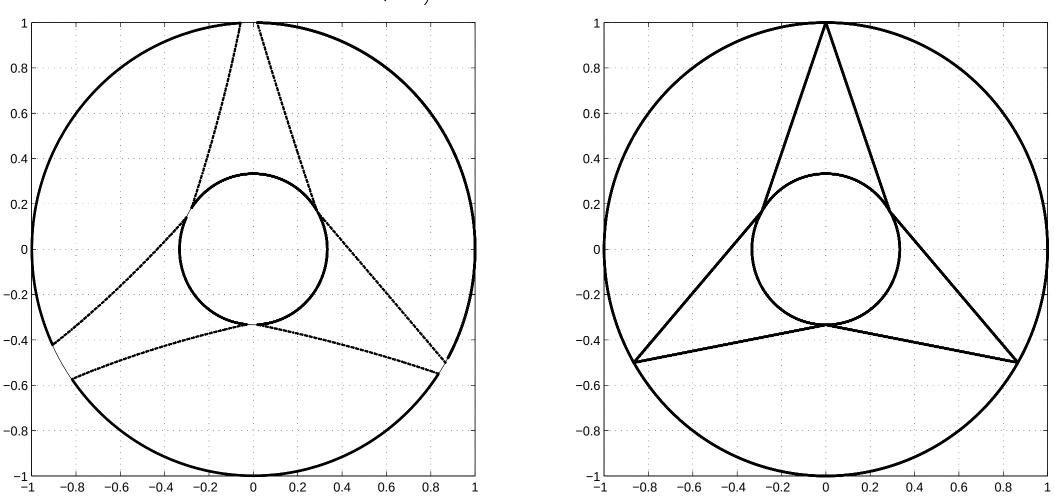
The sequence $\gamma_{a,\lambda} = \{\gamma_{a,\lambda}(n)\}$ for $\lambda = 40$ and $\lambda = 200$.



The symbol $a(\theta)$ and the function $\gamma_{a,\lambda}$ for $\lambda = 1$.



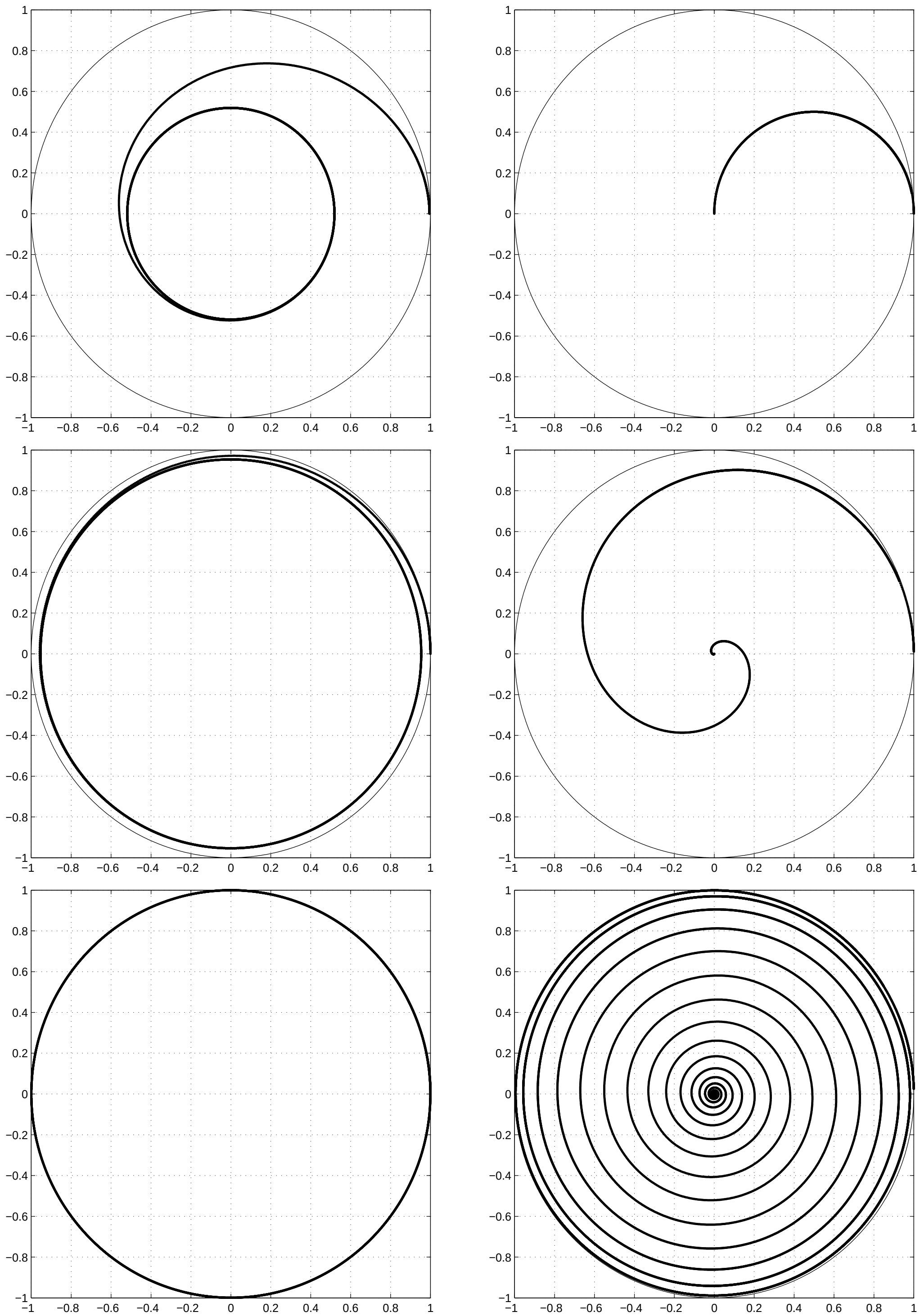
The function $\gamma_{a,\lambda}$ for $\lambda = 10$ and $\lambda = 70$.



The function $\gamma_{a,\lambda}$ for $\lambda = 500$ and the limit set $M_\infty(a)$.

Oscillating symbols

$$a_1(y) = (1+y)^i = e^{i\ln(1+y)} \quad \text{and} \quad a_2(y) = e^{iy}, \quad y \in [0, \infty)$$



The functions $\gamma_{a_1,\lambda}$ and $\gamma_{a_2,\lambda}$ for λ equals to 0, 10, and 1000.