The spectral function and the remainder in local Weyl's law: View from below

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X^n, $n \geq 2$ — compact manifold

g_{ij} — Riemannian metric

Δ — Laplace operator

$\Delta \phi_i = \lambda_i \phi_i, \quad \{\phi_i\}$ — orthonormal basis of eigenfunctions

$0 < \lambda_1 \leq \lambda_2 \leq \ldots$ — spectrum

Spectral function:

$$N_{x,y}(\lambda) = \sum_{\sqrt{\lambda_i} \leq \lambda} \phi_i(x) \phi_i(y)$$

If $x \neq y$, $N_{x,y}(\lambda) = O(\lambda^{n-1})$

If $x = y$, set $N_{x,y}(\lambda) := N_x(\lambda)$

Weyl's law:

$$N(\lambda) = C\, Vol(X)\, \lambda^n + R(\lambda),$$

$$R(\lambda) = O(\lambda^{n-1})$$

Local Weyl's law:

$$N_x(\lambda) = C\lambda^n + R_x(\lambda), \;\; R_x(\lambda) = O(\lambda^{n-1})$$

Remainder estimates are sharp (attained on a round sphere)

MAIN RESULTS: lower bounds for $N_{x,y}(\lambda)$ and $R_x(\lambda)$

Notation: $f_1(\lambda) = \Omega(f_2(\lambda)), f_2 > 0$ iff

$$\limsup_{\lambda \to \infty} \frac{|f_1(\lambda)|}{f_2(\lambda)} > 0$$

Theorem 1 If $x, y \in X$ are not conjugate along any shortest geodesic joining them, then

$$N_{x,y}(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}}\right)$$

On-diagonal version:

Set $u_j(x,x)$ — j-th local heat invariant

For example, $u_1(x,x) = \frac{\tau(x)}{6}$, where τ is scalar curvature

Denote $\kappa_x = \min\{j \geq 1 |\ u_j(x,x) \neq 0\}$.

If $u_j(x,x) = 0$ for all $j \geq 1$, set $\kappa_x = \infty$.

Theorem 2 If $n - 2\kappa_x - 1 > 0$ then

$$R_x(\lambda) = \Omega(\lambda^{n-2\kappa_x-1}).$$

If $n-4\kappa_x-1 < 0$, and X has no conjugate points, then

$$R_x(\lambda) = \Omega(\lambda^{\frac{n-1}{2}})$$

Example: flat square 2-torus

$$\lambda_j = 4\pi^2(n_1^2 + n_2^2), \quad n_1, n_2 \in \mathbf{Z}$$

$$\phi_j(x) = e^{2\pi i(n_1 x_1 + n_2 x_2)},\ x = (x_1, x_2)$$

$$|\phi_j(x)| = 1 \ \Rightarrow \ N(\lambda) \equiv N_x(\lambda)$$

Gauss's circle problem: estimate $R(\lambda)$

Theorem 2 $\Rightarrow$ $R(\lambda) = \Omega(\sqrt{\lambda})$

This is classical **Hardy–Landau bound**.

Theorem 2 $\Rightarrow$ Hardy–Landau bound for the *local* remainder on **any** surface without conjugate points.

Manifolds of negative curvature

Suppose sectional curvatures satisfy

$$-K_1^2 \leq K(\xi, \eta) \leq -K_2^2$$

Theorem (Berard '77) $R_x(\lambda) = O\left(\frac{\lambda^{n-1}}{\log \lambda}\right)$

Conjecture (Randol' 81) On a surface of constant negative curvature

$$R(\lambda) = O\left(\lambda^{\frac{1}{2}+\epsilon}\right)$$

Conjecture (attributed to ?) On a *generic* negatively curved surface

$$R(\lambda) = O(\lambda^{\epsilon}) \quad \text{for any} \quad \epsilon > 0.$$

Theorem On a negatively curved surface

$$R_x(\lambda) = \Omega(\sqrt{\lambda}).$$

This result was proved in an unpublished Ph.D. thesis of A. Karnaukh (Princeton, 1996) under the supervision of P. Sarnak.

It served as a starting point and a motivation for our work.

Thermodynamic formalism

G^t — geodesic flow on a unit tangent bundle SX. **Topological pressure** of $f : SX \to \mathbf{R}$:

$$P(f) = \sup_{\mu} \left(h_{\mu} + \int f d\mu \right),$$

μ is G^t-invariant, h_{μ} — measure–theoretic entropy.

Variational principle: $P(0) = h$,

h — **topological entropy** of G^t.

On negatively curved manifolds geodesic flows are *Anosov*.

$U(\xi)$ — *unstable* subspace of $T_\xi SX$

Sinai-Ruelle-Bowen potential

$$\mathcal{H}(\xi) = \frac{d}{dt}\Big|_{t=0} \ln\det dG^t|_{U(\xi)}$$

$P(-\mathcal{H}) = 0$ and the *equilibrium measure* (attaining the supremum) for $\mathcal{H}$ is the Liouville measure μ_L on SX. Thus

$$h_{\mu_L} = \int_{SX} \mathcal{H} d\mu_L$$

Off-diagonal:

Theorem 3. If X is negatively curved then for any $\delta > 0$ and $x \neq y$

$$N_{x,y}(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}} (\log \lambda)^{\frac{P(-\mathcal{H}/2)}{h} - \delta}\right)$$

Power of the logarithm is positive

$$\frac{P\left(-\mathcal{H}/2\right)}{h} \geq \frac{K_2}{2K_1},$$

and equals $\frac{1}{2}$ if curvature is constant.

On-diagonal:

Theorem 4. X — negatively curved. If $n \leq 5$ then for any $\delta > 0$

$$R_x(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}} (\log \lambda)^{\frac{P(-\mathcal{H}/2)}{h} - \delta}\right)$$

If $n \geq 6$ then

$$R_x(\lambda) = \Omega(\lambda^{n-3})$$

Note different asymptotics for small and large n.

Sketch of Proofs

Wave kernel on X:

$$e(t,x,y) \;=\; \sum_{i=0}^{\infty} \cos(\sqrt{\lambda_i}t)\phi_i(x)\phi_i(y)$$

Let $\psi \in C_0^\infty([-1,1])$, even, monotone decreasing on [0,1], $\psi \geq 0$, $\psi(0) = 1$. Fix $\lambda, T \gg 0$, consider the function

$$\frac{1}{T}\psi\left(\frac{t}{T}\right)\cos(\lambda t)$$

For $x, y \in M$, let

$$k_{\lambda,T}(x,y) = \int_{-\infty}^{\infty} \frac{\psi(t/T)}{T}\cos(\lambda t)e(t,x,y)dt$$

Off-diagonal case: $x \neq y$.

The following lemma is used in the proofs:

Lemma 5 If $N_{x,y}(\lambda) = o(\lambda^a), \ a > 0$ then

$$k_{\lambda,T}(x,y) = o(\lambda^a).$$

If $N_{x,y}(\lambda) = O(\lambda^a(\log\lambda)^b), \ a, b > 0$ then

$$k_{\lambda,T}(x,y) = O(\lambda^a(\log\lambda)^b).$$

Let us start with **Theorem 3**:

X — negatively curved.

Pretrace formula. Let $E(t, x, y)$ be the wave kernel on the universal cover M.

Given $x, y \in X$, we have

$$e(t, x, y) = \sum_{\omega \in \Gamma = \pi_1(X)} E(t, x, \omega y)$$

Given $x, y \in M$, define $K_{\lambda,T}(x, y)$ by

$$K_{\lambda,T}(x, y) = \int_{-\infty}^{\infty} \frac{\psi(t/T)}{T} \cos(\lambda t) E(t, x, y) dt$$

Then for $x, y \in X$

$$k_{\lambda,T}(x, y) = \sum_{\omega \in \Gamma} K_{\lambda,T}(x, \omega y)$$

Hadamard parametrix

Let $x, y \in M$, $r = d(x, y)$.

$$E(t,x,y) = \frac{1}{\pi^{\frac{n-1}{2}}}|t| \sum_{j=0}^{\infty} u_j(x,y) \frac{(r^2 - t^2)_-^{j - \frac{n-3}{2} - 2}}{4^j \Gamma(j - \frac{n-3}{2} - 1)}$$

modulo a smooth function.

Here $u_j(x, y)$ solve transport equations along the geodesic joining x and y.

Leading term asymptotics

Proposition 6 Let $x \neq y \in M, r = d(x,y)$. Then $K_{\lambda,T}(x,y)$ satisfies as $\lambda \to \infty$:

$$K_{\lambda,T}(x,y) = \frac{Q\lambda^{\frac{n-1}{2}}\psi(r/T)}{T\sqrt{g(x,y)r^{n-1}}} \sin(\lambda r + \phi_n) +$$

$$O(\lambda^{\frac{n-3}{2}}).$$

Here $g = \sqrt{\det g_{ij}}$ in normal coordinates centered at x,

$$\phi_n = \frac{\pi}{4}(3 - (n \bmod 4))$$

and $Q \neq 0$.

Proof of Theorem 3. Assume for contradiction that for some $\delta > 0$,

$$N_{x,y}(\lambda) = O\left(\lambda^{\frac{n-1}{2}} (\log \lambda)^{\frac{P(-\mathcal{H}/2)}{h} - \delta}\right).$$

Lemma 5 implies a similar bound for $k_{\lambda,T}(x,y)$.

Proposition 6 implies

$$k_{\lambda,T}(x,y) = \sum_{r_\omega < T} \frac{\lambda^{\frac{n-1}{2}} A\, \psi\left(\frac{r_\omega}{T}\right)}{T\sqrt{g(x,\omega y) r_\omega^{n-1}}} \sin(\lambda r_\omega + \phi_n)$$

$$+ O(\lambda^{\frac{n-3}{2}}) \exp(O(T)),$$

for some $A \neq 0$.

Consider the sum

$$S_{x,y}(T) = \sum_{r_\omega \leq T} \frac{1}{\sqrt{g(x, \omega y)\, r_\omega^{n-1}}}$$

It follows from results of Parry and Pollicott that

Theorem 7 As $T \to \infty$,

$$S_{x,y}(T) \geq C_0 e^{P\left(-\frac{\mathcal{H}}{2}\right) \cdot T}$$

Here $P\left(-\frac{H}{2}\right) \geq \frac{(n-1)K_2}{2}$.

Case $n \neq 3(\mathrm{mod}\,4) \Rightarrow \phi_n \neq 0(\mathrm{mod}\,\pi)$.

Dirichlet box principle $\Rightarrow$ can choose λ large so that

$$|e^{i\lambda r_\omega} - 1| < \epsilon, \ \epsilon \text{ small},$$

for all $r_\omega \leq T$. Then

$$|\sin(\lambda r_\omega + \phi_n)| \approx |\sin \phi_n| > 0.$$

For Dirichlet principle need

$$T \approx \frac{1}{h} \log\log\lambda$$

Thus, exponential bound in Theorem 7 yields log–improvement in Theorem 3.

Case $n = 3 (\mathrm{mod}\, 4) \Rightarrow \phi_n = 0 (\mathrm{mod}\ \pi)$.

Need a separate argument to establish

$$\sin(\lambda r_\omega) > \frac{\nu}{T}, \qquad \forall \omega : \frac{T}{\alpha} \leq r_\omega \leq T,$$

$\alpha > 0$ some constant.

Combined with Theorem 7, this contradicts Lemma 5 and proves Theorem 3 in all dimensions.

Proof of Theorem 1 Assume

$$N_{x,y}(\lambda) = o(\lambda^{\frac{n-1}{2}}).$$

Lemma 5 $\Rightarrow$ $k_{\lambda,T}(x,y) = o(\lambda^{\frac{n-1}{2}})$.

Work directly on X and adapt parametrix construction.

Let $x, y \in X$ not conjugate along any shortest geodesic $\Rightarrow$ finitely many shortest geodesics of length $r = d(x,y)$.

Also, there are no geodesics from x to y of length $l \in]r, r+\epsilon]$ for some $\epsilon > 0$.

Let $T = r + \frac{\epsilon}{2}$. Sum the parametrices along shortest geodesics and get

$$k_{\lambda,T}(x,y) = \beta\lambda^{\frac{n-1}{2}} \sin(\lambda r + \phi_n) + O(\lambda^{\frac{n-3}{2}}),$$

where β is a non-zero constant.

Choose a sequence $\lambda_k \to \infty$ such that

$$|\sin(\lambda_k r + \phi_n)| > \nu > 0$$

Contradiction with $k_{\lambda,T}(x,y) = o(\lambda^{\frac{n-1}{2}})$

On-diagonal case, $x = y$. Theorems 2 and 4 are proved similarly to Theorems 1 and 3 using the *on-diagonal* counterparts of Lemma 5 and Proposition 6.

The 0-th term of the parametrix on the diagonal cancels out with the main term in the Weyl's law.

Consider **Theorem 4** in more detail.

First on-diagonal term of the parametrix: $c\,\lambda^{n-3}$ (for $n > 3$).

Sum of the 0-th off-diagonal terms (by Theorem 3):

$$O\left(\lambda^{\frac{n-1}{2}}\,(\log\lambda)^{\frac{P(-\mathcal{H}/2)}{h}-\delta}\right)$$

Dimension $n \leq 4$:

$$n - 3 < \frac{n-1}{2},$$

so "diagonal<off-diagonal."

Dimension $n = 5$:

$$n - 3 = \frac{n-1}{2},$$

but "diagonal<off-diagonal" due to the power of log.

Dimension $n \geq 6$:

$$n - 3 > \frac{n-1}{2},$$

so "diagonal>off-diagonal."

Hence different bounds in Theorem 4 for $n \geq 6$ and $n \leq 5$.

Concluding remarks

• $R_x(\lambda) = \Omega(\sqrt{\lambda})$ in dimension 2. Together with the prediction $R(\lambda) = O(\lambda^\epsilon)$ on negatively curved surfaces this looks intriguing!

• Can one apply our method to estimate $R(\lambda)$ from below? We believe YES (in progress).