*Durham Simposium*

# Operators of Friedrichs with a non-trivial singular spectrum

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**Abstract**

A family of selfadjoint operators of the Friedrichs model is considered. These symmetric type operators have one singular point zero of order $m$. For every $m > 3/2$ one constructs a rank 1 perturbation from the class $Lip\, 1$ such that the corresponding operator has a sequence of eigenvalues converging to zero. Thus near the singular point there is no singular spectrum finiteness condition in terms of a modulus of continuity of a perturbation for these operators in case of $m > 3/2$.

## 1 Statement of the problem and Main result

Let us consider selfadjoints operators $S_m$ , $m > 0$, given by $S_m = \operatorname{sgn} t \cdot |t|^m \cdot + (\cdot\,, \varphi)\, \varphi$ on the domain of functions $u(t) \in L_2(\mathbb{R})$ such that $|t|^m u(t) \in L_2(\mathbb{R})$. Here $\varphi \in L_2(\mathbb{R})$ and $t$ is the independent variable. The action of the operator $S_m$ can be written as follows:

$$\big(S_m u\big)(t) = \operatorname{sgn} t \cdot |t|^m\, u(t) + \varphi(t) \int_{\mathbf{R}} u(x)\overline{\varphi(x)}\, dx\ . \tag{1.1}$$

The function $\varphi$ is assumed to satisfy the smoothness condition

$$|\,\varphi(t+h) - \varphi(t)| \le \omega\big(|h|\big)\ ,\ |\,h| \le 1\ , \tag{1.2}$$

where the function $\omega(t)$ (the modulus of continuity of the function $\varphi$) is monotone and satisfies a Dini condition

$$\omega(t) \downarrow 0 \quad \text{as} \quad t \downarrow 0\,, \quad \text{and} \quad \int_0^1 \frac{\omega(t)}{t}\, dt \,<\, \infty\,. \tag{1.3}$$

For the operators $S_m$ the absolutely continuous spectrum fills the real axis $\mathbb{R}$.The behavior of the singular spectrum of the operators $S_m$ is of interest to us. Note that we define the singular spectrum as the union of the point spectrum and the singular continuous one. The structure of the spectrum $\sigma_{sing}(S_1)$ (the singular spectrum of the operator $S_1 = t\, \cdot + (\cdot\,, \varphi)\, \varphi$ ) has been in detail studied. It is shown in the papers [1,2] that for this operator there exists an exact condition of the singular spectrum finiteness. Namely, if $\omega(t) = \mathrm{O}(\sqrt{t})$ as $t \to 0^+$, $\sigma_{sing}(S_1)$ consists of at most a finite number of eigenvalues of finite multiplicity (the singular continuous spectrum is missing). But if $\liminf \omega(t)/\sqrt{t} = +\infty$ as $t \to 0^+$, then one constructs examples showing that a nontrivial singular spectrum appears, in particular, the operator $S_1$ has accumulation points of eigenvalues. Note that the real appearance of a nontrivial singular spectrum in the Friedrichs model was for the first time shown by Pavlov and Petras (1970).

By using the simple change of variables $\operatorname{sgn} t \cdot |t|^m = x$, one can show that outside of any neighborhood of the origin the structure of the spectrum $\sigma_{sing}(S_m)$ is identical with the one of the operator $S_1$. At the same time in the talk it will be shown that for the operator $S_m$, $m > 3/2$, the behavior of the singular spectrum has quite different character in a neighborhood of the origin. In this case it turns out that the singular spectrum can appear for any modulus of continuity $\omega(t)$. And hence near zero there is not any condition of the singular spectrum finiteness in terms of the modulus of continuity of $\varphi(t)$ like for the operator $S_1$. Here we can also use the pointed change of variables but, since, for instance, $(\operatorname{sgn} t \cdot |t|^m)'_{|t=0} = 0$ for $m > 1$, it is not smooth (that is, not a diffeomorphism) near zero. In this sense zero is a singular point of the operators $S_m, m \neq 1$, and needs a special attention.

Observe that the actual modulus of continuity $\widetilde{\omega}(h) := \sup \{ |\varphi(t_1) - \varphi(t_2)| \ : \ |t_1 - t_2| < h \}$ of the function $\varphi$ always satisfies the additional constraint of semiadditivity $\widetilde{\omega}(t_1 + t_2) \leq \widetilde{\omega}(t_1) + \widetilde{\omega}(t_2)$ for all $t_1, t_2 \geq 0$.

**Theorem 1.1 (Main result)** *Let a nonnegative, monotone function $\omega(t)$, $t \geq 0$, be semiadditive: $\omega(t_1 + t_2) \leq \omega(t_1) + \omega(t_2)$ for all $t_1, t_2 \geq 0$. Then for any $m > 3/2$ one constructs a compactly supported function $\varphi$ satisfying the smoothness condition $|\varphi(t+h) - \varphi(t)| \leq \omega(|h|)$, $h \in \mathbb{R}$, and such that the corresponding operator $S_m = \operatorname{sgn} t \cdot |t|^m \cdot + (\cdot, \varphi)\varphi$ has a sequence of eigenvalues converging to zero.*

Note that the result of Theorem 1.1 can be formulated in terms of real zeros of some analytic functions. Define in the upper half plane an analytic function $M_m(z)$ in the following way

$$M_m(z) \ = \ 1 \ + \ \int_{-\infty}^{+\infty} \frac{|\varphi^2(t)|}{\operatorname{sgn} t \cdot |t|^m - z} \, dt \quad , \ \operatorname{Im} z > 0 \, .$$

It is easily shown that under our conditions the function $M_m(z)$ is continuously extended up to the real axis on the intervals $(-\infty; 0)$ and $(0; +\infty)$. Let us define for $\lambda \in \mathbb{R} \setminus \{0\}$ the value $M_m(\lambda) := M_m(\lambda + i\,0)$ and the roots set $N := \big\{ \lambda \in \mathbb{R} \setminus \{0\} : M_m(\lambda) = 0 \big\}$. Then we have the following inclusion $\sigma_{sing}(S_m) \subseteq N \cup \{0\}$. Further, the exact condition $\omega(t) = \mathrm{O}(\sqrt{t})$ as $t \to 0^+$ appears to guarantee that outside of any neighborhood of the origin there is at most a finite number of zeros of the function $M_m(\lambda)$. At the same time Theorem 1.1 means that for $m > 3/2$ the function $M_m(\lambda)$ can have a sequence of zeros converging to the origin for any monotone, nonnegative, and semiadditive function $\omega$ satisfying condition (1.3).

# 2 Construction of the function $\varphi$.

It is sufficient to prove Theorem 1.1 for $\omega(t) = C_\omega t$ with an arbitrary constant $C_\omega > 0$.

Let $\{u_n\}_{n=0}^{+\infty}$, $\{\varepsilon_n\}_{n=1}^{+\infty}$ be two sequences from the interval $(0; 10^{-1})$ satisfying the condition

$$u_n < \varepsilon_n < u_{n-1}/8 \, , \quad n = 1, 2, \ldots \tag{2.1}$$

On the real axis we define a sequence of functions $\varphi_n$ as follows

$$\varphi_n(t) \;:=\; \begin{cases} \omega\big(t - \dfrac{u_n}{2}\big) & , \;\; t \in [\tfrac{1}{2}u_n; \tfrac{3}{4}u_n] \\ \omega\big(u_n - t\big) & , \;\; t \in [\tfrac{3}{4}u_n; u_n] \\ 0 & , \;\; t \notin [\tfrac{1}{2}u_n; u_n] \end{cases} \qquad , \tag{2.2}$$

where $\omega(t) = C_\omega \, t$.

It will be shown that for any real–valued Lipschitz function $\gamma(t)$ compactly supported in the interval $(-\infty; -1)$ it is possible to select the sequences $u_n$ and $\varepsilon_n$, and a bounded sequence of nonnegative numbers $\{c^n\}_{n=1}^{+\infty}$ such that the points $\lambda_n := (u_n + \varepsilon_n)^m$ will be eigenvalues of the operator $S_m = \operatorname{sgn} t \cdot |t|^m \cdot + (\cdot \, , \varphi)\, \varphi$ with the function

$$\varphi(t) \;:=\; K \cdot \sum_{k=1}^{+\infty} (c^k)^{1/2} \, \varphi_k(t) \;+\; \gamma(t) \; . \tag{2.3}$$

Here $K > 0$ is a parameter. It is shown that for all $K$ large enough the function $\varphi$ satisfies the required smoothness condition $|\varphi(t+h) - \varphi(t)| \leq \omega(|h|)$ , $t, h \in \mathbb{R}$.

**Lemma 2.1** *For the points* $\lambda_n$ *to be eigenvalues of the operator* $S_m$ *, it is necessary and sufficient that*

$$\int_{-\infty}^{+\infty} \frac{|\varphi^2(t)|}{\operatorname{sgn} t \cdot |t|^m - \lambda_1} \, dt \;=\; -1 \; , \tag{2.4}$$

*and*

$$\int_{-\infty}^{+\infty} \frac{|\varphi^2(t)|}{(\operatorname{sgn} t \cdot |t|^m - \lambda_n)\,(\operatorname{sgn} t \cdot |t|^m - \lambda_{n+1})} \, dt \;=\; 0 \; , \qquad n = 1, 2, \ldots \tag{2.5}$$

Since $\varphi(t) = 0$ for $t > \lambda_1$ , it follows that

$$\alpha_m \;:=\; \int_{-\infty}^{+\infty} \frac{\varphi^2(t)}{\operatorname{sgn} t \cdot |t|^m - \lambda_1} \, dt \;<\; 0 \; . \tag{2.6}$$

Therefore, after solving the homogeneous system (2.5), the first equality (2.4) will be satisfied by replacing the function $\varphi$ by $\varphi / \sqrt{|\alpha_m|}$.

Substituting expression (2.3) for $\varphi(t)$ in (2.5), we obtain a system of linear equations for the unknowns $c^n$ :

$$\sum_{k=1}^{n-1} (-d_{nk}\, c^k) + d_{nn}\, c^n + \sum_{k=n+1}^{+\infty} (-d_{nk}\, c^k) = \gamma_n \; , \qquad n = 1, 2, \ldots \; , \tag{2.7}$$

with the coefficients

$$d_{nk} := K^2 \int\limits_{u_k/2}^{u_k} \frac{\varphi_k^2(t)}{|\,(t^m - \lambda_n)\,(t^m - \lambda_{n+1})|} \, dt \; , \tag{2.8}$$

and

$$\gamma_n := \int\limits_{-\infty}^{-1} \frac{\gamma^2(t)}{(|t|^m + \lambda_n)\,(|t|^m + \lambda_{n+1})}\, dt \,. \tag{2.9}$$

In the next section we show that the linear system (2.7) has a nonnegative solution in the space $l_\infty$ of bounded sequences.

# 3 Solution of the linear system.

**Lemma 3.1** *The coefficients $d_{nk}$ of the linear system (2.7) satisfy the following inequalities*

$$d_{nn} \;\geq\; K^2\, \frac{C_1}{\varepsilon_n^m\, u_n^{m-3}}\;, \tag{3.1}$$

$$\sum_{k=1}^{n-1} d_{nk} \;\leq\; K^2\, \frac{C_2}{u_{n-1}^{2m}}\;, \tag{3.2}$$

$$\sum_{k=n+1}^{+\infty} d_{nk} \;\leq\; K^2\, \frac{C_3\, u_{n+1}^3}{\varepsilon_n^m\, \varepsilon_{n+1}^m}\;, \tag{3.3}$$

*with some positive constants $C_1, C_2$, and $C_3$.*

We rewrite the system (2.7) in matrix form

$$(I + A)\,\vec{c} = f\;, \tag{3.4}$$

where the column vectors $\vec{c} = (c^1, c^2, \ldots)^T, f = (\gamma_1/d_{11}, \gamma_2/d_{22}, \ldots)^T$ and the infinite matrix $A$ has the entries $(A)_{nk} = (\delta_{nk} - 1)\cdot d_{nk}/d_{nn}$. The equation (3.4) will be considered in the Banach space $l_\infty$.

In the sequel we consider the sequences $u_n$ and $\varepsilon_n$ defined as follows

$$u_n = u_{n-1}^\alpha\;,\; \varepsilon_n = u_{n-1}^\beta\;,\; n = 1, 2, \ldots \tag{3.5}$$

with some $u_0 \in (0\,; 10^{-1})$ and $\alpha > \beta > 2$. It is evident that the inequality (2.1) is fulfilled.

**Lemma 3.2** *For every $m > 3/2$ one can find the numbers $\alpha, \beta$ satisfying the inequality $\alpha > \beta > 2$ such that $\|A\| < 1$ for all $u_0$ small enough.*

By virtue of the inequality $\|A\| < 1$, the equation (3.4) has a unique solution in $l_\infty$:

$$\vec{c} \;=\; (I + A)^{-1}\, f \tag{3.6}$$

It is easily seen that all the components $c^n$ of the vector $\vec{c} = (c^1, c^2, \ldots)^T$ are nonnegative.

# 4 Smoothness of the function $\varphi$.

**Lemma 4.1** *Suppose that $\omega$ is monotone and semiadditive ( and thus nonnegative). Then the function*

$$\psi(t) \ := \ \sum_{k=1}^{+\infty} (c^k)^{1/2}\, \varphi_k(t) \tag{4.1}$$

*satisfies the following smoothness condition*

$$|\,\psi(t+h) - \psi(t)| \ \leq \ \sup_k\, (c^k)^{1/2}\, \omega(|h|)\ , \quad t\,, h \in \mathbb{R}\,. \tag{4.2}$$

By (2.3) the function $\varphi(t) = K\psi(t) + \gamma(t)$, where the function $\gamma$ is assumed to satisfy a Lipschitz condition $|\,\gamma(t+h) - \gamma(t)| \leq C_\gamma\, |h|$, $h \in \mathbb{R}$. Since the functions $\psi$ and $\gamma$ have disjoint supports and $\omega(t) = C_\omega\, t$, we see that the following inequality

$$\begin{aligned} |\,\varphi(t+h) - \varphi(t)| \ &\leq \ K \sup_k\, (c^k)^{1/2} \cdot \omega(|h|) + C_\gamma\, |h| \\ &\leq \ \Big( K \sup_k\, (c^k)^{1/2} + C_\gamma / C_\omega \Big)\, \omega(\,|h|) \end{aligned}$$

holds for all $t\,, h \in \mathbb{R}$.

After finding the solution of the system (2.5), we satisfy the equation (2.4) replacing the function $\varphi$ by $\varphi/\sqrt{|\,\alpha_m|}$. This replacement corresponds to a passage from the functions $\psi$ and $\gamma$ to $\psi/\sqrt{|\,\alpha_m|}$ and $\gamma/\sqrt{|\,\alpha_m|}$ respectively. Therefore for the new function $\varphi$ the following smoothness condition will be fulfilled

$$|\,\varphi(t+h) - \varphi(t)| \ \leq \ \Big( K \sup_k\, (c^k)^{1/2}\, |\,\alpha_m|^{-1/2} + C_\gamma\, |\,\alpha_m|^{-1/2} / C_\omega \Big)\, \omega(\,|h|)\ , \quad h \in \mathbb{R}\ . \tag{4.3}$$

**Lemma 4.2** *The constant*

$$M \ := \ K \sup_k\, (c^k)^{1/2}\, |\,\alpha_m|^{-1/2} + C_\gamma\, |\,\alpha_m|^{-1/2} / C_\omega \tag{4.4}$$

*in the smoothness condition (4.3) satisfies the inequality $M \leq C/K$, and hence it is less than one for all $K$ large enough.*

As a result the function $\varphi$ satisfies the smoothness condition

$$|\,\varphi(t+h) - \varphi(t)| \ \leq \ \omega\big(|h|\big)\ , \ h \in \mathbb{R}\ ,$$

for all $K$ large enough. Theorem 1.1 thus is completely proved.

## Refrences


1. S. N. Naboko and S. I. Yakovlev, Conditions for the finiteness of the singular spectrum in a selfadjoint Friedrichs model, Funkcional Anal. i Priloen. **24** (1990), no.4, 88–89.

2. E. M. Dinkin, S. N. Naboko and S. I. Yakovlev, The finiteness conditions of the singular spectrum in the selfadjoint Friedrichs model, Algebra i Analiz, **2** (1991), 77–90.

3. B. S. Pavlov and S. V. Petras, The singular spectrum of the weakly perturbed multiplication operator, Funkcional Anal. i Priloen. **4** (1970), no.2, 54–61.



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