Spectral Problems Associated with Scattering by Unbounded Surfaces

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Funded by: EPSRC, EU
Transport Research Laboratory Ltd

Contents

My PhD Background - and a Continuing Interest

Figure 1: Outdoor noise measurements - University of Salford

A Very Different Length Scale
Surface Plasmon Polariton Band-Gap Structures

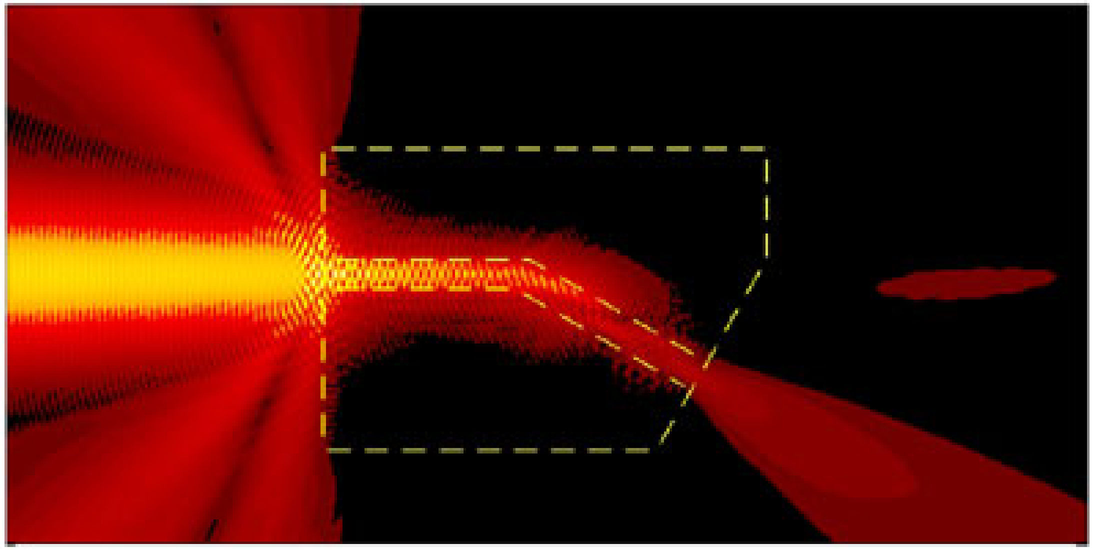


Figure 2: Calculated electric field magnitude 300 nm above an air-gold interface at wavelength 800 nm for a sharp 30° bend created by removing scatterers of height 50 nm, radius 125 nm in a SPPBG structure (Søndergaard & Bozhelvolnyi (2005)).

Other Similar Scattering Problems Include ...

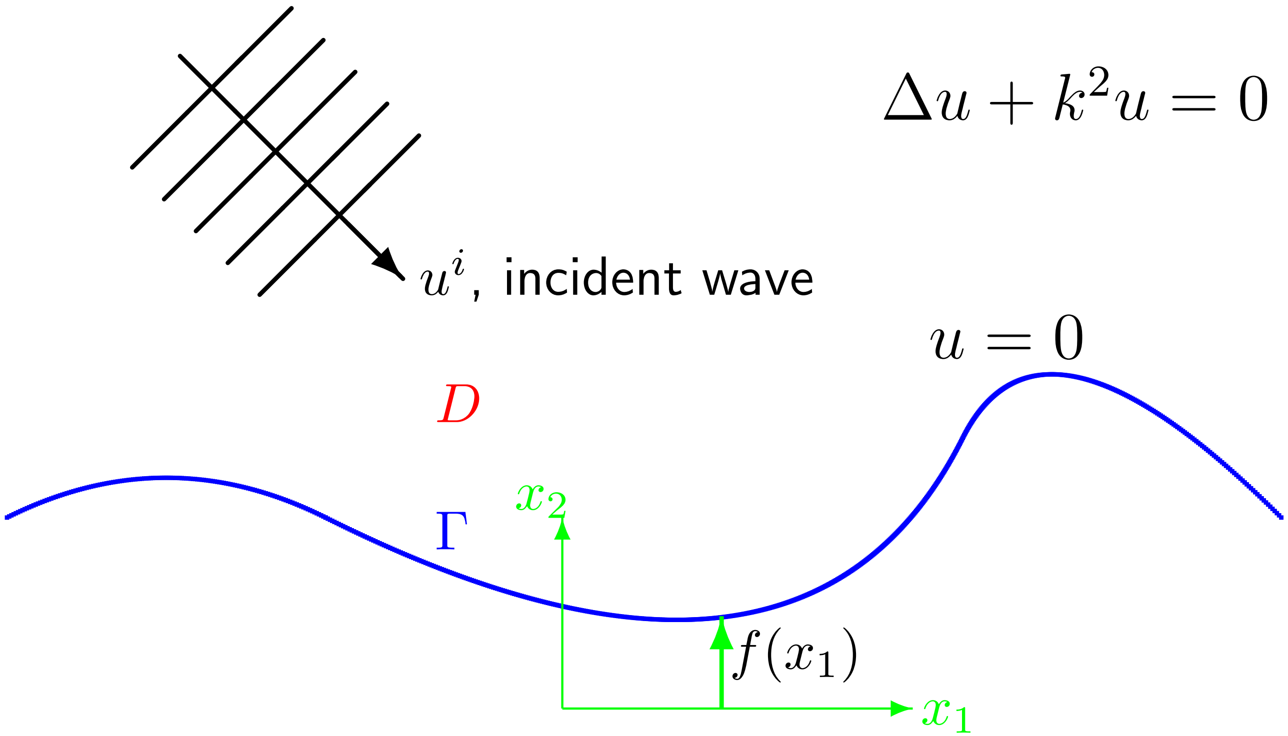
- Scattering of ground penetrating radar by buried surfaces (Rutherford Appleton Laboratory)
- Scattering of sonar waves from towed seismic transducers by the sea surface
- Scattering of radar waves by the sea surface
- Scattering of light by diffractive optics structures (e.g. the hologram on your passport)

Some Simple Mathematical Problems

Example 1 C-W, Ross, & Zhang (1999) [2D]

C-W, Heinemayer, Potthast (2005a,b) [3D]

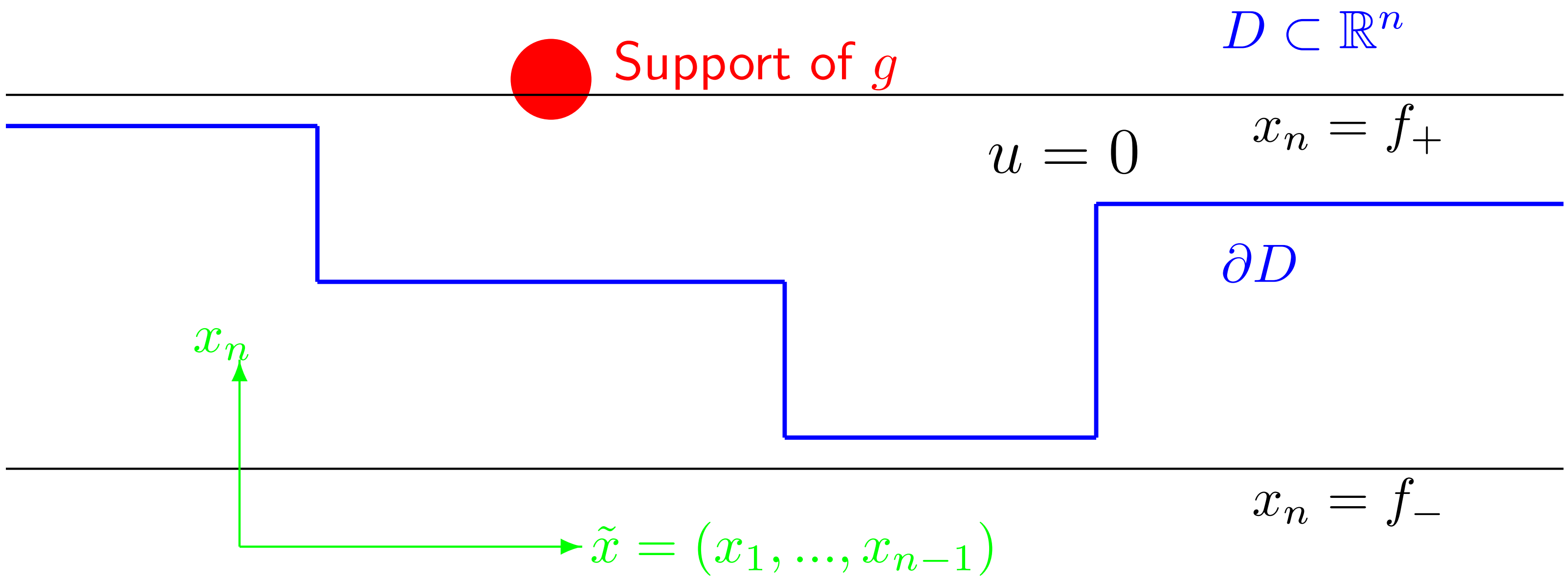
$$\Delta u + k^2 u = 0$$



f bounded and smooth (to simplify analysis of boundary integral equation methods)

Example 1a C-W & Monk (2005) [nD]

$$\Delta u + k^2 u = g$$



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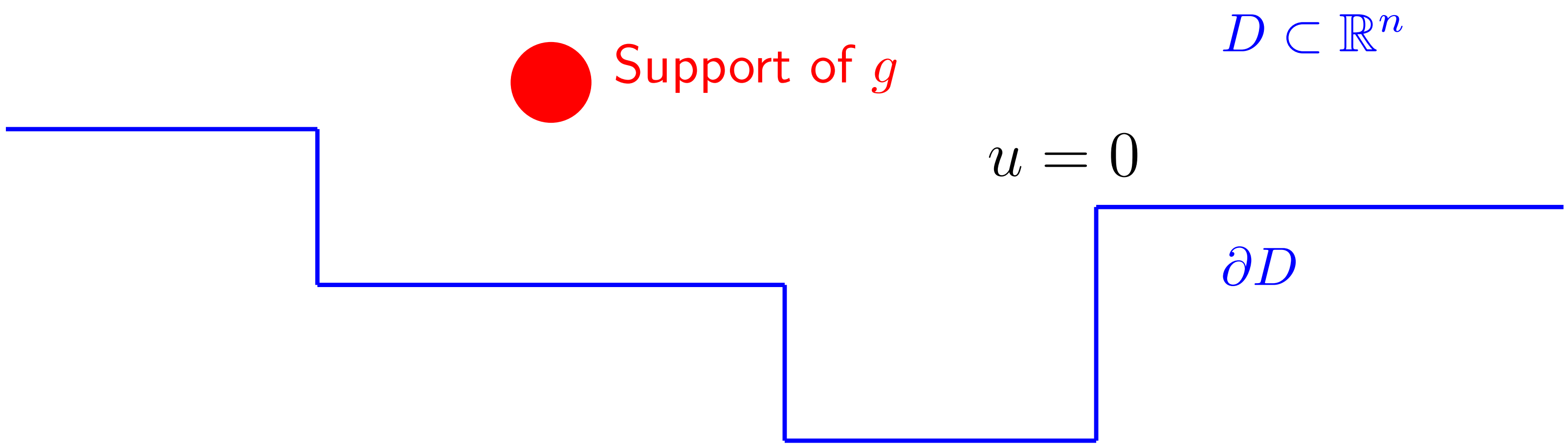
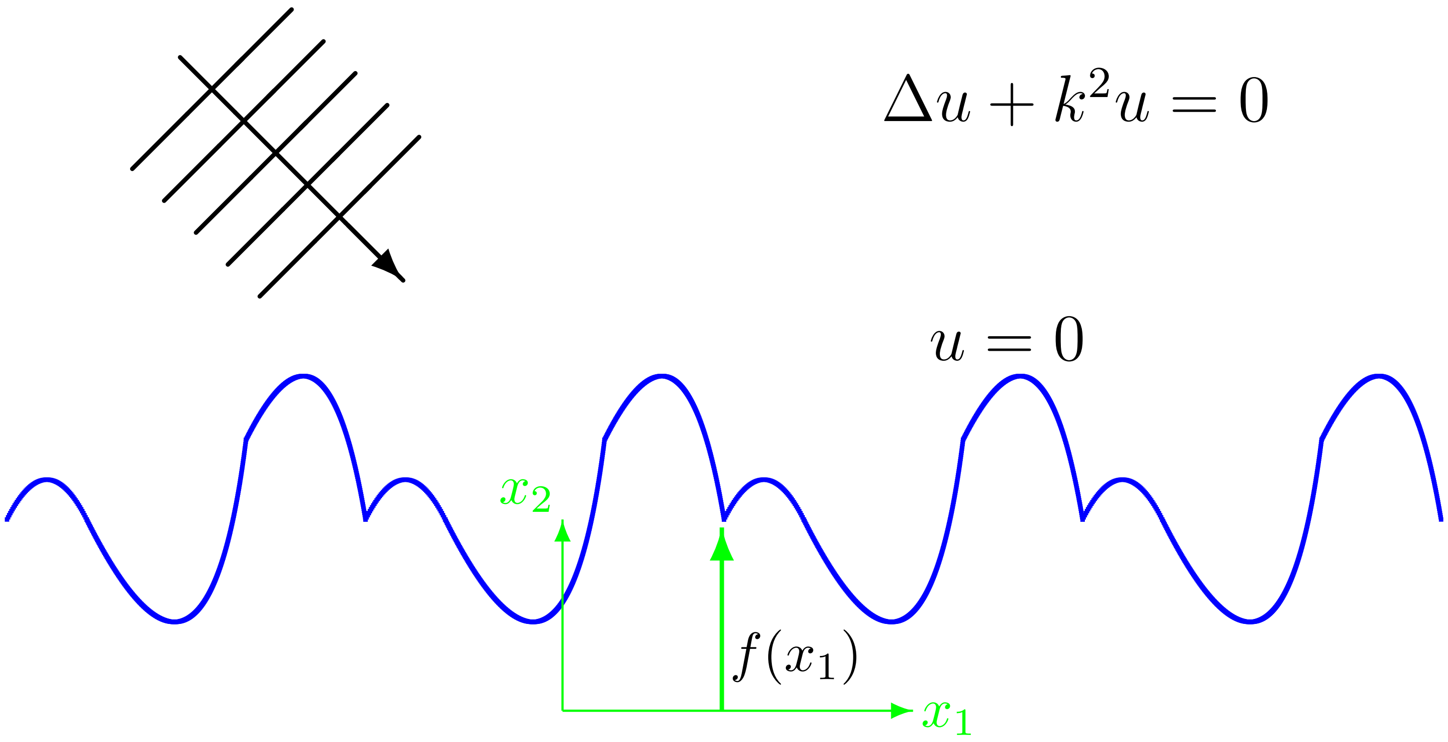


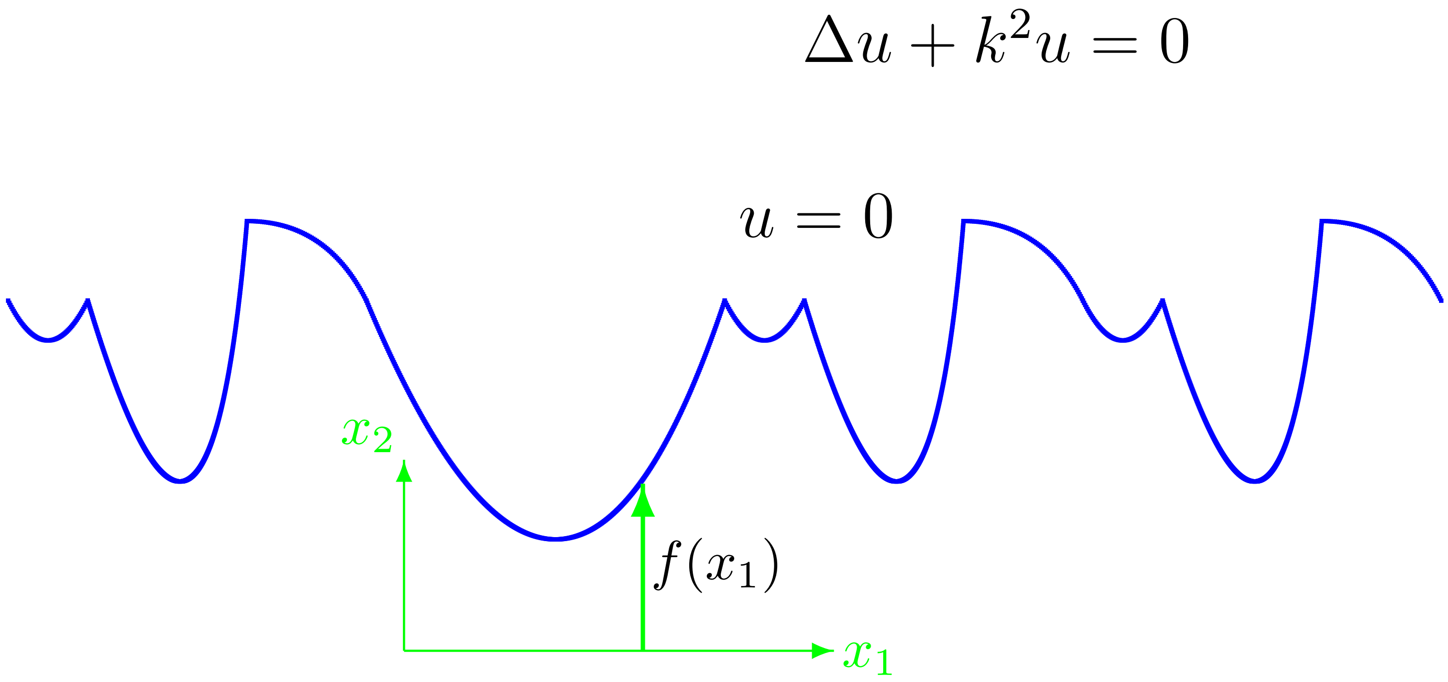
1. By Lax-Milgram unique tempered solution when $\Im k > 0$.
   What happens as $\Im k \to 0$?
2. Formulation for $k > 0$: radiation condition?
3. The spectrum is $[0, \infty)$. What sort of 'eigenfunctions' are possible?
4. What happens when ∂D is random?
5. Spectral properties of boundary integral equation formulations?

Special case: Diffraction Grating



f periodic

Special case: Locally Perturbed Diffraction Grating



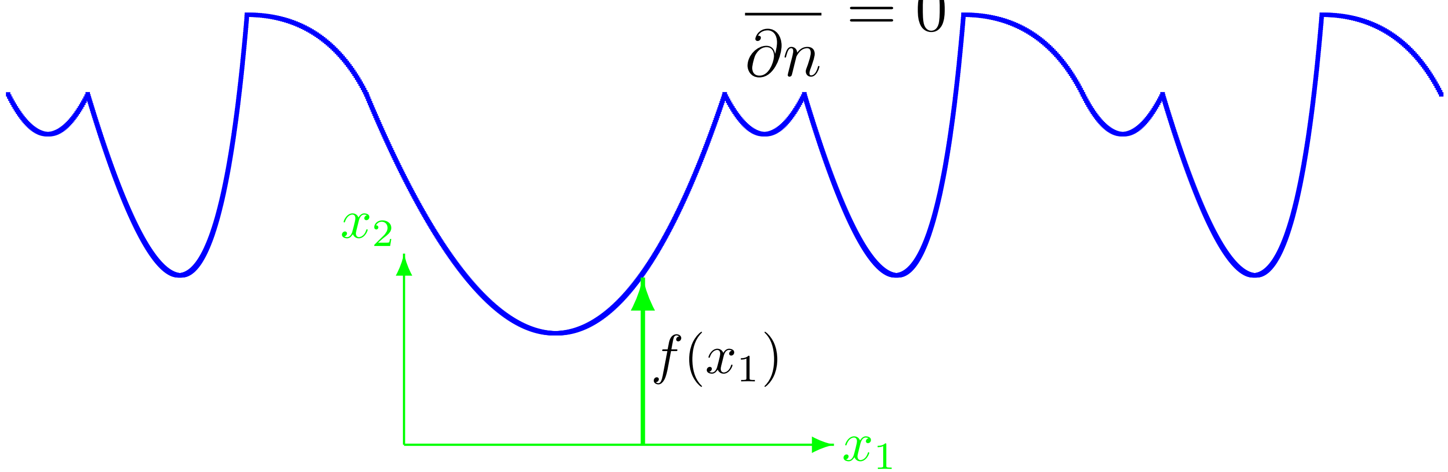
$f(x_1) =$ periodic outside finite interval

What solutions of this homogeneous problem are possible?

Special case: Locally Perturbed Diffraction Grating

$$\Delta u + k^2 u = 0$$

$$\frac{\partial u}{\partial n} = 0$$



$f(x_1) =$ periodic outside finite interval

What solutions of this homogeneous problem are possible?

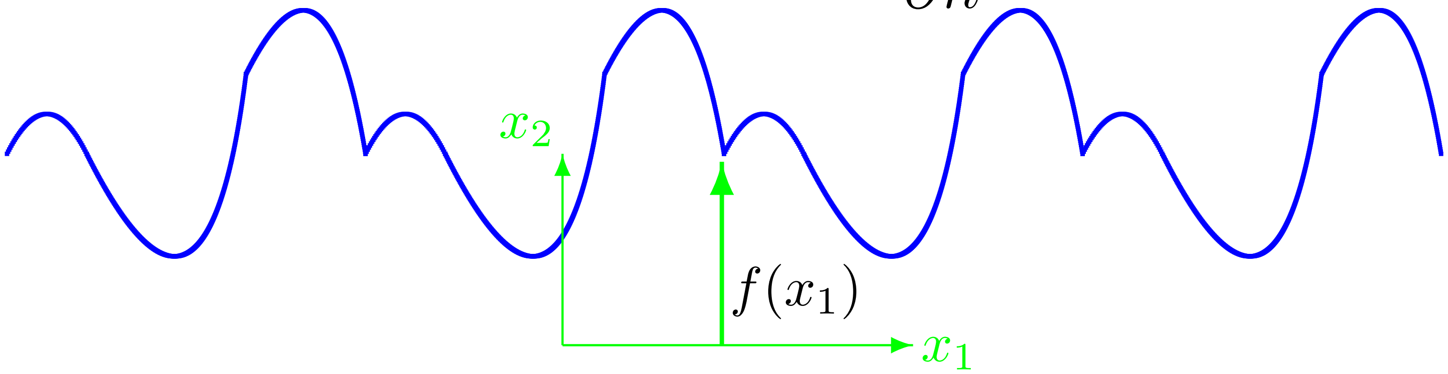
One interesting eigenfunction: the surface wave

Surface wave is one that decays exponentially with distance from the boundary.

Diffraction Grating Case

$$\Delta u + k^2 u = 0$$

$$\frac{\partial u}{\partial n} = 0$$

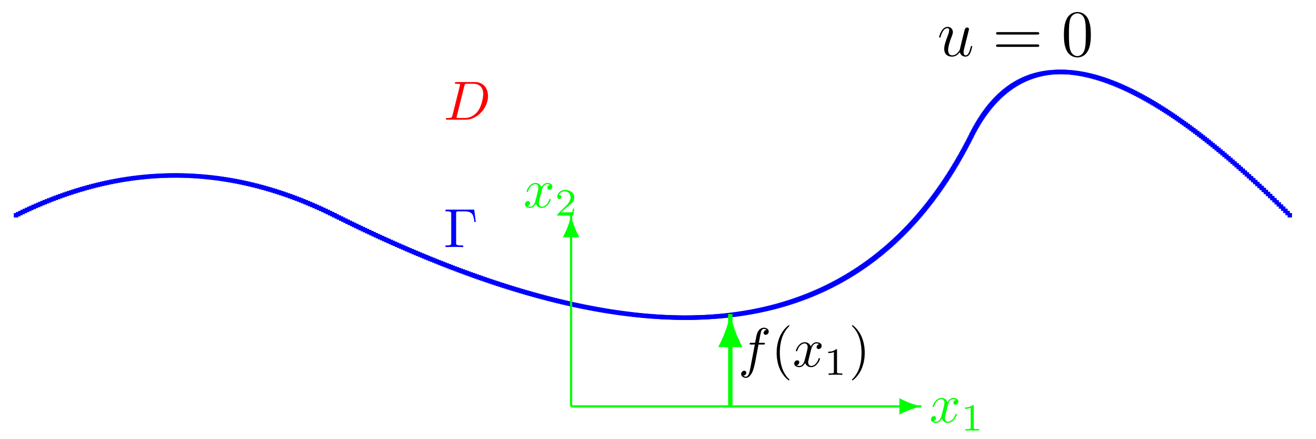


f periodic with period L

1. If $kL < \pi$ and $f \not\equiv$ constant, then there exists a surface wave solution that is quasi-periodic in the x_1-direction (Linton & McIver 2002).
2. If $kL > \pi$ there are no (rigorous) mathematical results. For periodic geometries and an air-metal interface there are theoretical justifications for the existence of a band-gap structure ('polaritonic crystals', Zayats et al., Physics Reports 2005). **Nothing appears (rigorously) known**.

General (not necessarily periodic) Case

$$\Delta u + k^2 u = 0$$



f bounded and at least piecewise smooth

1. There exists no surface wave in the 2D case (C-W & Zhang 1998).
2. If f is not the graph of a function then there is still no surface wave if $k(f_+ - f_-) < \sqrt{2}$ (C-W & Monk 2005).
3. If f is periodic (but not the graph of a function) then there is an example of a surface wave (Gotlib 2000).
4. **Nothing known for 3D case. No general criteria for existence of surface waves. Band gaps?**

True ($L^2(D)$) eigenfunctions

Surface Plasmon Polariton Band-Gap Structures

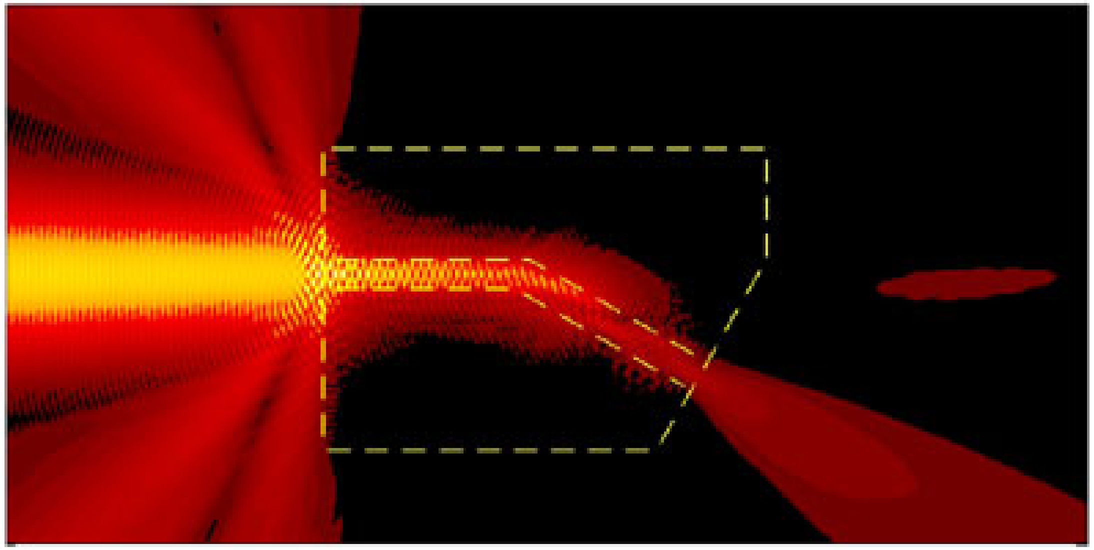


Figure 3: Calculated electric field magnitude 300 nm above an air-gold interface at wavelength 800 nm for a sharp 30° bend created by removing scatterers of height 50 nm, radius 125 nm in a SPPBG structure (Søndergaard & Bozhelvolnyi (2005)).

Locally Perturbed Diffraction Grating

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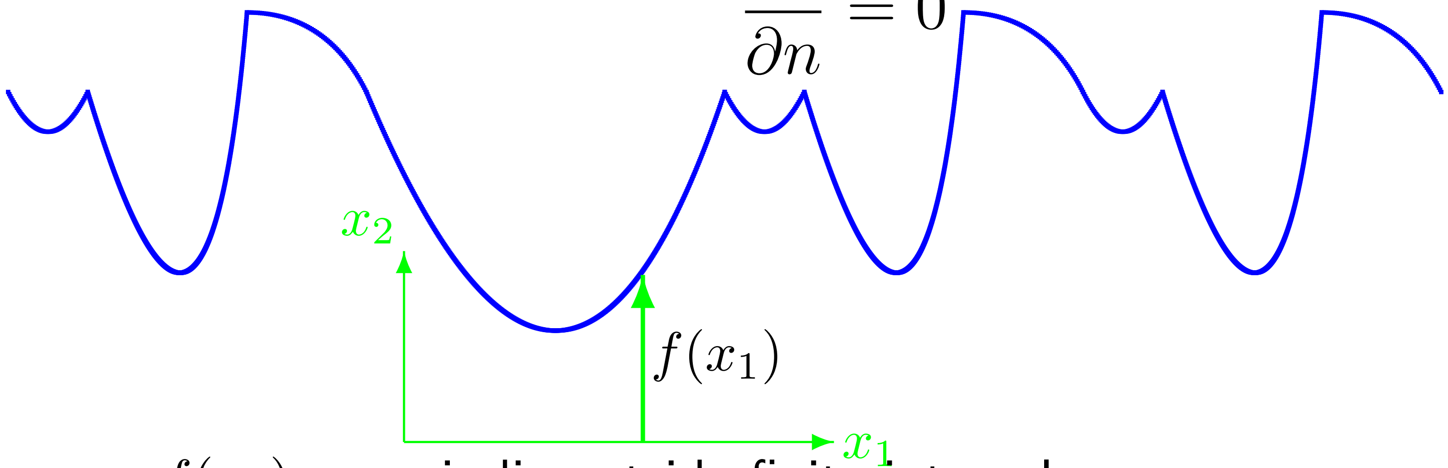
$$\frac{\partial u}{\partial n} = 0$$



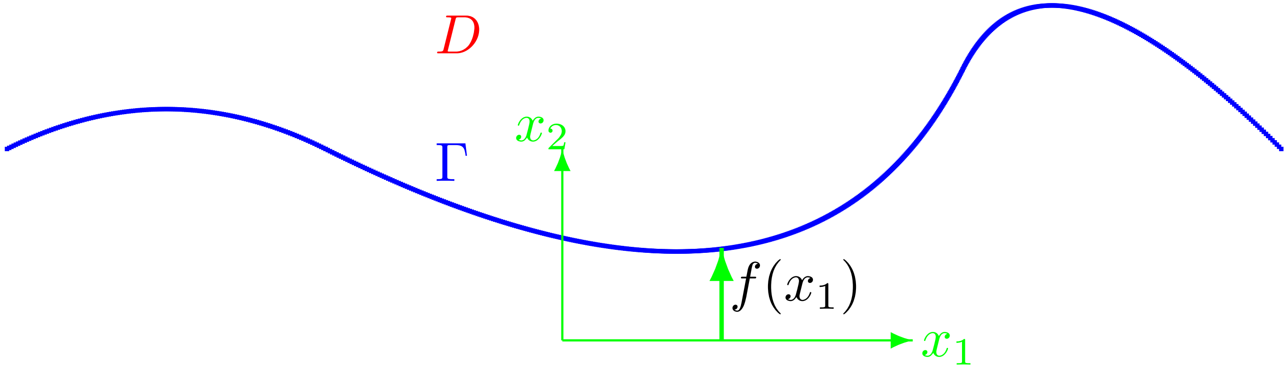
$f(x_1) =$ periodic outside finite interval

Can true eigenfunctions exist for this configuration? Nobody knows a proof of non-existence or existence of an eigenfunction even in the case when f is periodic with period L if $kL > \pi$.

f **random: Anderson Localization?**

$$\Delta u + k^2 u = 0$$

$$u = 0$$



f bounded and at least piecewise smooth

1. There exists no eigenfunction (in 2D or 3D) if $k(f_+ - f_-) < \sqrt{2}$ or if

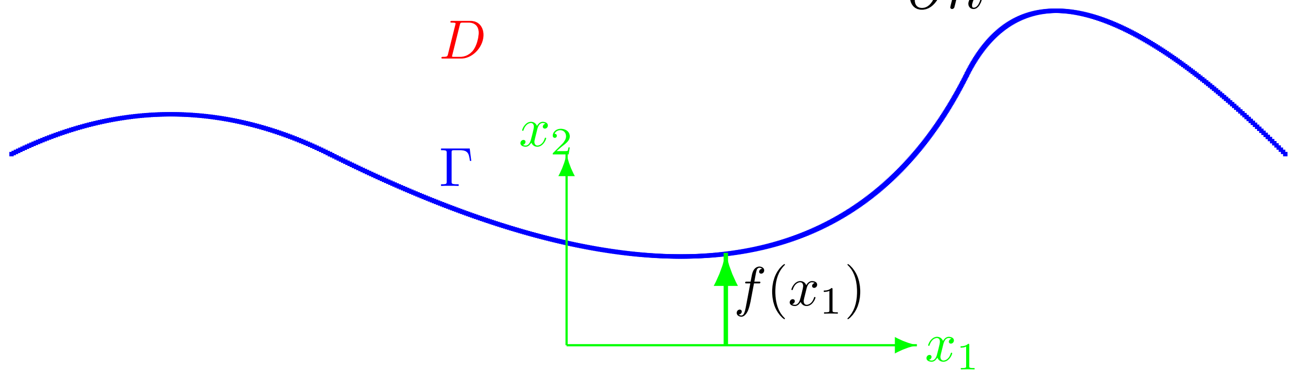
$$x \in D,\ s > 0 \Rightarrow x + se_n \in D$$

(C-W & Monk 2005)

f **random: Anderson Localization?**

$$\Delta u + k^2 u = 0$$

$$\frac{\partial u}{\partial n} = 0$$



f bounded and at least piecewise smooth

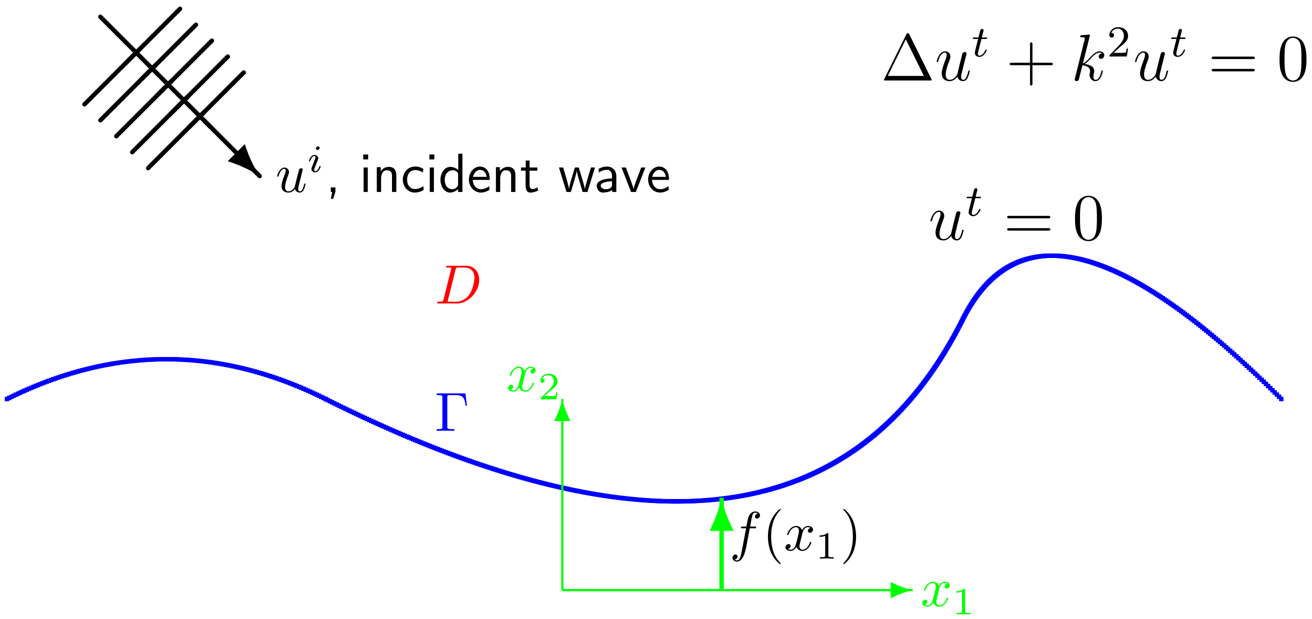
1. Maybe Anderson Localization happens here if f is random?

2. No rigorous results are known for surface scattering, but see de Monvel & Stollmann 2003 for an analogous configuration for the Schrödinger equation.

3. Significant numerical evidence of (approximate?) localization, e.g. Saillard (1994).

Formulation of Scattering Problem for $k > 0$

Mathematical Formulation for $k > 0$

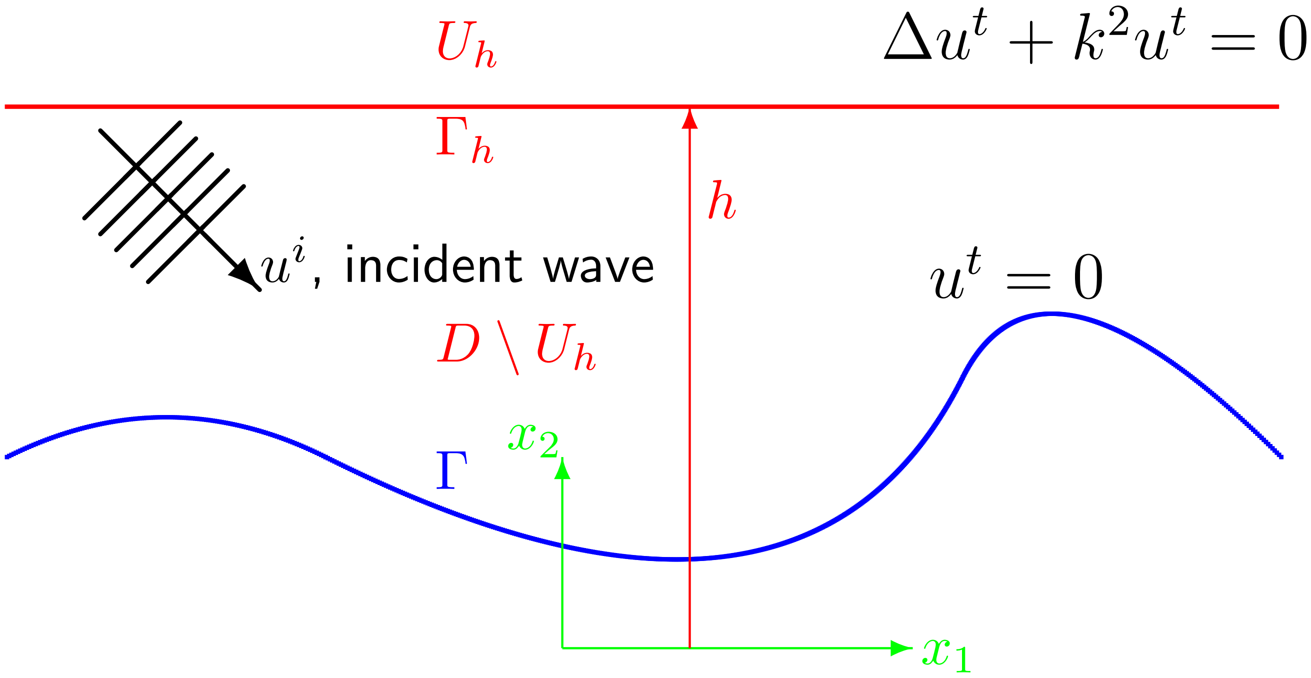


Assume $f \in BC(\mathbb{R}) := \{$bounded, continuous functions on $\mathbb{R}\}$, and at least piecewise smooth. Let $g := -u^i|_\Gamma \in BC(\Gamma)$.

BVP1. Given $k > 0$, $g \in BC(\Gamma)$, find scattered field $u \in C^2(D) \cap C(\bar{D})$ such that

$$\Delta u + k^2 u = 0 \text{ in } D, \quad u = g \text{ on } \Gamma,$$

and **...............?**



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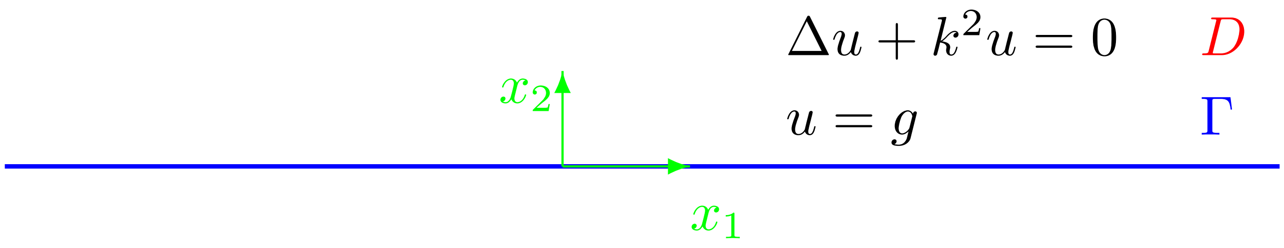
u is bounded in $D \setminus U_h$ for every $h > 0$, and u is **'outgoing'**.



Let

$$\Phi(x,y) := \frac{i}{4} H_0^{(1)}(k|x-y|), \quad x, y \in \mathbb{R}^2, \, x \neq y,$$

and define the Dirichlet Green's function

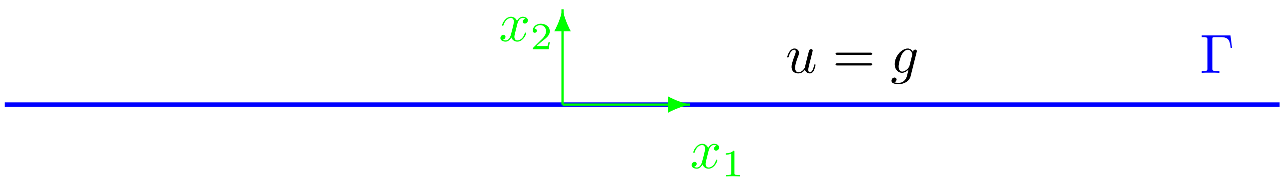
$$G(x,y) := \Phi(x,y) - \Phi(x,y'),$$

where

$$y = (y_1, y_2), \quad y' = (y_1, -y_2).$$

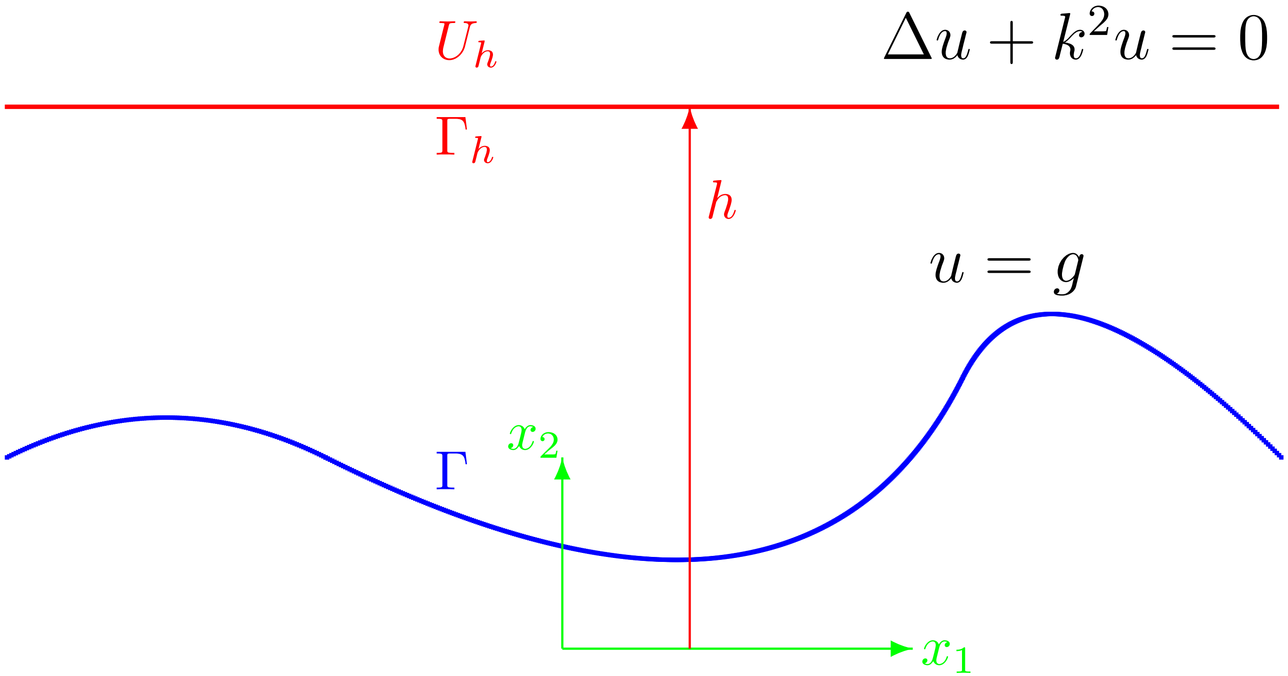
$$\Delta u + k^2 u = 0 \qquad D$$

$$u = g \qquad \Gamma$$



For $g \in BC(\Gamma)$ the correct solution satisfying limiting absorption condition is

$$u(x) = \int_\Gamma \frac{\partial G(x,y)}{\partial y_2} g(y) ds(y) = 2 \int_\Gamma \frac{\partial \Phi(x,y)}{\partial y_2} g(y) ds(y). \quad (*)$$



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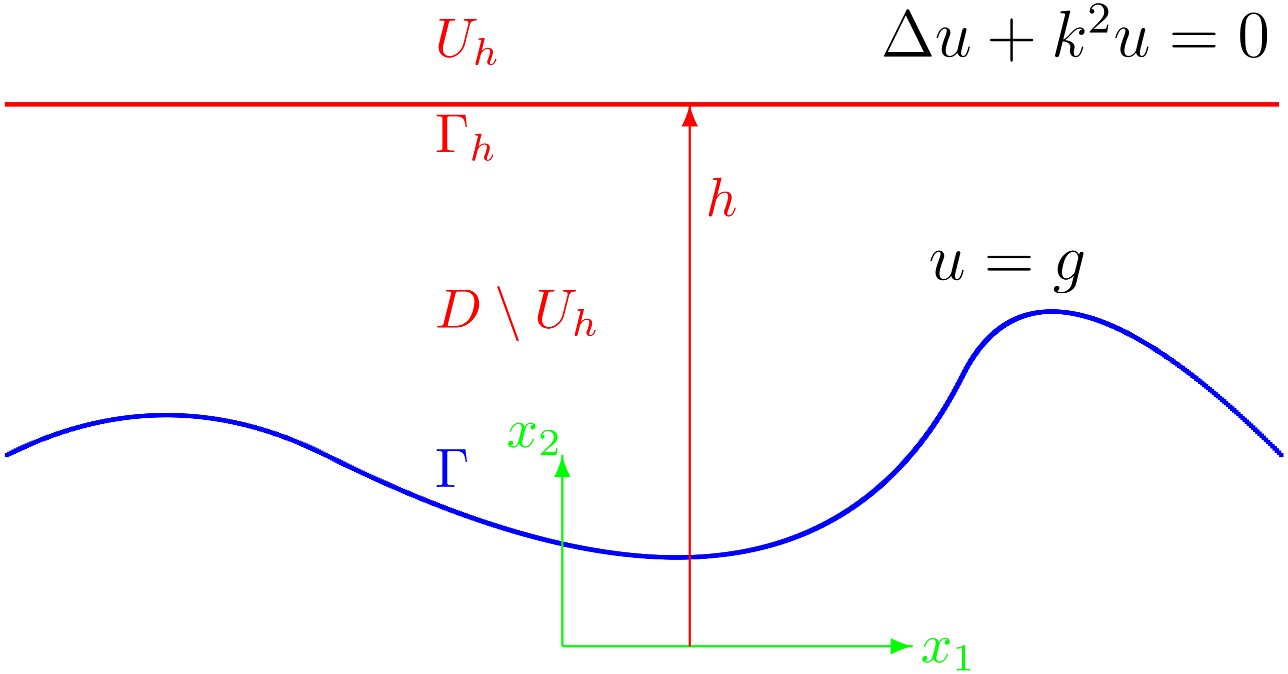


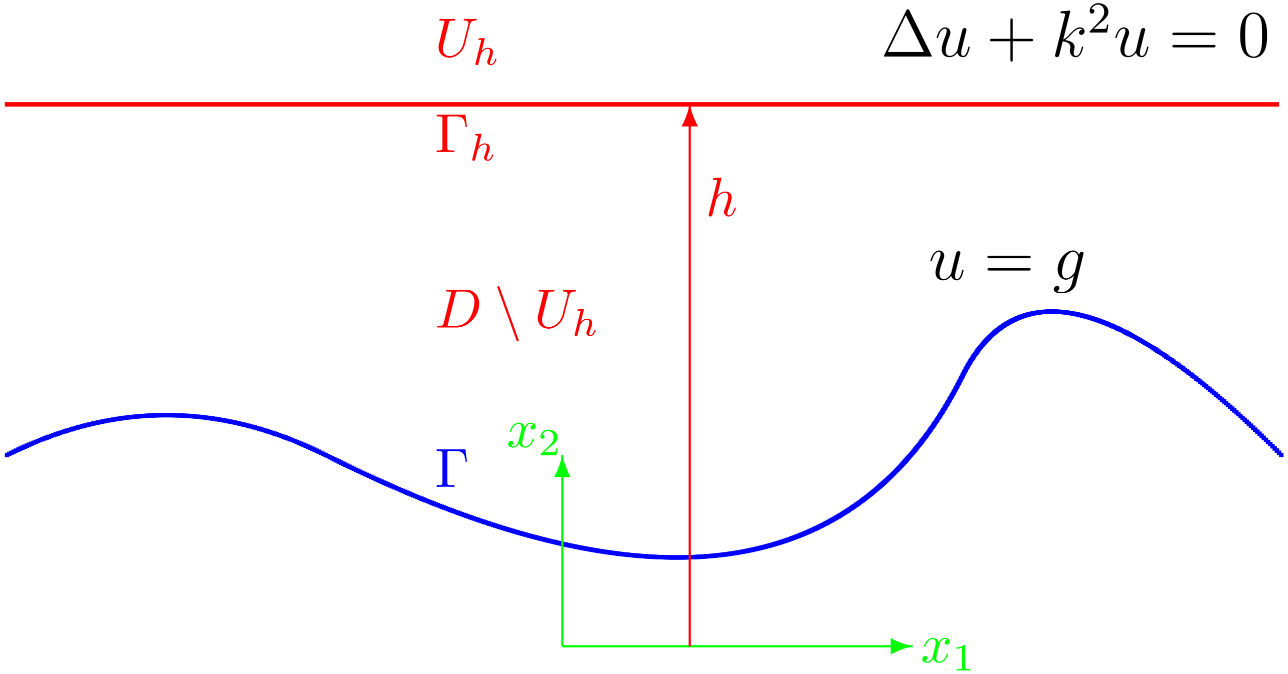
BVP1. Given $k > 0$, $g \in BC(\Gamma)$, find $u \in C^2(D) \cap C(\bar{D})$ such that

$$\Delta u + k^2 u = 0 \text{ in } D, \quad u = g \text{ on } \Gamma,$$

u is bounded in $D \setminus U_h$ for every $h > 0$, and, for some $h > 0$,

$$u(x) = 2 \int_{\Gamma_h} \frac{\partial \Phi(x,y)}{\partial y_2} u(y) ds(y), \quad x \in U_h.$$

We'll call this condition the Upward Propagating Radiation Condition (UPRC).



BVP1. Given $k > 0$, $g \in BC(\Gamma)$, find $u \in C^2(D) \cap C(\bar{D})$ such that

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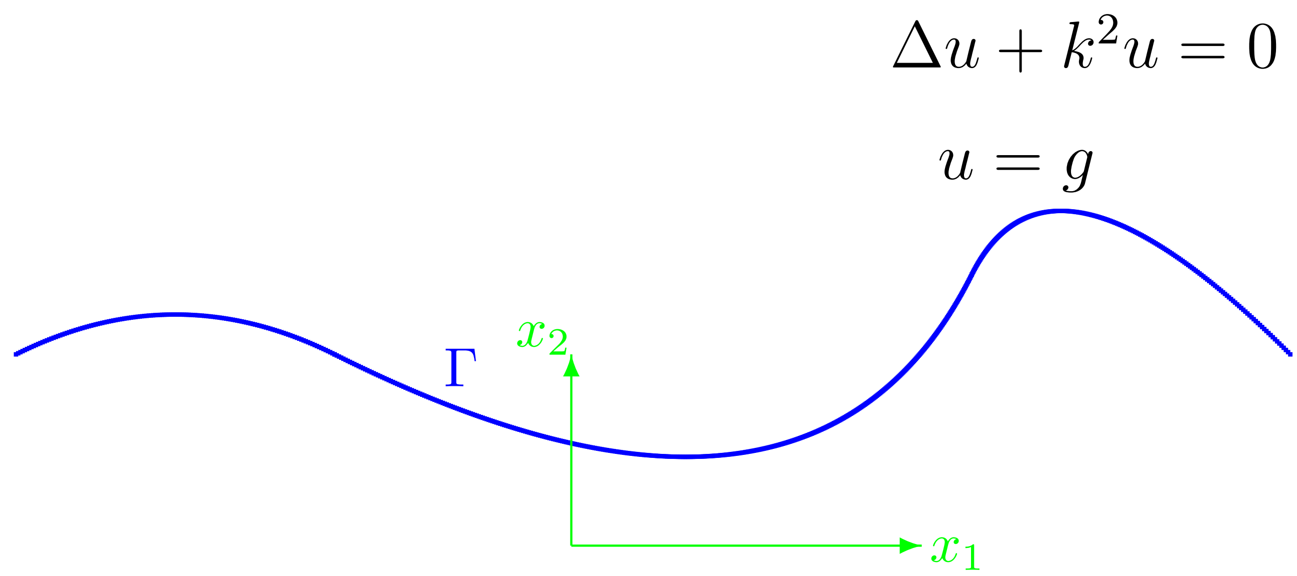
u is bounded in $D \setminus U_h$ for every $h > 0$, and, for some $h > 0$,

$$u(x) = 2 \int_{\Gamma_h} \frac{\partial \Phi(x,y)}{\partial y_2} u(y) ds(y), \quad x \in U_h.$$

Theorem. (C-W & Zhang 1998) This problem has at most one solution.

Boundary Integral Equation Formulation

Existence of Solution



Look for a solution as

$$u(x) = \int_\Gamma \left(\frac{\partial G(x,y)}{\partial n(y)} - i\eta G(x,y) \right) \phi(y) ds(y), x \in D,$$

where $\eta > 0$ is fixed coupling parameter and $\phi \in BC(\Gamma)$ is unknown density.

Theorem. u satisfies BVP1 provided

$$\phi(x) = 2g(x) - 2\int_\Gamma \left(\frac{\partial G(x,y)}{\partial n(y)} - i\eta G(x,y) \right) \phi(y) ds(y), x \in \Gamma.$$

In operator notation

$$\phi = \psi + K_f \phi, \quad \psi := 2g.$$

If the boundary is flat, i.e. f $\equiv$constant, then equation has form

$$\phi = \psi + \kappa_f * \phi$$

with $\kappa_f \in L^1(\mathbb{R})$. Then K_f has continuous spectrum and so is not compact.

Pick $c_1, c_2 > 0$ and let

$$B := \{f \in C^{1,1}(\mathbb{R}) : f \geq c_1, ||f||_{C_{1,1}(\mathbb{R})} \leq c_2\}.$$

Theorem. (Zhang & C-W 2003, Arens et al 2003)
There exists $f^* \in B$ such that, for every non-zero $\lambda \in \mathbb{C}$ and for $Y = BC(\Gamma)$ or $Y = L^p(\Gamma)$, $1 \leq p \leq \infty$, the following statements are equivalent:

(a) $\lambda \notin \Sigma_Y(K_{f^*})$;

(b) $\lambda \notin \Sigma_Y(K_f)$ **for all** $f \in B$;

(c) $\lambda \notin \Sigma^p_{BC(\Gamma)}$ **for all** $f \in B$.

As (c) holds for $\lambda = 1$ we get existence of solution to BIE and BVP.

(See talk by Lindner (tomorrow at 10.50!) for the operator theory behind this.)

The 3D Case: C-W, Heinemeyer, Potthast, 2005a,b

Theorem. If $g \in BC(\Gamma) \cap L^2(\Gamma)$ then u satisfies BVP1 provided

$$\phi(x) = 2g(x) - 2\int_\Gamma \left(\frac{\partial G(x,y)}{\partial n(y)} - i\eta G(x,y) \right) \phi(y) ds(y), x \in \Gamma.$$

In operator notation

$$\phi = \psi + K_f \phi, \quad \psi := 2g.$$

The operator is now strongly singular (not quite of Calderón-Zygmund type as the kernel is oscillatory). K_f is bounded as an operator on $L^2(\Gamma)$ and on $L^2(\Gamma) \cap BC(\Gamma)$ and invertible on each of these spaces with, by direct arguments (cf. Verchota 1984, Meyer & Coifman 2000),

$$||(I - K_f)^{-1}||_{L^2(\Gamma) \to L^2(\Gamma)} < 5(1+L)^2$$

where L is maximum surface slope, if $\eta := \kappa/2$.

Conclusions

- We've considered some problems of scattering by unbounded surfaces, localised in the x_n direction
- Quite a lot is known about the scattering problem and its integral equation formulation in one or two simple 2D and 3D cases
- A little is (rigorously) known about existence of 'surface wave eigenfunctions', and some non-existence results are known
- Many open problems

A Few Open Analysis Problems

- Is the 3D scattering problem ever well-defined for $k > 0$ and plane wave incidence?
- When do surface waves exist for the Neumann boundary condition (or Maxwell perfectly reflecting b.c.)? Are there band gaps?
- What is the correct limiting absorption principle when surface waves/eigenfunctions exist?
- Can we establish Anderson localization for surface scattering? (Done recently for a model problem in Schrödinger case (de Monvel & Stollmann 2003)).