Introduction to Supersymmetry: Mini-Notes and Errata for Chapter 1
Alan Reynolds
September 8, 2017

1 Introduction

What follows is the transcription of many of the notes that I wrote on little paper bookmarks and inserted into Chapter 1 of the book “Introduction to Supersymmetry” by Harald J. W. Müller-Kirsten and Armin Wiedemann [1]. Some of these are merely notes I made in order to clarify points for myself and to prevent myself from rechecking or reperforming calculations I have already checked or performed. Others make corrections to the text.

There are some difficulties with attempting to correct chapter related to choice of convention, with one choice leading to one set of corrections, while a different choice leads to a different set. We will use the following conventions, that are described further in the sister document [2].

- The complex conjugate representation consists of spinors of the form \( \bar{\Psi}_A \), i.e.
  \[ (\psi_A)^* = \bar{\Psi}_A \in \hat{F}. \]

- The \( \epsilon \)-matrices are given by
  \[ \epsilon^{\hat{A}\hat{B}} = \epsilon^{\hat{B}\hat{A}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \]
  \[ \epsilon_{\hat{A}\hat{B}} = \epsilon_{\hat{B}\hat{A}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \]

- Index position on \( \epsilon_{ijk} \) does not matter, i.e. \( \epsilon_{123} = \epsilon^{123} = \epsilon^{1}_{23} = \epsilon^{12}_{3} = 1. \)

- Index position on \( \epsilon_{\mu\nu\rho\sigma} \) does matter, with \( \epsilon^{0123} = +1. \) Indices are raised and lowered with the metric tensor, i.e. \( \eta^{\mu\nu} \) and \( \eta_{\mu\nu} \).

Positive line numbers, e.g. line 5, refers to the fifth line from the top of a page, while a negative line number counts up from the bottom. Equations and section titles are included in the count, while footnotes are not. Each item will be colour coded: black for a note; red for a correction; blue for a query or an instruction for me to investigate further in future. Green merely indicates something that I have checked. Red underlining in mathematical expressions is merely used to highlight the location of the correction.

I make these notes available to other students and researchers in case they may be of benefit when reading “Introduction to Supersymmetry”. Having access to a set of corrections for chapter one would certainly have saved me some time and effort! I hope they are useful and that I have not introduced too many new errors and typos of my own.

\[ ^* \text{This habit of mine has been known to strain the spines of a number of books.} \]

\[ ^{3} \text{Hence } \epsilon_{0123} = -1, \text{ unlike as suggested in page 26 of the book.} \]

\[ ^{3} \text{Hence ‘green’ notes are solely for my benefit!} \]
The notes

p8, l2: Note that $\Lambda^\rho_\mu = (\Lambda^T)^\rho_\mu$. Hence we may rearrange to get $\Lambda^T \eta \Lambda = \eta$. (See bottom of page.)

p9, l2: In more detail $\Upsilon^T = -\eta \Upsilon \eta$ is

$$(\Upsilon^T)^a_\beta = -\eta^\beta_\mu \Upsilon^\mu_\delta \eta^\delta_a.$$ 

If we equate $(\Upsilon^T)^a_\beta$ with $\Upsilon^a_\beta$ and then lower the $a$ index, we get

$$\Upsilon^a_\kappa \beta = -\Upsilon^a_\beta \kappa,$$

i.e. generators are real and antisymmetric. (Note that we cannot really discuss (anti)symmetry until the indices are either both up or both down.)

p9, l8: Note the lack of $-i$ in the exponent — the mathematician’s way of doing things. (We see the physicist’s way on p11.) For groups like $SU(n)$ we get generators that are anti-Hermitian. (Note that in the physicist’s approach, we get imaginary, anti-symmetric matrices — hence they are Hermitian.)

p9, l11: Here the metric has both indices down.

p10, l-6: Surely $\Lambda \in L^\perp$ maps the forward light cone on to the backward light cone and vice-versa.

p11, eq. 1.8: The factor of 1/2 is due, I guess, to the fact that there are only really six generators, but they are duplicated, i.e. we have $M_{12}$ and $M_{21} = -M_{12}$. With $\omega_{21} = -\omega_{12}$ we double count and hence, to compensate, we divide by two. (The minus sign is, I guess, purely conventional.)

p11, l-8: But it may look like $M_{0i}$ are anti-Hermitian (see pp13,14). Indeed, they are, but this is with the matrix indices in the up-down position and, with indices in these positions, it is perhaps unwise to talk of (anti)symmetry or (anti)Hermiticity. If we raise or lower an index then we find that all the generators are Hermitian as required — see p15. (The omission of matrix indices can easily lead to this sort of confusion.)

p11, eq 1.10→1.11: We have

$$\delta^\mu_\rho = -\frac{i}{2} \omega^{\rho\sigma} (M_{\rho\sigma})^\mu_\nu$$

for all antisymmetric $\omega^{\rho\sigma}$. Hence we cannot just cancel $\omega^{\rho\sigma}$. However, we can consider specific cases, e.g. $\omega^{12} = -\omega^{21} = 1$, $\omega^{ij} = 0$ otherwise. From this case we get

$$\delta^\mu_1 \eta_{i2} - \delta^\mu_2 \eta_{i1} = -\frac{i}{2} (M_{12} - M_{21})^\mu_\nu = -i (M_{12})^\mu_\nu,$$

i.e.

$$(M_{12})^\mu_\nu = i (\eta_{2i} \delta^\mu_1 - \eta_{1i} \delta^\mu_2).$$
But this reasoning can also be applied to $\omega^{13} = -\omega^{31} = 1$, ($\omega^{ij} = 0$ otherwise), etc. I.e.

$$(M_{\rho\sigma})^\mu = i(\eta_{\nu\delta}^\rho \delta^\mu_\sigma - \eta_{\rho\nu}^\delta \delta^\mu_\sigma),$$

which is (1.11).

Writing out the matrices $M_{\rho\sigma}$, we have

$$M_{01} = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

with $M_{02}, M_{03}$ similar and

$$M_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

with $M_{13}, M_{23}$ similar, i.e. $-i$ above the diagonal.

**p12, l2:** I find it clearer to go

$$-\frac{i}{2} \omega^{\rho\sigma}(M_{\rho\sigma})^\mu = \frac{1}{2} \omega^{\rho\sigma}(\eta_{\nu\delta}^\rho \delta^\mu_\sigma - \eta_{\rho\nu}^\delta \delta^\mu_\sigma) = \frac{1}{2} \omega^{\rho\sigma} \eta_{\nu\delta}^\rho \delta^\mu_\sigma + \frac{1}{2} \omega^{\rho\sigma} \eta_{\rho\nu}^\delta \delta^\mu_\sigma = \frac{1}{2} \omega^\mu_v + \frac{1}{2} \omega^\mu_v = \omega^\mu_v,$$

where, in the second line, we have exploited the antisymmetry of $\omega^{\rho\sigma}$. This way we do not have to worry about the odd looking $\omega^\mu_v = -\omega^\mu_v$.

**p12, l7:** Here we are returning to the mathematician’s definition of generators, i.e. no factor of $-i$ in the exponential. This switching back and forth is potentially confusing. (Though it is good to have all the calculational detail.)

**p12, bottom half:** Of course, we have already calculated the generators — we can simply exploit formula (1.11).

**p12, l3:** Intriguing how this looks like a field strength tensor from electromagnetism.

**p12, l1:** Note that the $M_j$ are different from the previously defined $M_{ij}$. We have

$$M_1 = -iM_{23} (= iM_{32}),$$
$$M_2 = iM_{13},$$
$$M_3 = -iM_{12},$$
$$N_j = iM_0.$$
p13, l13: If you are wondering where the antisymmetry went, remember that here the matrix indices \( \mu \) and \( \nu \) are considered to be in up-down formation. If we raise or lower an index, we get the antisymmetry back.

p13, l9: The relationship between \( J_l, K_l \) and \( M_{ij} \) follows on p14 — use my note on p12.

p14, l6: This should read*+  
\[ \frac{1}{2} \epsilon_{kij} M_{ij} = J_k. \]
If we then multiply both sides by \( \epsilon_{klm} \) and use the identity 
\[ \epsilon_{kij} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}, \]
we get  
\[ M_{ij} = \epsilon_{jk} J_k, \]
which matches what we have in (1.12).

p14, eq. 1.12: The matrix notation is potentially confusing here. Consider this more as a lookup table instead. So, for example, we see that the matrix \( M_{01} \) is the matrix \( -K_1 \). (The use of \( \mu \) and \( \nu \) here doesn’t help. Previously these were used as matrix indices. Here they are labels that indicate which matrix we are looking at. It might have been better to use \( i, j \) or \( \rho, \sigma \)).

Note an even more striking resemblance to the fields strength tensor of electromagnetism.

p15, eq. 1.13: From this point on, dropping the matrix indices would be most confusing — we wouldn’t know whether they were up-down or down-down.

p16, l3: Again, from now on, matrix indices, or at least their positions, will need to be explicitly given to avoid confusion.

Note that it is a combination of factors that leads to \( (K_i)_{\rho \sigma} \) being Hermitian:
1. \( (K_i)^{\rho}_{\sigma} \) is anti-Hermitian,
2. \( (K_i)^{\rho}_{\sigma} \) only has non-zero elements in positions \( 0i \) and \( i0 \),
3. These non-zero elements are purely imaginary.
A similar set of factors leads to \( (J_i)_{\rho \sigma} \) begin Hermitian

p17, l10→21: I have also performed this calculation with indices in the up-down positions, i.e. \( \ldots \alpha^\beta \). The same result is found.

p17, eq. 1.15: This one is correct. (See notes for p14.)

p17, l1: (1.14) should appear above the second equal sign, rather than the third.

p17, l1: The third equality arises because the metric is diagonal (hence \( \eta_{0j} = \eta_{00} = 0 \)) and \( M_{00} \) is taken to be the zero matrix.

p18, l5: The left hand side should read
\[ -i \eta_{ik} K_i + i \eta_{il} K_l. \]
Also the label (1.14) is above the wrong equal sign again.
18, eq. 1.20: This does agree with the commutation relation on
p13:

\[(1.20) \implies [J_k, K_l] = -i\epsilon_{kl}K_l = i\epsilon_{kl}K_l \implies \text{relation on p13},\]

where the last implication is obtained through simple index relabelling.

19, l6: Strictly speaking, \(K\) cannot be a vector under the Lorentz
group, since it is only a 3-vector. It is a vector under the rotation
group.

19, l9: In other words, non-homomorphic Lie algebras imply
non-homomorphic Lie groups, so this minus sign expresses a
difference not only in the algebras but also in the groups.

19, l11→12: Why non-unitary?

19, eq. 1.24: Checked.

19, l10: The set of generators \(S_i\) obey the commutation relations
of the Lie algebra of \(SU(2,\mathbb{C})\), while, separately, the generators \(T_j\)
do too.

19, l6: How can a set of real matrices possibly be a complex
vector space?

19, l1: Out of interest, what is \(so(1, 3;\mathbb{C})\) and how does it com-
pare with \(so(1, 3;\mathbb{R})^\mathbb{C}\)?

20, l4→6: I’ll take their word for this.

20→21: I should re-read this with a good text on ‘group theory’
to hand!

22, l12: We do not need to know the details about semi-direct
products here.

22, l3: What part of the argument requires the representation to
be faithful?

22→25: Why do we work with a general representation? Well,
we do not have a natural representation to start from, unlike the
situation with just Lorentz transformations. Lorentz transforma-
tions may be considered to act linearly on displacement (or in-
deed position) vectors. Poincaré transformations act on positions,
which do not form a vector space, or at least not one upon which
Poincaré transformations act linearly. (Consider translations in
particular.) Hence the use of \(g(\Lambda, a)\).

23, l3: Note that we are not using just any (infinitesimal) parameter-
ization of the Lorentz part of the transformation — we use the
\(\omega\) defined by (1.6) on p10, i.e. \(\Lambda = 1 + \omega\). Hence

\[\Lambda^{-1}\Lambda'\Lambda = \Lambda^{-1}(1 + \omega')\Lambda = 1 + \Lambda^{-1}\omega'\Lambda,\]

i.e. the infinitesimal parameters for \(\Lambda^{-1}\Lambda'\Lambda\) are \(\Lambda^{-1}\omega'\Lambda\).
Mixing index notation with matrix notation does lead to a little confusion here. Might it be easier to start with an alternative for (1.30), i.e.

$$g(\Lambda, a) = \mathbb{1}_V - \frac{i}{2} \omega^{\rho \gamma} M_\rho^\gamma + i a_\mu P^\mu$$

or something?

Here we use the fact that the $\Lambda$ belong to $SO(1,3)$. In detail, we use $\Lambda^\top \eta \Lambda = \eta$ from p9 to get

$$\eta_{\alpha \gamma} \Lambda^\top \delta^\gamma = \eta_{\alpha \delta},$$
$$\eta_{\alpha \delta} \Lambda^\top \alpha = \eta_{\alpha \delta},$$
$$\eta_{\alpha \delta} \Lambda^\top \delta = \delta_{\alpha \delta},$$

i.e.

$$(\Lambda^{-1})^\beta_{\alpha} = (\Lambda^\top)^{\beta}_{\alpha} = \Lambda^\top_{\alpha} \delta^\beta.$$

$p23$, l-1: Typo: There is an odd gap between the indices on $\omega^\rho_{\nu \sigma}$.

$p24$, l13: Here we see an alternative way to ‘cancel’ $\omega_{\rho \sigma}$. Recall that if $\omega_{\rho \sigma}$ is antisymmetric only, then we cannot simply cancel the $\omega_{\rho \sigma}$. We can, however, state that

$$\omega_{\rho \sigma} M^{\rho \sigma} = \omega_{\rho \sigma} W^{\rho \sigma}$$

for all antisymmetric $\omega_{\rho \sigma}$ only, then we cannot simply cancel the $\omega_{\rho \sigma}$. We can, however, state that

$$M[\rho \sigma] = W[\rho \sigma],$$

where the square brackets denote taking the antisymmetric part. Furthermore, if we can arrange for both $M^{\rho \sigma}$ and $W^{\rho \sigma}$ to be antisymmetric, then we can cancel $\omega_{\rho \sigma}$. Lines 15–18 are arranging this antisymmetry, allowing for the cancellation of $\omega_{\rho \sigma}$ to produce line 20.

$p25$, l6: Change $p^\nu$ to $P^\nu$.

$p25$, l8–9: Again, we arrange for everything to be antisymmetric before cancelling $\omega_{\mu \nu}$.

$p26$, l11: Change this to $\epsilon^{0123} = +1$.  

$p27$, l5–7: Just note that $P^\rho$ and $P^\sigma$ commute, so $P^\rho P^\sigma$ is symmetric. Multiplying by the totally antisymmetric tensor therefore gives zero. In detail

$$\epsilon_{\nu \rho \sigma \mu} P^\rho P^\sigma = \frac{1}{2} (\epsilon_{\nu \rho \sigma \mu} P^\rho P^\sigma + \epsilon_{\nu \rho \sigma \mu} P^\rho P^\sigma)$$
$$= \frac{1}{2} (\epsilon_{\nu \rho \sigma \mu} P^\rho P^\sigma + \epsilon_{\nu \rho \sigma \mu} P^\rho P^\sigma)$$
$$= 0.$$

In the first line we have merely split $\epsilon_{\nu \rho \sigma \mu} P^\rho P^\sigma$ into two halves and then relabelled the dummy indices ($\rho \leftrightarrow \sigma$) in the first term. In the second line, we have moved $P^\rho$ past $P^\sigma$ (since they commute) and used $\epsilon_{\nu \rho \mu} = -\epsilon_{\nu \rho \mu}$.
p27, l-9→-7: Lines −9 and −8 are an odd mixture. Note that
\[ \epsilon^a \beta_\gamma \delta = -\epsilon_\beta^a \gamma \delta, \]
i.e. \( \epsilon \) is still antisymmetric upon exchanging
indices, but each index’s position (up or down) must be main-
tained.\(^8\)

p27, l-4: \( \eta^{a\beta} \) is annihilated upon contraction with \( \epsilon^{a\beta\gamma\delta} \). (Symmet-
ric \( \times \) antisymmetric.)

p28, l-10: We might well call \( I \) ‘manifestly’ Lorentz invariant, being
constructed from objects that look a lot like tensors. However, I
have not yet convinced myself that these objects are
tensors and, as such, I would like to check
\[ [M^\mu_\nu, I] = 0 \]
via calculation. Thus
far this has proved awkward.

p27: Calculations checked.

p28, l-7→-6: This formula for \( \epsilon^\mu_{a\beta\gamma} \epsilon_{\mu\rho\sigma\tau} \) is out by a factor of \(-1\).
E.g., if \( a = \rho = 1, \beta = \sigma = 2 \) and \( \gamma = \tau = 3 \) then the left hand
side is
\[ \epsilon^0_{123} \epsilon_{\mu23} = \epsilon^0_{123} \epsilon_{0123} = \epsilon_{0123} \epsilon_{0123} = 1, \]
while the right hand side is
\[ \eta_{11} \eta_{22} \eta_{33} = (-1)(-1)(-1) = -1. \]

Similarly,
\[ \epsilon^\mu_{012} \epsilon_{\mu012} = \epsilon^3_{012} \epsilon_{3012} = -\epsilon_{3012} \epsilon_{3012} = -1, \]
while
\[ \eta_{00} \eta_{11} \eta_{22} = (1)(-1)(-1) = 1. \]

This error explains the otherwise mysterious appearance of the
overall minus sign on line -4, which corrects the subsequent
calculations.

p29, l-14: Also uses \[ [P_\tau, P_\sigma] = 0 \] to get \( P_\tau P_\sigma \) symmetric. This, com-
bined with the antisymmetry of \( M^{\tau\sigma} \), is what leads to \( P_\tau P_\sigma M^{\tau\sigma} \)
vanishing.

p29, l-17: Checked. I would say ‘similarly’ rather than ‘in the same
way’.

p29, l-5→-1: Since the commutators have the same structure, this
is essentially the same as at the top of page 26. (Note that we do
not need \[ [W_\mu, W_\nu]. \])

p30, l-18→-12: On the second line of this calculation, we can already
see that each term has \( \epsilon_{\mu\rho\sigma\tau} \) contracted (on two indices) with
something symmetric, and thus that the overall result must be
zero.

p31, l-4: \( S^{jk} \) is just \( M^{jk} \). Since \( j \) and \( k \) are both spacelike indices, this
is a generator associated with rotation.

p31, eq. 1.44, 1.45: Checked.

p31, eq. 1.45: \( S^i \) appears to be just \( J^i \) from page 14.
Taking careful note of our change of convention to $\epsilon_{0123} = +1$ and the correction to page 14, we get

$$\frac{1}{2} \epsilon^{ijk}M_{jk} = \frac{1}{2} \epsilon_{ijk}M_{jk} = I_i = -J^i,$$

so that

$$\frac{1}{2} \epsilon^{ijk}p_iM_{jk} = -p_iJ^i = pJ.$$ 

Note that we continue to sum over repeated indices here, even when they are not up-down or down-up.

$$\frac{1}{2} \epsilon_{ijk}p_iM_{jk} = -pJ.$$

I suspect this should read

$$p^\mu = (p^0, 0, 0, p^0).$$

As written in the book, we get

$$0 = \omega_\mu p^\mu = \omega_0 p^0 + \omega_3 p^3 = \omega^0 p^0 - \omega^3 p^3,$$

but $p^3 = -P_3 = -P_0$ so that $\omega^0 = -\omega^3$ (though we do have $\omega_0 = \omega_3$ instead).

To avoid issues, it might be best to avoid the use of $P_0$ (or $P^0$) entirely and instead write something like

$$p^\mu = (P, 0, 0, P).$$

$D^{(1)}(M) = UD^{(2)}(M)U^{-1}$ must hold for all $M$, with just the single matrix $U$, for the representations to be equivalent.

I like the use of the phrase “self-representation” rather than “fundamental representation” — mathematicians seem to use “fundamental representation” to mean something different to physicists.

Good to see explicit use of $\dot{1}$ and $\dot{2}$.

Replace with $\psi_A =: \overline{\psi}_A \in \bar{F}$.

Note that if we use equation (1.63) or its replacement directly in calculations then we can end up with rather haphazard index notation. This can be fixed by using a ‘identity matrix in disguise’ to fix the index structure. See page 110, my corrections to that page and the sister document [2].

Again, it must not be possible to find a single $2 \times 2$ matrix $C$ such that

$$M = CM^*C^{-1}$$

for all $M$ in $SL(2, \mathbb{C})$. To prove that no such $C$ exists, we start by assuming that there is such a $C$. Scaling $C$ by an overall factor

$^+$ Note that we continue to sum over repeated indices here, even when they are not up-down or down-up.

$^\dagger$ The lectures I attended used “fundamental representations”.

$^\ddagger$ This is a key change and it took quite a while for me to be sure of its necessity. See the sister document [2] for further details.

$^{11}$ I make no claims as to the elegance of this proof!
\( \lambda \) does not change \( CM \cdot C^{-1} \), so we may assume that \( \det C = 1 \).
Consider now a specific \( M \),
\[
M = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.
\]
Then\(^{13}\)
\[
CM^*C^{-1} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} C_{22} & -C_{12} \\ -C_{21} & C_{11} \end{pmatrix} = \begin{pmatrix} -i(C_{11}C_{22} + C_{12}C_{21}) & \cdot \end{pmatrix},
\]
and we note that we must have \( C_{11}C_{22} + C_{12}C_{21} = -1 \). Combining this with \( \det C = C_{11}C_{22} - C_{12}C_{21} = 1 \), we find that \( C_{11}C_{22} = 0 \) and \( C_{12}C_{21} = -1 \). We now repeat for a different \( M \in SL(2, \mathbb{C}) \). Consider
\[
M = \begin{pmatrix} 2i & 0 \\ 0 & -i/2 \end{pmatrix}.
\]
We now get
\[
CM^*C^{-1} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} -2i & 0 \\ 0 & i/2 \end{pmatrix} \begin{pmatrix} C_{22} & -C_{12} \\ -C_{21} & C_{11} \end{pmatrix} = \begin{pmatrix} -2iC_{11}C_{22} - \frac{i}{2}C_{12}C_{21} & \cdot \end{pmatrix}.
\]
Since we already know that \( C_{11}C_{22} = 0 \), this forces us to have
\[
-\frac{i}{2}C_{12}C_{21} = 2i,
\]
i.e. \( C_{12}C_{21} = -4 \neq -1 \). We have a contradiction and therefore no such \( C \) can exist.

**p34, eq. 1.64:** Recall that if a contravariant\(^{14}\) vector transforms under \( A \), i.e. \( x \rightarrow x' = Ax \), then a covariant vector transforms under \( (A^{-1})^\top \), i.e. \( y \rightarrow y' = (A^{-1})^\top y \).\(^{15}\) This explains the motivation for considering covariant and contravariant spinors and for using index location to denote this.

**p35, eq. 1.65:** Again, there should be a single \( \epsilon \) matrix that works with all \( M \in SL(2, \mathbb{C}) \). It is worth asking what happens if we consider \( M \in SO(1,3; \mathbb{R}) \) instead. Well, then \( M^\top \eta M = \eta \) implies that \( (M^{-1})^\top = \eta M^\top \eta^{-1} \), so we use \( \eta \) to raise/lower indices rather than \( \epsilon \), precisely as we are used to doing! (See (1.72) for raising spinor indices with \( \epsilon \).)

**p35, l.9→6:** These should be
\[
\begin{align*}
a &= M_{11} \epsilon_{11} \epsilon_{22} - M_{12} \epsilon_{11} \epsilon_{21} + M_{21} \epsilon_{12} \epsilon_{22} - M_{22} \epsilon_{12} \epsilon_{21} \\
b &= -M_{11} \epsilon_{11} \epsilon_{12} + M_{12} \epsilon_{11}^2 - M_{21} \epsilon_{12}^2 + M_{22} \epsilon_{11} \epsilon_{12} \\
c &= M_{11} \epsilon_{21} \epsilon_{22} - M_{12} \epsilon_{21}^2 + M_{21} \epsilon_{21} \epsilon_{22} - M_{22} \epsilon_{21} \epsilon_{22} \\
d &= -M_{11} \epsilon_{12} \epsilon_{21} + M_{12} \epsilon_{11} \epsilon_{21} - M_{21} \epsilon_{12} \epsilon_{22} + M_{22} \epsilon_{11} \epsilon_{22}.
\end{align*}
\]
\(^{13}\) Here we keep only those elements of the matrix used in the proof.

\(^{14}\) I may need to switch covariant and contravariant around!

\(^{15}\) If we choose to write covariant vectors as rows, then we get \( y \rightarrow y' = yA^{-1} \). (However, note that attempting to fix a rigid relationship between index location and the direction (row or column) of the vector being considered is unwise and can lead to difficulties. (See [2].))
The equations that follow are also incorrect, but the conclusions of page 36 are correct.

**p36, l12:** Up to this point, all indices have been ‘down’ and index location (up or down) has not been a factor. However, from now on, index location is important. Some confusion results from the shift. The $e$ in $e Me^{-1}$ considered on page 35 becomes $e^{AB}$, so what we actually do is we choose $e^{12} = 1$ and $e^{21} = -1$. $e_{AB}$ is then the inverse of $e^{AB}$, i.e. $e^{-1} \rightarrow e_{AB}$.

**p36, l18:** There is an odd mixture of notation here. On line 18, $e^{AB}$ is a matrix — rather than a matrix component and we are simply performing matrix multiplication. There is no summation convention at use here.

**p36, eq. 1.68:** Note that the Kronecker delta does not obey the usual index raising/lowering rules (see (1.72)\(^{-17}\). If it were the case that $\delta^{AC} = e^{AB} e_{CD} \delta^{D}\) then we would get

$$\delta^{AC} = e^{AB} e_{CB} = -e^{AB} e_{BC} = -\delta^{CA}$$

**p37, eq. 1.70:** Note that this means that we also should not apply index raising/lowering rules to $M^{B}_{A}$ either — otherwise we get

$$M^{F}_{E} = -(M^{-1\top})^{-1}_{E}$$

**p37, eq. 1.72:** It is vital that the ordering of the indices on $e^{AB}$ is precisely as given, since $e^{BA} = -e^{AB}$. Hence

$$\psi^{A} = e^{AB} \psi_{B} = \psi_{B} e^{AB}$$

**p37, eq. 1.76a:** As before, there is some choice in $\bar{\epsilon}$, i.e. we may scale it by $\lambda$ (possibly -1). But by choosing as shown, we get

$$e^{AB} = \bar{\epsilon}^{AB}$$

(which is easier on the memory) and, more importantly, $\bar{\psi}^{A} = (\psi^{A})^{*}$, which is convenient.\(^{18}\)

**p39, l13:** While * combined with $\dagger$ is $\ddagger$, I find it better to consider the two operations separately. Then we get an inversion and a transposition for contravariant\(^{19}\) quantities and a complex conjugation for dotted quantities. Also, to clarify, it is $\bar{\psi}^{B}$ that transforms under $(M^{\dagger})^{-1}$.\(^{19}\)

**p42, l11:** We certainly don’t have to consider $\bar{\psi}_{A}$ as rows and $\bar{\psi}^{A}$ as columns. After all, we could have considered $\bar{F}$ as the vector space of dotted spinors with lower indices, $\bar{\psi}_{A} \in \bar{F}$ instead, in which case $\bar{F}^{*}$ is the dual space (containing $\bar{\psi}^{A}$). However, it is typical to write the Dirac spinor as a column formed from $\psi_{A}$ and $\bar{\psi}^{A}$.

While it can be handy at times to think in terms of matrices, it is not always convenient. (As a simple example, try writing the index raising/lowering formulae as matrix formulae using the suggested conventions for rows and columns, without adding transposition symbols.)
page 42, l-8→1:  Replace (1.90) with
\[ \delta_A^B (\psi_B)^* = \overline{\psi}_A^B, \]
where \( \delta_A^B \) is, numerically, the identity index.\(^{20}\) Similarly, (1.91) becomes
\[ \delta_A^B (\overline{\psi}_B)^* = \psi_A^B, \]
while the equation on line -1 and that on line 2 of page 43 become
\[ \psi^A = (\overline{\psi}^B)^* \delta_B^A, \]
and
\[ \overline{\psi}^A = (\psi^B)^* \delta_B^A. \]

p42, footnote: Reference to eq (1.199) doesn’t seem right. See p53→54.

p43, l5: Yes, there are only 4 real degrees of freedom here, but can we really take \( \psi_A \) and \( \overline{\psi}^A \) to be real?

p43, l10: I might have used \( f \) rather than \( \phi \), in order to better distinguish between elements of vector spaces (and their duals) and maps between them.

p43, l11: The dual map is also known as the pullback or as the transpose map. In terms of vectors, if \( \phi \) maps column vectors in \( V \) to column vectors in \( W \), then \( \phi^* \) maps row vectors back again, i.e. from \( W^* \) to \( V^* \).

p43, l3: Proof of linearity is straightforward.
\[ \phi^*(a \psi + \chi)(v) = (a \psi + \chi)(\phi(v)) \]
\[ = a \psi(\phi(v)) + \chi(\phi(v)) \]
\[ = a \phi^*(\psi)(v) + \phi^*(\chi)(v), \]
so
\[ \phi^*(a \psi + \chi) = a \phi^*(\psi) + \phi^*(\chi). \]

Note that linearity of \( \phi \) is not needed above. (If \( \phi \) is not linear, we find that \( \phi^* \) as defined does not map to \( V^* \), i.e. not to linear functionals. E.g. if \( \phi(2v) = 4 \phi(v) \) then
\[ \phi^*(\psi)(2v) = \psi(\phi(2v)) = \psi(4 \phi(v)) \]
\[ = 4 \psi(\phi(v)) = 4 \phi^*(\psi)(v) \]
and \( \phi^*(\psi) \) is not linear.)

p44, l1→4: Note that the matrix terminology is already creaking here. Suppose we consider vectors in \( V \) and \( W \) to be column vectors, with vectors in the dual space being row vectors (represented, for example, by \( f^\top \)). We are given \( x \in V \) and matrix \( A \) such that \( x' = Ax \in W \). We are also given \( f^{\top} \in W^* \) and we must find the dual map that sends \( f^\top \) to \( f^{\top} \in V^* \), such that
\[ f^{\top} x = f^{\top} x' = f^{\top} A \psi. \]
Clearly, to get $f^\top$ from $f'^\top$, we multiply by $A$ on the right. We can take the transpose, to get $f = A^\top f'$ so that our dual map is given by $A^\top$ as suggested in the text. But note that $A^\top$ is then applied to a column vector, which does not look like a member of $W^\ast$. (If this is unclear, set 
\[
x = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]
and $f'^\top = (\alpha' \beta')$).

p44, l13→17: The exposition seems confused here. I think that, since we want to examine spinor transformation laws, we want to look at the inverse of the dual map, i.e. we have 
\[
f : F \to F',
\]
where I use a dash to indicate the space of transformed spinors (which is, of course, just $F$) and we want to examine 
\[
(f^\ast)^{−1} : F^\ast \to F'^\ast.
\]

p45: Since we have changed (1.63) and are now using matrices such as $\delta^A_B$ to fix the index notation, rather than $\sigma^a$ and $\bar{\sigma}^\alpha$, we need to redo the calculations of this page. For the first calculation, we get 
\[
\bar{\psi}'_A = \delta^A_B (\psi'_A)^\ast \\
= \delta^A_B (M^B_A \psi_B)^\ast \\
= \delta^A_B (M^B_A)^\ast \psi_B \\
= \delta^A_B (M^B_A)^\ast \delta_B^B \psi_B.
\]

We then simply define 
\[
(M^\ast)^B_A : = \delta^A_B (M^B_A)^\ast \delta_B^B. \tag{1}
\]
Note that, numerically, this just means that matrix $M^\ast$ is simply the complex conjugate of $M$.

We continue to see how $\bar{\psi}'_A$ transforms. We get 
\[
\bar{\psi}'^A = e^{AB} \bar{\psi}'_B \\
= e^{AB} \delta^B_B (M^B_A)^\ast \delta_C^C \bar{\psi}_C \\
= e^{AB} \delta^B_B (M^B_A)^\ast \delta_C^C e_{CD} \bar{\psi}_D \\
= e^{AB} (M^B_A)^\ast e_{CD} \bar{\psi}_D,
\]
where we define \[^{21}e^{AB} := e^{AB} \delta^B_B\] and similarly for $e_{CD}$. Numerically, we find the same matrix as given at the bottom of page 45. Performing the matrix multiplication, we find that 
\[
(e^{AB} (M^B_A)^\ast e_{CD}) = \begin{pmatrix} 0 & 1 & m_1^{1s} & m_1^{2s} \\ -1 & 0 & m_2^{1s} & m_2^{2s} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
= \begin{pmatrix} m_2^{2s} & -m_2^{1s} \\ -m_1^{2s} & m_1^{1s} \end{pmatrix},
\]
\[^{21}\text{Do not attempt to use these versions of } e \text{ with mixed indices to raise or lower indices!}\]
which is $M^{*-1\top}$ (according to the standard definition), where we have used the fact that $M \in SL(2,\mathbb{C})$, so has determinant of 1.

**p46, figure 1.2:** The change to equation (1.63) means that figure 1.2 requires a number of changes.

**p46→p47:** There is no longer any need for unconventional definitions of $M^*$ or $M^{*-1\top}$. Much of p47 can be omitted.

**p47, l-9:** I am not convinced that the derivation of $M^*$ here shows that complex conjugation cannot be described as a linear map. However, I have already performed the explicit calculation suggested — see my notes for page 34.

**p49, bottom half:** The exposition is somewhat odd here. It gives the impression that we might be about to prove that the components must be Grassmann. All we are actually doing is noting that, with non-Grassmann components, $(\psi\psi)$ is zero and, since we do not want this to be zero, we postulate Grassmann components.

(Note that $(\psi\psi)^2$ is always zero (see p115), i.e. $(\psi\psi)$ is not an ordinary number when $\psi$ is Grassmann.)

**p49, bottom half:** (So why do we want $(\psi\psi) \neq 0$?)

**p50, l3:** After fixing the index placement, the second expression should read $\chi_B^\dagger \delta^C_B \psi_C$.

**p50, l11:** Includes a step omitted from line 3, that is $(\epsilon^\top)^B_A = -\epsilon^{BA}$.

**p50, l14:** It would be better if this read “if $\theta$ is a Grassmann spinor”.

**p50, eq. 1.99:** Note that $\theta^1 \theta^2 = \theta_1 \theta_2$ and $\bar{\theta}_1 \bar{\theta}_2 = \bar{\theta}_1^\dagger \bar{\theta}_2^\dagger$.

**p51, l3→6:** Each of these is actually (1.99) in disguise. The sign difference between (1.100a) and (1.100b) looks odd at first — after all, are we not just lowering indices? However, we do not lower indices on $\epsilon^{AB}$ by using $\epsilon$-matrices. For example

$$\epsilon_{CA} \epsilon_{DB} \epsilon^{AB} = -\epsilon_{CA} \epsilon_{DB} \epsilon^{BA} = -\epsilon_{CA} \delta_D^A = -\epsilon_{CD} \neq \epsilon_{CD}.$$  

**p51, l8→10:** These are the standard identities for the two dimensional Levi-Civita symbol in disguise. Recall that, for the 2d Levi-Civita symbol, we have

$$\epsilon_{ab} \epsilon_{cd} = \delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}.$$  

For the $\epsilon$ matrices, the sign is reversed since $\epsilon^{AB} = -\epsilon_{AB}$.

**p51, l15:** This should read

$$-\frac{1}{2} (\theta^B \theta^A - \theta^A \theta^B) = \theta^A \theta^B.$$  

**p52, l1:** I.e.

$$(\theta \theta) \psi_B = \theta^A \theta_A \psi_B \quad \text{Here...} - \theta^A \psi_B \theta_A \quad \text{...and here.} \quad \psi_B \theta^A \theta_A = \psi_B (\theta \theta).$$
As \(\sigma^i\) have been defined with indices up, it would be better to give (1.104b) also with indices up, particularly because, while the raising and lowering of the index has not yet been discussed, it will turn out that we do this using the metric \(\eta_{\mu\nu}\). So, (1.104b) becomes
\[
[\sigma^i, \sigma^j] = 2i\epsilon^{ij}_k \sigma^k. \tag{2}
\]

As it happens, the only one of these equations that appears to be, technically, wrong, is (1.104b), according to the conventions that we use. For if we start with (2), with \(\epsilon_{ijk} = \epsilon_{i}^{jk} = \epsilon_{ijk}\) etc., then we get
\[
[\sigma_i, \sigma_j] = [-\sigma^i, -\sigma^j] = [\sigma^i, \sigma^j] = 2i\epsilon^{ij}_k \sigma^k = 2i\epsilon_{ijk} \sigma^k = -2i\epsilon_{ijk} \sigma^k.
\]

See my notes on pages 57–58 for whether \(\sigma^\mu\) is in the adjoint representation and whether we must have this index structure.

Note that the order of the indices on \(\sigma^\mu\) are mysteriously reversed. However, this is correct and matches other sources. Also, a bar appears on \(\sigma\). This indicates that the \(\epsilon\)-matrices do not perform index raising/lowering on \(\sigma\)-matrices in the usual way.\(^{22}\)

It is unclear to me here what is meant by “another formulation” or why we should be particularly interested in \(\sigma^{\mu\top}\). ((1.107) is consistent with (1.106a).)

We can express the right hand side of line 8 as a matrix product,
\[-\epsilon^{AB} (\sigma^\mu)_{\dot{B}B} \epsilon_{\dot{B}A} = -\epsilon \sigma^\mu \epsilon.\]

However, note that this expression, overall, has two upper indices in the order \(A\dot{A}\), while the left hand side of line 8 has the indices in the order \(\dot{A}A\). So we may perform the matrix arithmetic simply, but must remember to take the transpose at the end\(^{23}\), i.e.
\[
\sigma^{\mu\top} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}
\]

so that
\[
\sigma^{\mu} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}^\top = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.
\]

(The approach in the text is probably less prone to error, but is perhaps overly fussy, even for me.)

A complex linear combination — we have four complex dimensions.

What is referred to here as the ‘completeness relation’ is otherwise known as ‘resolving the identity’ and it means something like the relation
\[
\sum_k |k\rangle \langle k| = 1
\]

\(^{22}\) Indeed, we will have little use for raising/lowering the spinor indices on these objects. It is better to simply use \(\sigma^\mu\) and \(\sigma^\mu\) with spinorial indices only in their natural positions.

\(^{23}\) Of course, in the case of \(\sigma^1\), the result is symmetric, but taking the transpose is important for the case of \(\sigma^2\).
from basic quantum mechanics, which allows us to write any state as a linear combination of $|k\rangle$, i.e.

$$|\psi\rangle = \sum_k \langle k|\psi\rangle |k\rangle.$$ 

If we multiply (1.110) on the right by $A_{BB}$ then we get

$$A_{AA} = \left(\frac{1}{2} (\sigma_\mu)^{\mu B} A_{BB}\right) \sigma_\mu^{AB}.$$

Here $\sigma_\mu^{AB}$ takes the role of the basis vector $|k\rangle$ while $\frac{1}{2} (\sigma_\mu)^{\mu B} A_{BB}$ takes the role of the coefficient $\langle k|\psi\rangle$.

p55, eq. 1.110: This is the first place in the book where we have $\sigma$ with the $\mu$ index down. It is fairly obvious what is meant, i.e. $\sigma_\mu = \eta_{\mu\nu} \sigma^\nu$.

p56, l-11: Trivially checked.

p56, l-2: 

$$x_\mu \sigma^\mu = \left( \begin{array}{cc} x^0 - x^3 & -x^1 + ix^2 \\ -x^1 - ix^2 & x^0 + x^3 \end{array} \right).$$

p57, l-10: The matrix needs correcting to that given above. Note that we do still get $\det X = x^2$ though, where $x^2$ denotes $x$ squared, not the second component of $x$.

p57, l-5: The Lie algebra associated with $SL(2,\mathbb{C})$ is the vector space of traceless matrices, not $\mathbb{H}(2,\mathbb{C})$. Hence this is not the adjoint representation as I am familiar with it from lectures.

p58, l-2: Note that, if $M$ is unitary then we do get the adjoint action $X \to MXM^{-1}$. What we seem to have is the adjoint representation of $SU(2)$, but extended to cover $SL(2,\mathbb{C})$.

One may ask, since the Lie algebra for $SL(2,\mathbb{C})$ is the space of traceless matrices, whether tracelessness is an invariant under this action. If $M \in SU(2)$ then tracelessness is preserved$^{24}$, but this is not true for all $M \in SL(2,\mathbb{C})$. For example, consider

$$M = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \in SL(2,\mathbb{C}), \quad X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathbb{H}(2,\mathbb{C}),$$

with $\text{Tr} X = 0$. Then

$$MXM^\dagger = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 4 & 5 \end{pmatrix},$$

which is certainly not traceless.

p58, l-9: When $\sigma$-matrices were introduced, they were just matrices. We now consider them as belonging to a particular representation of $SL(2,\mathbb{C})$ and it is this consideration that leads to the index structure. It is then the link to $L^\dagger_\mu$ that reveals that considering these matrices in this way is a useful thing to do and that this index structure is useful.

$^{24}$ Indeed, trace is conserved.
p58, l-13: Replace "the matrices $\sigma^\mu$ therefore map $\tilde{F}$ into $F$, and similarly the matrices $\overline{\sigma}^\mu$ map $F$ onto $\tilde{F}$."

p58, l-10: Using equations (1.113) and (1.116).

p61, l4: Uses the fact that $SL(2, \mathbb{C})$ is connected.

p61, footnote: If $M$ were Hermitian, then it would be unitarily diagonalizable. However, we only know that $M \in SL(2, \mathbb{C})$ and some such $M$ are not diagonalizable at all.

p61, l10 $\rightarrow$ p62, l15: This entire proof needs to be replaced, since it is not necessarily the case that $M$ is unitarily diagonalizable. For such a replacement, see my sister document [2].

p63, l4: Explicitly verified!

p64, l1: Typo: Replace 'restriced' with 'restricted'.

p64, l4$\rightarrow$5: Apparently, the universal cover is a cover that is simply connected. A universal cover also covers any connected cover. Intuitively, $SL(2, \mathbb{C})$ seems simply connected and hence qualifies, reversing the logic of these lines.

p64, l4$\rightarrow$5: Why is this information of particular interest? If it is important then I should fill in the details.

p64, l7: The second $\mu$ index should be down.

p64, l8$\rightarrow$1: Easily checked.

p65, l16: Typo: Replace 'identiy' with 'identity'.

p65, l13: An explicit definition of Hermitian conjugation as it applies to spinors might be beneficial.

p65, l13: Should the components of spinors be considered as operators? Perhaps we should have been dealing with Hermitian conjugation, rather than complex conjugation, all along, i.e. $\langle A^B \psi \rangle = \langle \psi | A \rangle$.

p65, l12$\rightarrow$1: It is not necessary to use Hermitian conjugation here — using the simple complex conjugate works fine. However, it is important to remember that the complex conjugate of the product, $a \beta$, of Grassmann numbers is $(a \beta)^* = \beta^* a^*$, even if $a$ and $\beta$ are both real. Replacing the Hermitian conjugation in (1.135) with complex conjugation, the proof goes as follows. First we note that

$$\langle \phi \overline{\sigma}^\mu \overline{x} \rangle^* = (\phi^A \overline{\sigma}^\mu_{AB} \overline{x}^B)^* = \overline{x}^B \overline{\sigma}^\mu_{AB} \phi^A,$$

using the property mentioned above. Then we simply note that $\overline{x}^B = \lambda^B$, $\phi^A = \overline{\phi}^A$, and that the complex conjugate of Hermitian matrix $\sigma^\mu$ is simply its transpose, to get:

$$\langle \phi \overline{\sigma}^\mu \overline{x} \rangle^* = \lambda^B \overline{\sigma}^\mu_{BA} \overline{\phi}^A = (\lambda \overline{\sigma}^\mu \phi).$$

Similarly,

$$(\theta \phi)^* = (\theta^A \phi_A)^* = \phi_A^* \theta^A = \overline{\phi}_A \overline{\theta}^A = (\overline{\phi} \overline{\theta}),$$

using (1.98).

---

35 After some thought, I suspect that the Hermitian conjugation given here can probably be replaced with complex conjugation — see below.

36 If $a$ and $\beta$ are both real (but Grassmann), then this leads to the rather odd result that $(a \beta)^* = -a \beta$, much like an imaginary (but non-Grassmann) number.

37 If we want to be fussy about index notation, we use $\lambda^B = \lambda^B$, $\phi^A = \delta^A_A \tilde{\phi}$ and $\sigma^{\mu*}_{AB} = \delta^A_A \sigma^{\mu}_{BC} \delta^B_B$ to get

$$\langle \phi \overline{\sigma}^\mu \overline{x} \rangle^* = \lambda^B \overline{\delta}^B_B \delta^A_A \sigma^{\mu}_{BC} \delta^B_B \delta^A_A \overline{\phi}^A = (\lambda \overline{\sigma}^\mu \phi).$$
p66, l2: “As might be expected from the index structure.”

p66, l-9: The proof of the first part of proposition 1.30 is missing. This is just as well, since the first part is only true if \( \mu = \nu \). To show this, one may work through the calculations using 

\[
\theta_A = \begin{pmatrix} a \\ b \end{pmatrix}, \quad \bar{\theta}_A = \begin{pmatrix} a^* & b^* \end{pmatrix}, \quad \theta^A = \begin{pmatrix} b, -a \end{pmatrix}, \quad \bar{\theta}^A = \begin{pmatrix} b^*, -a^* \end{pmatrix},
\]

for different values of \( \mu \) and \( \nu \). Perhaps more simply, we may note that the second part of proposition 1.30 (which is correct) implies that \( (\theta^\mu)^{A\bar{A}} \bar{\theta}^A \) is not identically zero. But then, for the first part of the proposition to be true, we must get \( (\sigma^\mu)_{\bar{A}A} \bar{\sigma}^A \) zero whenever \( \mu \neq \nu \). This is clearly nonsense.

p66, l-3: Using \((1.100a)\) and \((1.100b)\).

p67, l-9: Awkward notation. Note that the result is a \( 2 \times 2 \) matrix

p67, l-1: Physicist’s convention again.

p68, l2→4: \( M \) is doing double duty here, as both \( M(\Lambda) \in SL(2, \mathbb{C}) \) and as \( M^{\mu\nu} \), the generators of the Lorentz group self-representation.

p68, l4: And note that \( \frac{1}{4!} \sigma^{\mu_1 \cdots \mu_4} = \mathbb{1} \).

p68, l9→10: “Trivial self-representation” is a troubling phrase, since there is such a thing as the trivial representation, which is not the self-representation.

p68, l7→13: This is just a recapitulation of stuff from section 1.1. In particular, line 11 does not come from line 6.

p68, l-10→8: I have not yet confirmed these statements.

p68, l-6: \( J_i \) and \( \bar{K}_i \) are generators in this ‘adjoint’ representation that correspond with the \( J_i \) and \( K_i \) from the self-representation.

p68, l-5: \( J \) and \( K \) are defined on page 13, but in terms of \( M_l \). To get them in terms of \( M^{\mu\nu} \) we need to use the equations just before equation (1.12).

p68, l-4: Just before (1.12) on page 14 we have \( K_i = -M_{0i} \).

p68, l-1: I therefore get

\[
\bar{\sigma}^i = \frac{i}{2} = \sigma^i \sigma^0.
\]

p68, l-2→1: We keep the \( \sigma^0 \) here to keep the spinor indices straight, i.e. to get generators with 2 undotted indices, down then up. (After all, numerically, \( \sigma^0 \) is just the identity.)

p69, l2→5: We are combining things that typically have Greek indices taking values from 0 to 3 (i.e. \( \epsilon^{\mu\nu\rho\sigma}, M^{\mu\nu}, \sigma^{\mu\nu}, \sigma^\mu \)) with things that have Latin indices taking values from 1 to 3 (\( \epsilon_{ijk} \)). The former have indices that are raised or lowered using the metric \( \eta_{\mu\nu} \), while the latter are usually written with indices down, and if they are raised, no change is made to the value. Mixing these two types of object is potentially confusing and risky, especially if we start raising and lowering indices on \( \epsilon \) and particularly because we are using a \(+---\) metric signature.
p69, l5: This suggests that index position (up or down) does not matter on $\epsilon^{ijk}$. Contrast this with how $\epsilon ^ {\mu \nu \rho \sigma}$ is treated (see note for pages 72–74) and $e^{AB}$.

p69, l10: Not quite the Hodge dual as I know it, due to the $\frac{1}{2}$. However, this does seem to be just what is needed to make $(\ast)^2 = 1 \ast\sigma$ for 2-forms, assuming index position on $\epsilon ^ {\mu \nu \rho \sigma}$ matters. Taking the dual of (1.140), we get

$$\frac{1}{2!} \epsilon _ {\lambda \mu \nu } e ^ {\mu \nu } = - \frac{1}{4} \epsilon _ {\lambda \mu \nu } \epsilon ^ {\mu \nu \rho \sigma } \sigma _ {\rho \sigma}.$$  

Now using $\epsilon _ {\lambda \mu \nu } e ^ {\mu \nu \rho \sigma } = -2(\delta _ {\mu } ^ {\beta \mu } - \delta _ {\nu } ^ {\beta \nu })$, this becomes

$$\frac{1}{2!} \epsilon _ {\lambda \mu \nu } \sigma ^ {\mu \nu } = \frac{1}{2} (\delta _ {\nu } ^ {\beta \nu } - \delta _ {\nu } ^ {\beta \nu }) \sigma _ {\rho \sigma} = \frac{1}{2} (\sigma _ {\lambda \beta } - \sigma _ {\lambda \nu }) = \sigma _ {\lambda \nu},$$

where we use the antisymmetry of $\sigma _ {\lambda \nu}$.

p69, 1-3→2: Note that $\sigma _ {j} = - \sigma ^ {j}$ (lowering a spacelike index). Of course

$$[\sigma _ {j} , \sigma _ {k}] = [ - \sigma ^ {j} , - \sigma ^ {k}] = [\sigma ^ {j} , \sigma ^ {k}] = 2i e ^ {ij} \sigma _ {k} = -2i e ^ {ik} \sigma _ {j}.$$  

p69, l2→1: Note that this calculation is only correct if $e ^ {\mu \nu \rho \sigma}$ is defined with $e ^ {0123} = +1$ and not $e _ {0123} = +1$.

p70, l3: Using the convention that the position of indices (up or down) on $\epsilon _ {ijk}$ does not affect the value, we have

$$[\sigma _ {j} , \sigma _ {k}] = 2i e ^ {ij} \sigma _ {k} = -2i e ^ {ik} \sigma _ {j}.$$  

So we get

$$\sigma ^ {ij} = - \frac{i}{4} [\sigma _ {j} , \sigma _ {j}] = - \frac{i}{4} (-2i e ^ {ij} \sigma _ {k}) = - \frac{1}{2} i e ^ {ij} \sigma _ {k}.$$  

p70, l6: Using $e ^ {0123} = +1$, we have $e ^ {0ij} = \epsilon _ {ijk}$ and $e ^ {ij0} = - \epsilon _ {ijk}$. We then get

$$\frac{1}{2} \epsilon ^ {ijk} \sigma _ {k} = - \frac{1}{2} \epsilon ^ {ijk} \sigma _ {k} = \sigma ^ {ij}$$  

and the proof works.

p70, l13→14: As on page 69, this only works if $e ^ {0123} = +1$.

p70, l3: As on line 3, there is a sign error. Again, we should have

$$\sigma ^ {ij} = - \frac{i}{4} [\sigma _ {j} , \sigma _ {j}] = - \frac{i}{4} (-2i e ^ {ij} \sigma _ {k}) = - \frac{1}{2} i e ^ {ij} \sigma _ {k}.$$  

p70, l1: As on line 6, we get

$$\frac{1}{2} \epsilon ^ {ijk} \sigma _ {k} = - \frac{1}{2} \epsilon ^ {ijk} \sigma _ {k} = \sigma ^ {ij}.$$

p71, l4: We have already proven (1.141a) — it was (1.109b) in proposition 1.20, page 54.
**pp72→76:** Proposition 1.35 depends on proposition 1.36, which in turn depends on propositions 1.37 and 1.38. So these pages are, in effect, one long and rather dry proof.

**p72, eqs. 1.144, 1.145:** Alas, using $\epsilon^{0123} = +1$ means that we must change the signs of the last terms to get

$$
\text{Tr}[\sigma^{\mu\nu} \sigma^{\rho\sigma}] = \frac{1}{2}(\eta^{\mu\rho} \eta^{\nu\sigma} - \eta^{\mu\nu} \eta^{\rho\sigma}) - \frac{i}{2} \epsilon^{\mu\rho\nu\sigma},
$$

$$
\text{Tr}[\sigma^{\mu\nu} \sigma^{\rho\sigma}] = \frac{1}{2}(\eta^{\mu\rho} \eta^{\nu\sigma} - \eta^{\mu\nu} \eta^{\rho\sigma}) + \frac{i}{2} \epsilon^{\mu\rho\nu\sigma}.
$$

**p72, l13→21:** Change the sign of every term involving $\epsilon^{\alpha\beta\gamma\delta}$.

**p73, eq. 1.146:** Again, using $\epsilon^{0123} = +1$ means that we must change the sign of the last term to get

$$
\text{Tr}[\sigma^{\mu} \sigma^{\nu} \sigma^{\rho} \sigma^{\sigma}] = \frac{1}{2} (\eta^{\mu\nu} \eta^{\rho\sigma} + \eta^{\mu\rho} \eta^{\nu\sigma} - \eta^{\mu\nu} \eta^{\rho\sigma} + i \epsilon^{\mu\nu\rho\sigma}).
$$

**p76, l7:** Using $\epsilon^{0123} = +1$ we find that $c = i$.

**p76, l-7:** Using $\epsilon^{0123} = +1$, we find that this expression is $-24i$ and so $c = i$.

**p77, l1:** Note that $\epsilon$ is shorthand for $\epsilon_{AB}$ here, i.e. $\epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

**p77, l3:** Not proved in the book.

We want to show that $(\epsilon \sigma^{\mu\nu})^\top = \epsilon \sigma^{\mu\nu}$. Well,

$$(\epsilon \sigma^{\mu\nu})^\top = \sigma^{\mu\nu} \epsilon^\top = \frac{i}{4} (\sigma^\mu \sigma^\nu - \sigma^\nu \sigma^\mu)^\top \epsilon^\top.$$

From (1.107) we have

$$\sigma^\mu = \epsilon \sigma^\mu \epsilon^\top = \epsilon^\top \sigma^\mu \epsilon,$$

since $\epsilon^\top = -\epsilon$. Therefore

$$(\epsilon \sigma^{\mu\nu})^\top = \frac{i}{4} (\epsilon \sigma^\mu \epsilon^\top \sigma^\nu - \epsilon \sigma^\nu \epsilon^\top \sigma^\mu)^\top \epsilon^\top = \frac{i}{4} \epsilon (\epsilon^\top \sigma^\nu \epsilon \sigma^\mu - \epsilon^\top \sigma^\mu \epsilon \sigma^\nu) e^\top = \frac{i}{4} \epsilon (\overline{\sigma^\mu} \sigma^\nu - \sigma^\nu \sigma^\mu) = \epsilon \sigma^{\mu\nu}.$$

**p77, l8:** (1.107) in matrix notation is $\sigma^{\mu\nu} = \epsilon \sigma^{\mu\nu} \epsilon^\top$ (via $-\epsilon \sigma^{\mu\nu} \epsilon$). We use $\sigma^\mu = \epsilon \sigma^\mu \epsilon^\top$, which is simply the transpose.

**p78, l5:** The proof that $\frac{1}{2}(\psi \sigma_\mu \overline{\chi})$ gives the components of a vector is in proposition (1.29), not (1.33).

**p79, l10:** I need a better understanding of the notation used in proposition 1.41.
We cannot use equation (1.100b), that is
\[ \theta_A \theta_B = \frac{1}{2} \epsilon_{AB}(\theta \theta), \]
here, since this only involves only one spinor.

An alternative proof is as follows. First note that \( \frac{1}{2}(\psi_A \chi_B - \psi_B \chi_A) \) is clearly antisymmetric in \( A \) and \( B \) and hence must be a multiple of \( \epsilon_{AB} \), i.e.
\[ \frac{1}{2}(\psi_A \chi_B - \psi_B \chi_A) = c \epsilon_{AB}. \]

Multiply both sides by \( \epsilon^{BC} \) to get
\[ c \delta_A^C = \frac{1}{2}(\psi_A \chi_B - \psi_B \chi_A) \epsilon^{BC} = -\frac{1}{2}(\psi_A \chi_C - \psi_C \chi_A). \]

Take the trace to get
\[ 2c = \frac{1}{2}(\psi_A \chi_A - \psi_A \chi_A) = \frac{1}{2}(\psi_A \chi_A + \chi_A \psi_A) = \frac{1}{2}(\psi \chi - \chi \psi) = (\psi \chi). \]

Hence \( c = \frac{1}{2}(\psi \chi) \) as required.

Terms are combined via a simple relabelling of dummy indices, e.g.
\[ (\sigma^\mu \bar{\sigma}_\mu)^B_A (\sigma^\nu \bar{\sigma}_\nu)^D_C = (\sigma^\mu \bar{\sigma}_\mu)^B_A (\sigma^\nu \bar{\sigma}_\nu)^C_D, \]
switching the dummy indices \( \mu \) and \( \nu \).

I have only skimmed this page.

(1.152g) appears to have a sign error, regardless of our conventions regarding \( \epsilon \)-matrices and (1.63). For
\[ \Phi_{AB} - \Phi_{BA} = c \epsilon_{AB} \]
implies that
\[ \epsilon^B_A (\Phi_{AB} - \Phi_{BA}) = 2c, \]
from which we get
\[ c = \frac{1}{2} (\Phi_B^B - \Phi_B^B) = \Phi_B^B = -\Phi_B^B. \]

Hence
\[ \Phi_{AB} - \Phi_{BA} = \Phi_C^C \epsilon_{AB}. \]

Similarly
\[ \Phi^{AB} - \Phi^{BA} = c \epsilon^{AB} \]
implies that
\[ \epsilon_{BA} (\Phi^{AB} - \Phi^{BA}) = 2c, \]
from which we get
\[ c = \frac{1}{2} (\Phi_B^B - \Phi_B^B) = \Phi_B^B = -\Phi_B^B. \]

Hence
\[ \Phi^{AB} - \Phi^{BA} = -\Phi_C^C \epsilon^{AB}. \]
p84: Replace all occurrences of $M^{*-1}$ with $M^{*-1\top}$. Note that $M^{*-1}$ does not even have the correct index structure:  

$$M_A^B \to (M^*)_A^B \to (M^{*-1})_A^B \to (M^{*-1\top})_A^B.$$ 

p85, l14: I do not see how this “demonstrates explicitly the irreducibility of the representation space under the parity transformation”. Furthermore, I do not believe that this is true. If $\psi_B$ and $\psi^b$ are numerically equal then they remain equal under the parity transformation, giving us an invariant subspace.

It is probably true that, under the combination of parity with the Lorentz transformations, we have an irreducible representation. However, line 14 does not demonstrate this.

p85, l15–17: I would be tempted to reverse this, to say that if our theory also has parity as a symmetry, then it will be convenient to use Dirac spinors. If not, then we must (either explicitly or via the use of projection operators such as $\frac{1}{2}(1 + \gamma^5)$) use the Weyl spinors.  

p85, l8: The sections that follow seem to assume some prior knowledge of Dirac spinors.

p87, l4–2: Here, ‘Hermitian conjugate’ seems to be taken very much in the ‘matrix’ sense, i.e. complex conjugate transpose.

p88, l9: This second verification of (1.164) seems almost designed to cause confusion, perhaps due to its use of a mixture of matrix and index formalisms. It is best skipped.  

p89, l8–9: These essentially just say that, numerically,  

$$(\sigma^H)^{AD} = (\sigma^H)_{AD}$$

and  

$$(\tau^H)_{AD} = (\tau^H)^{AD}.$$ 

In the end, these are merely used as a stepping stone to the use of (1.108) to get  

$$(\bar{\sigma}^0)_{AB} = (\sigma^0)_{AB},$$

$$(\bar{\sigma}^1)_{AB} = -(\sigma^1)_{AB}.$$ 

However, I always read (1.108) as  

$$(\bar{\sigma}^0)^{AB} = (\sigma^0)_{AB}$$

and  

$$(\bar{\sigma}^1)^{AB} = -(\sigma^1)_{AB}$$

anyway, making these new objects redundant.  

p91, l1: (1.169) is hardly more compact than  

$$\sigma^\mu \partial_\mu \bar{\psi} = 0, \quad \bar{\sigma}^\mu \partial_\mu \phi = 0!$$

p92, l5: It is equation (1.49) that defines the helicity operator.

---

1 In earlier pages, $M^*$ was simply the complex conjugate, not the Hermitian conjugate, i.e. it did not include a transposition. I assume that the same should hold here. Also, $M^*$ is certainly not the weird version described on pages 45–46.

2 See the appendix of [2] for a more detailed discussion of how the matrix operations of complex conjugation, transposition and inversion affect the index structure.

3 I have a fairly low level of confidence in this statement. I ought to think on this a little more.

4 It also seems ‘empty’ — the new proof is neither more formal nor more explanatory.

5 In these expressions, A must match with $\Lambda$ and $D$ must match $\bar{D}$.

6 See sidenote -38- about this seeming ‘empty’. Also, as mentioned in sidenote -22-, it is probably best to only consider the $\sigma$-matrices with their spinor indices in their natural position and to ignore these new objects.
Moreover, it is worth adding some further explanation. We have\(^{41}\)
\[ \lambda = \frac{P J}{P_0}, \]
where \( J_i = \frac{1}{2} \epsilon_{ijk} M_{jk} \) (see (1.12)). But in the spinor representation, \( M_{jk} \) becomes \( \sigma_{jk} \) and we get
\[ J^i = \epsilon^i_{jk} \sigma^{jk} = \frac{1}{2} \sigma^i \sigma^0. \]
(See (1.139a).) Now note that numerically, \( \sigma^0 \) is just the identity, so we get
\[ J^i = \frac{1}{2} \sigma^i. \]
But now
\[ \lambda = \frac{P J}{P_0} \]
becomes
\[ \lambda = \frac{1}{2} \frac{p \sigma}{|p|} = \frac{1}{2} (\hat{p}, \sigma) = \frac{1}{2} (\sigma, \hat{p}). \]
Hence we are discussing the helicity operator in the spinor representation.

In the non-relativistic limit, Dirac’s equation,
\[ (i \gamma^\mu \partial_\mu - m) \psi = 0, \]
i.e.
\[ (i \gamma^0 \partial_0 + i \vec{\gamma} \vec{\nabla} - m) \psi = 0, \]
becomes
\[ (i \gamma^0 \partial_0 - m) \psi = 0, \]
i.e.
\[ i \partial_0 \psi = \gamma^0 m \psi. \]
Since \( i \partial_0 \) gives the energy and \( \gamma^0 \) is diagonal in the Dirac representation, we have diagonalized the energy.

At first, it looked to me as if the \( \alpha^i \) would be anti-Hermitian, being the product of a Hermitian matrix \( \gamma^0 \) and an anti-Hermitian matrix \( \gamma^i \). However, it does not work out that way. Indeed
\[ \alpha^i = \gamma^0 \gamma^i \]
leads to
\[ (\alpha^i)^\dagger = (\gamma^i)^\dagger (\gamma^0)^\dagger = -\gamma^i \gamma^0 = \gamma^0 \gamma^i = \alpha^i. \]

Here \( \gamma^0 \gamma^0 = 1 \) is inserted after \( \Psi^\dagger \), while the whole thing is multiplied by \( \gamma^0 \) on the right.

Set
\[ C_D = i \gamma_0^D \gamma_0^2 = -i \gamma^2_D \gamma_0^0 = i \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}. \]
This makes no difference to equations (1.189), (1.191) etc. It does, however, allow us to fix page 111 and still get
\[ (\Psi^W)_A = \begin{pmatrix} \phi_A \\ \bar{\phi}_A \end{pmatrix}. \]
rather than
\[(\Psi^c_W)_a = -\left(\begin{array}{c} \Phi_A \end{array}\right)\]

(See notes for page 111.)

\textbf{p106, l-1:} Remove the first minus sign from this line.

\textbf{p107, l9→10:} Use \(C_D = i\gamma^0_D \gamma^2_D\) (see note for page 102) to get
\[C_W = i\gamma^0_W \gamma^2_W = \left(\begin{array}{cc} -iv^2 & 0 \\ 0 & iv^2 \end{array}\right).\]

(This makes no difference to (1.194) etc., but fixes page 111.)

\textbf{p107, l13:} The first matrix should be \(\left(\begin{array}{cc} -1 & \sigma^0 \\ \sigma^0 & -1 \end{array}\right)\).

\textbf{p107, l13→15:} These lines now become
\[C_W = \frac{i}{2} \left(\begin{array}{cc} -1 & \sigma^0 \\ -\sigma^0 & -1 \end{array}\right) \left(\begin{array}{cc} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{array}\right) \left(\begin{array}{cc} -1 & -\sigma^0 \\ \sigma^0 & -1 \end{array}\right)\]
\[= \frac{i}{2} \left(\begin{array}{cc} -\sigma^2 \sigma^0 + \sigma^0 \sigma^2 & \sigma^2 + \sigma^0 \sigma^2 \sigma^0 \\ -\sigma^0 \sigma^2 \sigma^0 - \sigma^2 \sigma^0 \sigma^0 & \sigma^0 \sigma^2 - \sigma^2 \sigma^0 \sigma^0 \end{array}\right)\]
\[= \frac{i}{2} \left(\begin{array}{cc} -2\sigma^2 & 0 \\ 0 & 2\sigma^2 \end{array}\right) = \left(\begin{array}{cc} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{array}\right)\]

\textbf{p107, l7:} This is not the correct index structure the \(C_W\). The correct index structure is actually
\[\left(\begin{array}{cc} ()^{AB} & ()^A_B \\ ()^B_A & ()^{AB} \end{array}\right)\].

\textbf{p108, l6:} Note that this argument can be used to show that \(C = -C^{-1}\) in any representation. This will be useful when we look at problems with page 109. (The arguments for \(C^\top\) and \(C^\dagger\) do not generalize in this way, as they depend on properties of our particular \(X\) matrix.)

\textbf{p108, l8:} For (1.187) read (1.192). The reference to (1.182) is also wrong — perhaps this should be (1.173b).

\textbf{p109, l7:} The reference to (1.184) is wrong — perhaps it should be (1.189). However, see the next note.

\textbf{p109, l6→7:} (1.189) states that
\[C\gamma^\mu \gamma^\nu C^{-1} = -\gamma^\mu,\]
which gives
\[C^{-1}\gamma^\mu C = \gamma^\mu \gamma^\nu.\]

At this point one might be tempted to replace (1.196) with
\[C^{-1}\gamma^5 C = (\gamma^5)^\top\]
and (1.197) with
\[C^{-1}(\gamma^5 \gamma^\mu) C = (\gamma^5 \gamma^\mu)^\top.\]
However, our note for page 108 suggests that \( C = -C^{-1} \) in any representation. We therefore get
\[
C^{-1} \gamma^5 C = (-C) \gamma^5 (-C^{-1}) = C \gamma^5 C^{-1},
\]
etc., and all is well.

**p110, eq. 1.199:** Replace with something like
\[
\psi^A = \overline{\psi}^B \delta_A^B, \quad \phi^A = \overline{\phi}^B \delta_A^B.
\]
(Note that these are merely a repeat of formulae from pp. 42, 43.)

**p110, eq. 1.200:** This is not quite (1.63), even as given in the book. Having changed (1.63) anyway, we could replace with
\[
\Psi_A = \overline{\psi}^A,
\]
which is our new (1.63) or with
\[
\psi^A = \overline{\psi}^A,
\]
which is a more direct counterpart to equation (1.200) as given.

**p110, l15:** If \( \Psi_W = \begin{pmatrix} \phi_A \\ \overline{\phi}^A \end{pmatrix} \), then \( \Psi_W^\dagger \) is not equal to \( \begin{pmatrix} \phi^A \star & \overline{\psi}^A \end{pmatrix} \).
Instead, we get \( \Psi_W^\dagger = \begin{pmatrix} \phi_A^* & \overline{\psi}^A \end{pmatrix} \). Line 15 therefore becomes
\[
\Psi_W = \Psi_W^\dagger \gamma^0_W = \begin{pmatrix} \phi_A^* & \overline{\psi}^A \end{pmatrix} \begin{pmatrix} 0 & I_{2 \times 2} \\ I_{2 \times 2} & 0 \end{pmatrix} = \begin{pmatrix} \phi_A^* & \overline{\psi}^A \end{pmatrix} = \begin{pmatrix} \psi^A & \overline{\phi}^A \end{pmatrix}
\]
with the desired end result obtained legitimately.

**p110, eq. 1.201a:** Again, \( \Psi_W \) is not equal to \( \begin{pmatrix} \phi^A \star & \overline{\psi}^A \end{pmatrix} \). However, this equation is not so easy to fix. To get the index structure to work correctly, the sister document [2] suggests replacing \( \gamma^0 \) with the (numerically equal) matrix \( ^{44} \)
\[
\gamma^0_W = \begin{pmatrix} 0 & 0 \\ \delta_A^B & \delta_A^B \end{pmatrix}.
\]
Equation (1.201a) then becomes
\[
\Psi_W = \Psi_W^\dagger \gamma^0_W = \begin{pmatrix} \phi_A^* & \overline{\psi}^A \end{pmatrix} \begin{pmatrix} 0 & \delta_A^B \\ \delta_A^B & 0 \end{pmatrix} = \begin{pmatrix} \phi_A^* & \overline{\psi}^A \end{pmatrix} = \begin{pmatrix} \psi^A & \overline{\phi}^A \end{pmatrix}.
\]

**p110, eq. 1.201b:** Having found that \( \Psi_W = \begin{pmatrix} \psi^A & \overline{\phi}^A \end{pmatrix} \), we must have \( ^{44} \)
\[
\Psi_W^\dagger = \begin{pmatrix} \psi^A & \overline{\phi}^A \end{pmatrix}.
\]

\(^{44}\) The alternative suggestion of [2], that is, fixing the index structure of \( \gamma^0 \) via the insertion of an ‘identity matrix in disguise’ might be better, as this would fix the index structure in a representation independent way.

\(^{40}\) Alternatively, we can stop worrying so much about index structure.

\(^{44}\) If we insist that \( \psi^A \) and \( \overline{\phi}^A \) are rows, then we might prefer to write
\[
\Psi_W^\dagger = \begin{pmatrix} \psi^A & \overline{\phi}^A \end{pmatrix}.
\]
However, it is certainly not the case that \( \psi^A \) is equal to \( \overline{\phi}^A \). Transposition does not lower indices — at least not without the help of some \( \epsilon \)-matrices!
The matrix \((i\sigma^2\sigma^0)_A^B\) may be calculated directly, i.e.
\[
(i\sigma^2\sigma^0)_A^B = i(\sigma^2)_{AC}(\sigma^0)^C_B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^B_A
\]
and not \(\delta_A^B\). Similarly
\[
(i\sigma^2\sigma^0)^A_\dot{B} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^A_\dot{B}
\]
Since we have also changed the matrix \(C\), we correct the entire calculation to
\[
(\Psi^c_W)_a = (C_W)_{ab}(\Psi^T_W)_b = \begin{pmatrix} -i(\sigma^2)_{AB} & 0 \\ 0 & -(i\sigma^2)^{AB} \end{pmatrix} \begin{pmatrix} \psi^b_B \\ \phi^\dot{B} \end{pmatrix}.
\]
Numerically, \(-i(\sigma^2)_{AB}\) is just \(\epsilon_{AB}\), while \(-i(\sigma^2)^{AB}\) is just \(\epsilon^{A\dot{B}}\). We therefore get,
\[
(\Psi^c_W)_a = \begin{pmatrix} \psi_A \\ \phi^\dot{A} \end{pmatrix},
\]
as desired.

References
