Single Mathematics A

Basic course notes for term 1

2015/16

Some course information

These brief and basic notes are provided to help you produce a comprehensive and comprehensible set of notes of your own. They are not a replacement for textbooks which you will be expected to consult for further details as necessary: the University Library and many College libraries have copies of the books recommended for the course.

The notes are *not* replacements for your attendance at lectures. There will be material (mainly explanations and examples) covered in lectures which is not included in these notes.

These notes were originally written by John Bolton and Iain MacPhee; various changes and additions have been made by Patrick Dorey, Olaf Post and Wojtek Zakrzewski.

Homework and tutorials: problems will be set every week and you are required to attempt them and hand in their solutions. These will be assessed and returned at the weekly tutorials which you should attend. Further problems will be set for tutorials. Details of tutorial groups will be sorted out in the first week of Michaelmas term and made available on the noticeboards in the Mathematics building and on the course webpages. It is important that you attempt, at the very least, all the set questions. The main reason for this is that practice is necessary in order to master the mathematical techniques covered.

Web pages & DUO

Information about the course and additional material is also available through DUO and via the Maths Dept pages at

http://maths.dur.ac.uk/teaching/Auxiliary *(click on the lecturer's name).*

In particular, a record of the work set appears on the course webpage and, after you have handed in your attempts at the set questions, their solutions also appear there. This does mean, of course, that work handed in late without prior arrangement with the lecturer cannot be considered as counting towards the continuous assessment element of the module.

Chapter 1

Preliminaries

This part of the course is mostly to get you warmed up after the summer break.

1.1 Algebra

1.1.1 Algebraic manipulation

Example 1.1. Simplify $\dfrac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a-b}}$. Well, $\dfrac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a-b}} = \dfrac{\frac{b+a}{ab}}{\frac{1}{a-b}} = \dfrac{(a+b)(a-b)}{ab} = \dfrac{a^2-b^2}{ab}$.

Example 1.2. Simplify $\dfrac{\sqrt{3}-1}{4-2\sqrt{3}}$. Well, $\dfrac{\sqrt{3}-1}{4-2\sqrt{3}} = \dfrac{(\sqrt{3}-1)(4+2\sqrt{3})}{(4-2\sqrt{3})(4+2\sqrt{3})}$

$$= \frac{4\sqrt{3}-2\sqrt{3}-4+6}{16-12} = \frac{2+2\sqrt{3}}{4} = \frac{1}{2}(1+\sqrt{3}).$$

Example 1.3. Find the roots of $2x^2+5x+2$. Either use the quadratic formula or, much better, factorise as follows.

$$2x^2+5x+2 = (2x+1)(x+2),$$

so the roots are $x = -1/2$ and $x = -2$.

Example 1.4. Find the solutions of $4x^2+12x+7 = 0$. Either use the quadratic formula or, perhaps better, complete the square as follows.

$$0 = 4x^2+12x+7 = (2x+3)^2-2.$$

So, $2x+3 = \pm\sqrt{2}$, so that $x = \frac{1}{2}(-3\pm\sqrt{2})$.

1.1.2 The binomial theorem

The Binomial theorem is useful to expand things like

$$(a+b)^2 = a^2+2ab+b^2,$$

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3,$$

$$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4.$$

The **binomial coefficients** are given by $\dbinom{n}{k} = \dfrac{n!}{k!(n-k)!}$. They give the number of ways of choosing k objects from a collection of n objects, and are pronounced as 'n

choose k'. (Sometimes you will also see the notations ${}^nC_k$, ${}_nC_k$ or even $C(n,k)$ for these same numbers.) Binomial coefficients may be worked out using Pascal's triangle, the top part of which is shown on the right below. Each number in Pascal's triangle is obtained by adding the two numbers diagonally above it.

$$\begin{array}{ccccccccc} & & & & \binom{0}{0} & & & & \\ & & & \binom{1}{0} & & \binom{1}{1} & & & \\ & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\ & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\ \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \end{array} \quad = \quad \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

A quick look at the triangle suggests some simple properties of the binomial coefficients, all of which are easily checked from the initial definition:

$$\binom{n}{0} = \binom{n}{n} = 1\,; \quad \binom{n}{1} = \binom{n}{n-1} = n\,; \quad \binom{n}{k} = \binom{n}{n-k}\,.$$

We can also verify that the coefficients obey the basic 'additive' property of Pascal's triangle:

$$\begin{aligned} \binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} \\ &= \frac{n!}{(k-1)!(n-k)!}\left[\frac{1}{n-k+1} + \frac{1}{k}\right] \\ &= \frac{n!}{(k-1)!(n-k)!}\left[\frac{k+(n-k+1)}{(n-k+1)k}\right] = \frac{(n+1)!}{k!(n-k+1)!} = \binom{n+1}{k}\,. \end{aligned}$$

The name given to the coefficients themselves comes from the following theorem.

Theorem 1.1 (The binomial theorem)**.**

$$\begin{aligned} (a+b)^n &= a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots \binom{n}{k}a^{n-k}b^k + \ldots + b^n \\ &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k\,. \end{aligned}$$

Example 1.5. Express $(1-x)^5 + (1+x)^5$ as a polynomial in x. Well,

$$\begin{aligned} (1-x)^5 + (1+x)^5 &= (1-5x+10x^2-10x^3+5x^4-x^5) + (1+5x+10x^2+10x^3+5x^4+x^5) \\ &= 2(1+10x^2+5x^4). \end{aligned}$$

WHY is theorem 1.1 true? A quick argument is to observe that $(a+b)^n = (a+b)\dots(a+b)$, and the coefficient of $a^{n-k}b^k$ is the number of ways of choosing b from precisely k of the n brackets.

Alternatively, the result can be built up starting from the simplest case. First we note that the theorem clearly holds for $n = 1$. Now assume that it is true for n. Then

$$\begin{aligned}
(a+b)^{n+1} &= (a+b)(a+b)^n \\
&= (a+b)\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k \\
&\qquad \text{(using the assumption that the result is true for } n\text{)} \\
&= \sum_{k=0}^{n}\binom{n}{k}a^{n-k+1}b^k + \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{k+1}
\end{aligned}$$

Now let's rewrite the second sum by substituting $j = k+1$, adjusting the upper and lower limits of the sum to make sure the same set of terms is counted:

$$\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{k+1} = \sum_{j=1}^{n+1}\binom{n}{j-1}a^{n-j+1}b^j$$

(this substitution is very similar to the substitution rule for integrals). Hence

$$\begin{aligned}
(a+b)^{n+1} &= \sum_{k=0}^{n}\binom{n}{k}a^{n-k+1}b^k + \sum_{j=1}^{n+1}\binom{n}{j-1}a^{n-j+1}b^j \\
&= a^{n+1} + \sum_{k=1}^{n}\binom{n}{k}a^{n-k+1}b^k + \sum_{j=1}^{n}\binom{n}{j-1}a^{n-j+1}b^j + b^{n+1} \\
&= a^{n+1} + \sum_{k=1}^{n}\left[\binom{n}{k} + \binom{n}{k-1}\right]a^{n-k+1}b^k + b^{n+1} \\
&= a^{n+1} + \sum_{k=1}^{n}\binom{n+1}{k}a^{n-k+1}b^k + b^{n+1} \\
&= \sum_{k=0}^{n+1}\binom{n+1}{k}a^{n-k+1}b^k.
\end{aligned}$$

To get from the first line to the second I separated off the first term of the first sum, and the last term of the second; to get from the second to the third line, I just rename j into k, and to get from the third line to the fourth, I used the 'additive' property of binomial coefficients that was proved on the last page. For the last equality, note that $k = 0$ and $k = n+1$ corresponds to the remaining terms a^{n+1} and b^{n+1}. It is now easy to see that this is exactly the binomial theorem for $n+1$.

Since we already decided that the theorem holds for $n = 1$, we now have it for $n = 2$, and then for $n = 3, 4, 5$, and so on to *all* positive integers, and the result has been proved.

1.1.3 Proof by induction

The argument just given might seem something of a sledgehammer to crack a nut, but it is a nice example of a general method known as **mathematical induction**. Suppose we want to prove some result is true for all positive integers n. If it can be shown that the result holds for $n = 1$, and also that, for all $N \geq 1$, *if* the result is true for $n = N$, then it must also be true for $n = N + 1$, then it follows that the result is true for *all* positive n. (Think of a line of dominoes falling over one after the other.)

The example in the last subsection was quite elaborate (and more complicated than you'd see in an exam), so here's a simpler one to show how the method works:

Example 1.6. Prove that $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$.

Proof. Let's prove this by induction. For $n = 1$ the statement is $\sum_{k=1}^{1} k = 1 = \frac{1}{2}1(1+1)$, which is clearly true. Now suppose that the statement holds for $n = N$. Then for $n = N + 1$,

$$\begin{aligned}
\sum_{k=1}^{N+1} k &= \sum_{k=1}^{N} k + (N+1) \\
&= \frac{1}{2}N(N+1) + (N+1) \\
&= (N+1)(\frac{1}{2}N+1) = \frac{1}{2}(N+1)(N+2)
\end{aligned}$$

which is exactly the statement we're after for $n = N + 1$. Hence, by induction, the result holds for all positive integers n. ∎

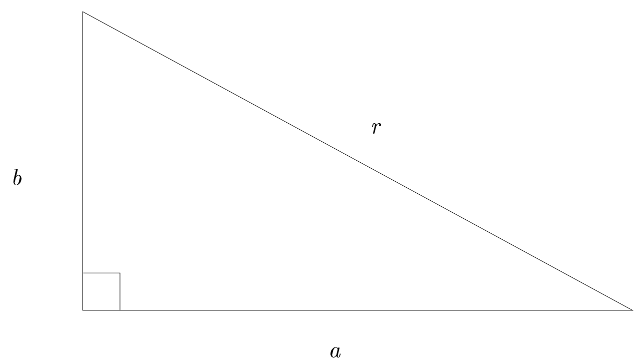
You can find many other examples of induction in introductory textbooks. In chapter 4 we'll see another method of proof which is also very useful: proof by contradiction.

1.2 Trigonometry

1.2.1 Pythagoras's Theorem.

For a right angled triangle as shown

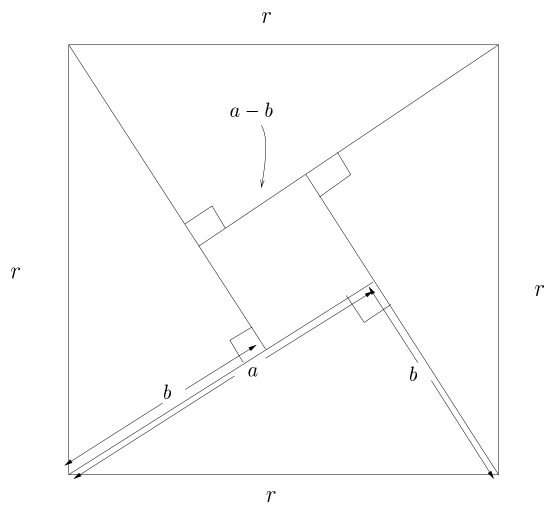
$$a^2 + b^2 = r^2.$$



To see this arrange four copies of the triangle inside a square as shown. Then the area of the big square is equal to the area of the small square plus four times the area of the triangle. Recalling that the area of the triangle is half the base times the perp height, and noting that the small square has side length $a - b$, we get

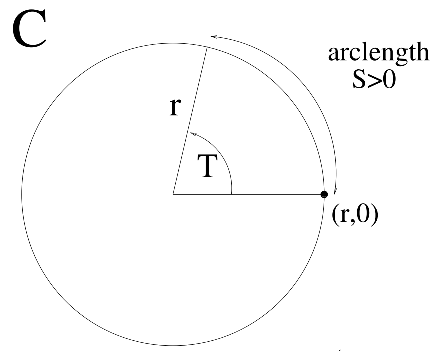
$$r^2 = (a-b)^2 + 4(\frac{1}{2}ab).$$

The right hand side of the above quickly simplifies to $a^2 + b^2$, which gives the result.



1.2.2 Trigonometric functions

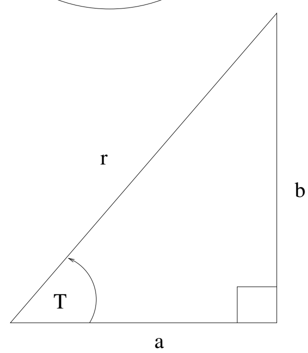
Let C be the circle centre 0 radius r in the plane. This has equation $x^2 + y^2 = r^2$. Then the angle $\theta = s/r$ radians, where we have travelled from $(r, 0)$ anticlockwise along C for an arclength distance $s > 0$. For $s < 0$ travel from $(r, 0)$ clockwise along C for a distance $-s$.
The circumference of the circle is $2\pi r$; so one complete revolution is angle $\theta = 2\pi$ radians.



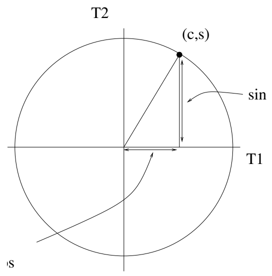
Now drop a perpendicular and consider the right angled triangle shown. We put

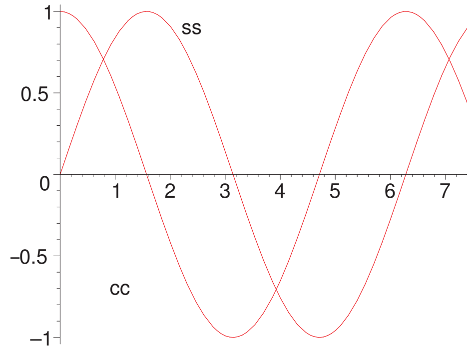
$$\cos\theta = a/r \text{ and } \sin\theta = b/r.$$

NB $r > 0$, but a, b are allowed to be positive or negative.



Graphs Since the definitions of sin and cos are ratios, they are independent of the size of the circle. So we choose a circle of radius $r = 1$.



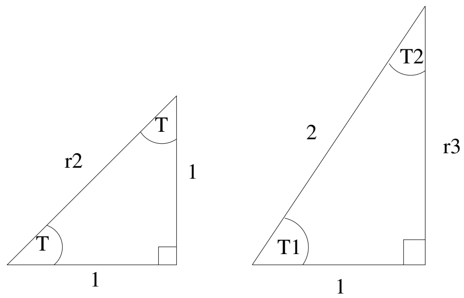


The trigonometric functions $\sin\theta$ and $\cos\theta$ are important for waves or periodic behaviour.

Easily proved properties

1. $\cos^2\theta + \sin^2\theta = 1$. (Use Pythagoras's theorem)
2. $\cos 0 = 1, \quad \sin 0 = 0, \quad \cos(\pi/2) = 0, \sin(\pi/2) = 1.$
3. $\cos(-\theta) = \cos\theta, \quad \sin(-\theta) = -\sin\theta.$
4. $\cos(\theta + 2\pi) = \cos\theta, \quad \sin(\theta + 2\pi) = \sin\theta.$

Some useful triangles



The first triangle shows that $\sin(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$; the second that $\cos(\pi/3) = \sin(\pi/6) = 1/2$ and $\cos(\pi/6) = \sin(\pi/3) = \sqrt{3}/2$.

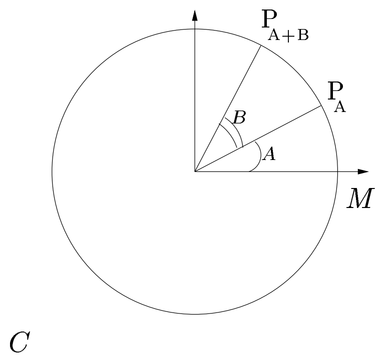
1.2.3 The addition formula for $\cos(\alpha + \beta)$

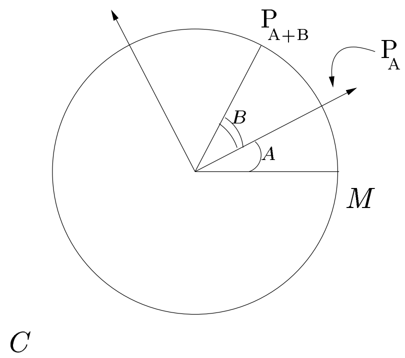
$$\boxed{\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

Proof. Let C be the unit circle. Then $P_{\alpha+\beta}$ has coordinates $(\cos(\alpha+\beta), \sin(\alpha+\beta))$ and M has coordinates $(1, 0)$, so the square of the distance apart of M and $P_{\alpha+\beta}$ is

$$(\cos(\alpha + \beta) - 1)^2 + (\sin(\alpha + \beta) - 0)^2. \qquad (*)$$

Now take the different coordinate system as shown below right.





The coordinates of M with respect to the new coordinate system are $(\cos(-\alpha), \sin(-\alpha))$ while those of $P_{\alpha+\beta}$ are $(\cos\beta, \sin\beta)$. Thus the square of the distance apart of M and $P_{\alpha+\beta}$ is equal to

$$(\cos\alpha - \cos\beta)^2 + (-\sin\alpha - \sin\beta)^2.$$

If we equate this with $(*)$ then the result drops out (using $\cos^2 + \sin^2 = 1$). ∎

Other formulae $\quad \sin\alpha = \cos\left(\alpha - \frac{\pi}{2}\right), \qquad \cos\alpha = \sin\left(\frac{\pi}{2} - \alpha\right).$

Proof. To prove the first one take $\beta = -\pi/2$ in the addition formula for cos, and to prove the second one replace α in the first one by $\frac{\pi}{2} - \alpha$. ∎

1.2.4 The addition formula for $\sin(\alpha + \beta)$

$$\boxed{\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta}$$

Proof.

$$\begin{aligned}\sin(\alpha+\beta) &= \cos\left(\alpha + \beta - \frac{\pi}{2}\right)\\ &= \cos\alpha\cos\left(\beta - \frac{\pi}{2}\right) + \sin\alpha\sin\left(\frac{\pi}{2} - \beta\right)\\ &= \cos\alpha\sin\beta + \sin\alpha\cos\beta.\end{aligned}$$

∎

Example 1.7. Find $\cos(11\pi/4)$.

$$\text{Well, } \cos(11\pi/4) = \cos(2\pi + 3\pi/4) = \cos(3\pi/4) = \cos(\pi/2 + \pi/4)$$
$$= \cos(\pi/2)\cos(\pi/4) - \sin(\pi/2)\sin(\pi/4) = -\sin(\pi/4) = -1/\sqrt{2}.$$

Example 1.8. Draw the graph of $\sin x + \cos x$.
Idea: Try to write $\sin x + \cos x = R\sin(x + \alpha)$ for some constants R and α. We need R and α so that

$$\sin x + \cos x = R\sin(x+\alpha) = R(\sin x\cos\alpha + \cos x\sin\alpha).$$

Comparing coefficients of $\sin x$ and of $\cos x$, we need

$$1 = R\cos\alpha, \qquad 1 = R\sin\alpha.$$

If we square and add these equations we see that we can take $R = \sqrt{2}$, and looking again at the equations, we can take $\alpha = \pi/4$. So we see that

$$\sin x + \cos x = \sqrt{2}\sin(x + \pi/4).$$

This latter expression is much easier to plot.

For completeness, we note that

$$\tan x = \frac{\sin x}{\cos x}, \quad \operatorname{cosec} x = \frac{1}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \cot x = \frac{1}{\tan x}.$$

1.2.5 Derivatives

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\cos x = -\sin x.$$

Example 1.9. $\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \sec^2 x = \frac{1}{\cos^2 x}.$

Exercise Show that $\frac{d}{dx}\sec x = \sec x\,\tan x.$

1.2.6 Inverse trigonometric functions

For $-1 \le x \le 1$, $\arcsin x$ is equal to that angle θ in the range $-\pi/2 \le \theta \le \pi/2$ such that $\sin\theta = x$. This means that

$$\boxed{\sin(\arcsin x) = x, \qquad -\pi/2 \le \arcsin x \le \pi/2}$$

defines $\arcsin x$. So, for example, since $\sin(\pi/4) = 1/\sqrt{2}$, it follows that $\arcsin(1/\sqrt{2}) = \pi/4$.

The inverse cosine can be defined in a similar manner – the only point to watch is that to make the function single-valued we must select a different range for its values. More precisely, for $-1 \le x \le 1$, $\arccos x$ is equal to that angle θ in the range $0 \le \theta \le \pi$ such that $\cos\theta = x$:

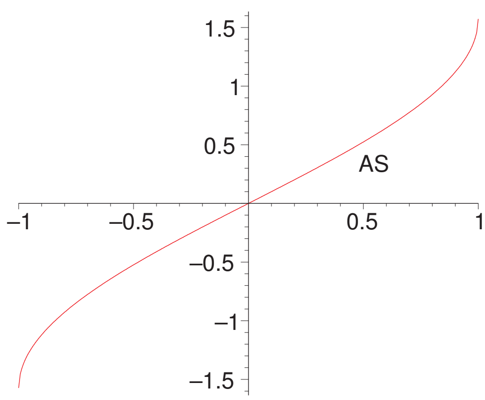
$$\boxed{\cos(\arccos x) = x, \qquad 0 \le \arccos x \le \pi}$$

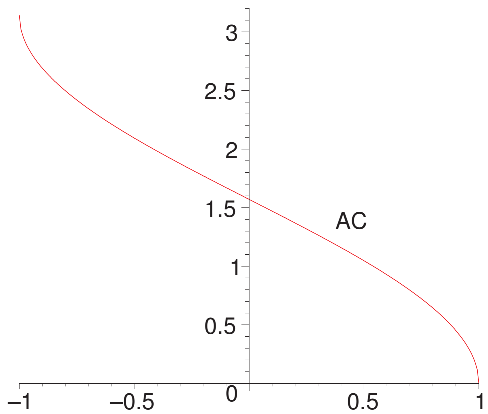
For example, since $\cos(\pi) = -1$, it follows that $\arccos(-1) = \pi$.

Finally, since tan takes values between $-\infty$ and $+\infty$, its inverse is defined for *all* real numbers x (and not just for $-1 \le x \le 1$). Again taking care with the range, for $-\infty < x < \infty$, $\arctan x$ is the angle θ in the range $-\pi/2 < x < \pi/2$ such that $\tan\theta = x$:

$$\boxed{\tan(\arctan x) = x, \qquad -\pi/2 < \arctan x < \pi/2}$$

The graphs of arcsin and arccos are shown below – you should check that you understand why they look the way that they do.





Derivatives of inverse trigonometric functions:

Differentiating $\sin(\arcsin x) = x$ using the chain rule, we get

$$\cos(\arcsin x)\arcsin' x = 1,$$

so that

$$\sqrt{1-\sin^2(\arcsin x)}\ \arcsin' x = 1,$$

ie

$$\sqrt{1-x^2}\ \arcsin' x = 1,$$

ie

$$\arcsin' x = \frac{1}{\sqrt{1-x^2}}\,.$$

Similarly (exercise!), it can be shown that

$$\arccos' x = \frac{-1}{\sqrt{1-x^2}}\ , \quad \arctan' x = \frac{1}{1+x^2}\ ,$$

Chapter 2

Integration

2.1 Two types of integral

2.1.1 Integration as the reverse of differentiation

Definition If $F(x)$ is a continuously differentiable function such that $F'(x) = f(x)$, then $F(x)$ is an *indefinite integral* (or *antiderivative*) of $f(x)$.

Example 2.1. If $f(x) = 3x^2$ then $F(x) = x^3$ is an indefinite integral of $f(x)$, as is $x^3 + 7$.

Note If $F(x)$, $G(x)$ are indefinite integrals of $f(x)$ then $G(x) = F(x) + c$ for some constant c.

In fact, this follows easily because

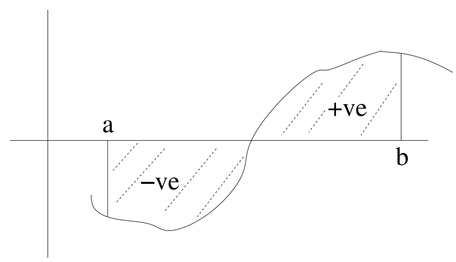
$$(F(x) - G(x))' = F'(x) - G'(x) = f(x) - f(x) = 0,$$

so that $F(x) - G(x)$ is constant.

Notation Write $\int f(x)\,dx$ for an indefinite integral of $f(x)$. We see from the above Note that it is defined up to an additive constant.

2.1.2 Integration as the area under a graph

If $a < b$, the *definite integral* of $f(x)$ on the interval $a \leq x \leq b$ is given by the (signed) area under the graph of $f(x)$ between $x = a$ and $x = b$ (shaded in the diagram). Here, areas below the x-axis are counted negatively.



Notation Write $\int_a^b f(x)\,dx$ for the definite integral of $f(x)$ on the interval $a \leq x \leq b$. Also, we let $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$. Unlike the indefinite integral, the definite integral is uniquely defined.

Notice that the variable 'x' in the expression $\int_a^b f(x)\,dx$ is a 'dummy variable' in that it can be replaced by any other symbol without affecting the value of the expression. For instance, $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$. As a specific example $\int_1^3 x^2\,dx = \int_1^3 t^2\,dt$.

2.2 The fundamental theorem of calculus

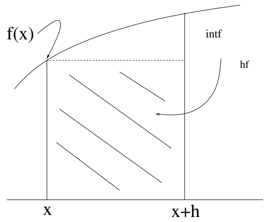
The relation between the definite and indefinite integral is provided by the following.

Theorem 2.1 (The fundamental theorem of calculus)**.** Let $f(x)$ be a continuous function and let $A(x) = \int_a^x f(t)\,dt$. Then $A(x)$ is continuously differentiable and $A'(x) = f(x)$ (ie $A(x)$ is an indefinite integral of $f(x)$).

Idea of proof $A'(x) = \lim\limits_{h\to 0} \dfrac{A(x+h) - A(x)}{h}$

$$= \lim_{h\to 0} \frac{1}{h}\left(\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)dt = f(x).$$



Since indefinite integration is just the reverse of differentiation, indefinite integrals are often quite easy to find. The following result is really useful because it enables us to use indefinite integrals to evaluate definite integrals. In fact, everything we're going to do here depends on the following consequence of the Fundamental Theorem of Calculus.

Let $\mathbf{F(x)}$ be any indefinite integral of $\mathbf{f(x)}$ (ie $\mathbf{F'(x) = f(x)}$). Then

$$\boxed{\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{f(x)\,dx = F(b) - F(a)}}$$

Proof The fundamental theorem of calculus shows that $A(x)$ is an indefinite integral of $f(x)$. Since both $F(x)$ and $A(x)$ are indefinite integrals of $f(x)$ it follows from the result on indefinite integrals that, for some constant c,

$$A(x) = F(x) + c. \qquad (*)$$

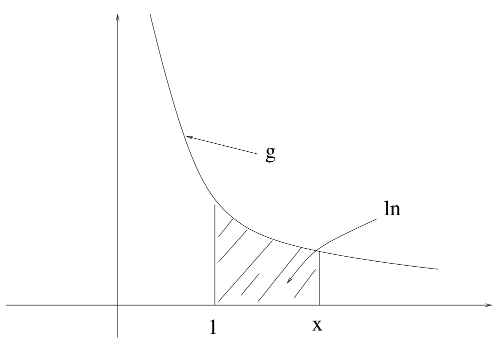
Putting $x = a$ gives $0 = A(a) = F(a) + c$, so that $c = -F(a)$. Putting $x = b$ in $(*)$ now gives the required result since $\int_a^b f(x)\,dx = A(b) = F(b) - F(a)$.

2.3 Natural logarithm and the exponential

2.3.1 Natural logarithm

For $x > 0$, let

$$\ln x = \int_1^x \frac{1}{t}\,dt\,.$$

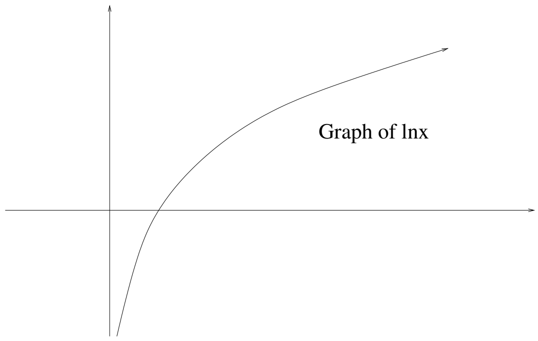


Properties

1. $\ln 1 = 0$,
2. $\ln x > 0$ for $x > 1$,
3. $\ln(1/x) = -\ln x$, or, more generally,

$$\ln(ax) = \ln a + \ln x$$

4. $\ln x < 0$ for $0 < x < 1$,
5. $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$,
6. $\ln(x/a) = \ln x + \ln(1/a) = \ln x - \ln a$,
7. $\ln(x^n) = n \ln x$.



2.3.2 Inverse ln function

Just like trigonometric functions have inverse functions, so does $\ln x$. So, let $\exp x$ be that number $y > 0$ such that $\ln y = x$. So,

$$\boxed{\ln(\exp x) = x}$$

Then (3) above shows that

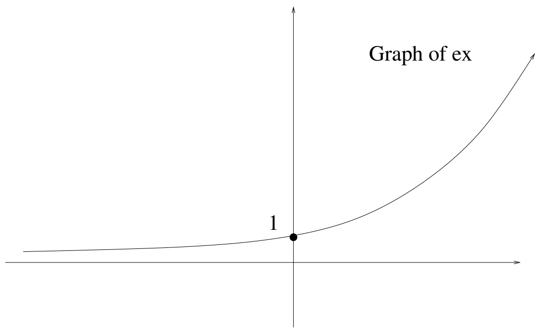
$$\exp(a + b) = \exp a \exp b.$$

Because of this we often write e^x rather than $\exp x$.

Properties

1. Let $e = \exp 1$. Then $\ln e = \ln(\exp 1) = 1$, (e=2.7182818....)
2. $e^{\ln x} = x$, $\ln(e^x) = x$,

   If $a > 0$, we put $a^x = e^{x \ln a}$. Then
3. $\ln(a^x) = x \ln a$.



2.3.3 The derivative of $\mathbf{e^x}$

Differentiate $\ln(e^x) = x$ using the chain rule. We get $\dfrac{1}{e^x}\dfrac{d}{dx}(e^x) = 1$, ie

$$\frac{d}{dx}(e^x) = e^x.$$

Example 2.2. Find all solutions to

$$\ln(x + 3) = 1 + \ln x.$$

Apply exp to both sides to get

$$x + 3 = e^{1+\ln x} = e^1 e^{\ln x} = ex.$$

So $x(1 - e) = -3$, ie

$$x = \frac{3}{(e - 1)}.$$

Example 2.3. Find all solutions to

$$e^{(x^2)} = (e^x)^2. \quad (= e^x e^x = e^{2x})$$

Apply ln to both sides to get

$$x^2 = 2\ln(e^x) = 2x.$$

So, $x(x-2) = 0$, ie

$$x = 0 \text{ or } x = 2$$

Example 2.4. Differentiate $y = e^{3x}\ln x$. Using the product rule and the chain rule,

$$\frac{dy}{dx} = 3e^{3x}\ln x + \frac{e^{3x}}{x}.$$

Exercise Write y as a function of x (ie solve for y in terms of x) when $e^y - 2e^{-y} = x$.

Ans: $y = \ln\left(\frac{x+\sqrt{x^2+8}}{2}\right)$

2.4 Hyperbolic functions

For any real number x we put

$$\boxed{\sinh x = \tfrac{1}{2}\left(e^x - e^{-x}\right), \quad \cosh x = \tfrac{1}{2}\left(e^x + e^{-x}\right)}$$

We also have

$$\tanh x = \frac{\sinh x}{\cosh x}, \quad \text{cosech} x = \frac{1}{\sinh x}, \quad \text{sech } x = \frac{1}{\cosh x}, \quad \coth\ x = \frac{1}{\tanh x}.$$

These are the hyperbolic functions, and here are some properties.

1. $\sinh 0 = 0$, $\cosh 0 = 1$.

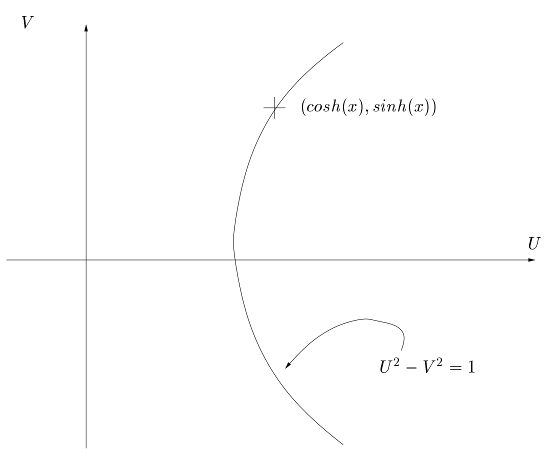
   Proof. $\sinh 0 = (e^0 - e^0)/2 = (1-1)/2 = 0$, while $\cosh 0 = (e^0+e^0)/2 = (1+1)/2 = 1$. ∎

2. $\cosh^2 a - \sinh^2 a = 1$.

   Proof. $\cosh^2 a - \sinh^2 a = \frac{1}{4}(e^a + e^{-a})^2 - \frac{1}{4}(e^a - e^{-a})^2 = \frac{1}{4}4 = 1.$ ∎

The reason for their names

The functions $\sinh x$ and $\cosh x$ are called the hyperbolic functions because of (2), which shows that, as x goes from $-\infty$ to ∞, then $(\cosh x, \sinh x)$ traces out (the right hand side of) the hyperbola $u^2 - v^2 = 1$ in the (u, v)-plane.



Physical interpretation. The graph of $\cosh x$ gives the shape of a chain or cable hanging under gravity (see Figure 2.1).

3. $0 < \sinh x < \cosh x$ for $x > 0$.

   Proof. The above inequalities hold when $0 < e^x - e^{-x} < e^x + e^{-x}$. The second of these inequalities always holds since e raised to any power is positive. The first inequality holds when $e^{-x} < e^x$ which holds when $e^{2x} > 1$, which holds when $x > 0$. ∎

4. $\sinh(-x) = -\sinh x$, $\cosh(-x) = \cosh x$.

   Proof. $\sinh(-x) = \dfrac{e^{-x} - e^x}{2} = -\dfrac{e^x - e^{-x}}{2} = -\sinh x$. The proof for $\cosh x$ is similar. ∎

5. $\sinh' x = \cosh x$, $\cosh' x = \sinh x$.

   Proof. $\sinh' x = \dfrac{d}{dx}\left(\dfrac{e^x - e^{-x}}{2}\right) = \dfrac{e^x + e^{-x}}{2} = \cosh x$. The proof for $\cosh x$ is similar. ∎

6. $\sinh x$ is everywhere increasing, and $\cosh x$ is increasing for $x > 0$.

   Proof. $\sinh' x = \cosh x$ (by (5)), which is always positive (by (3) and (4)). The proof that $\cosh x$ is increasing for positive x is similar. ∎

7. $e^x = \cosh x + \sinh x$, $e^{-x} = \cosh x - \sinh x$.

   Proof. $\cosh x + \sinh x = \dfrac{(e^x + e^{-x})}{2} + \dfrac{(e^x - e^{-x})}{2} = e^x$. The second result is proved similarly. ∎

8. $\cosh(a + b) = \cosh a \cosh b + \sinh a \sinh b$
   $\sinh(a + b) = \sinh a \cosh b + \cosh a \sinh b$.

   Proof. $\cosh a \cosh b + \sinh a \sinh b = \dfrac{1}{4}(e^a + e^{-a})(e^b + e^{-b}) + \dfrac{1}{4}(e^a - e^{-a})(e^b - e^{-b}) = \dfrac{1}{4}(2e^{(a+b)} + 2e^{-(a+b)}) = \cosh(a + b)$. The other relation is proved similarly. ∎

9. $\sinh 2a = 2 \sinh a \cosh a$, $\cosh 2a = \cosh^2 a + \sinh^2 a$.

   Proof. Take $a = b$ in (8). ∎

10. $1 - \tanh^2 a = \text{sech}^2 a$, $\coth^2 a - 1 = \text{cosech}^2 a$.

    Proof. $1 - \tanh^2 a = 1 - \dfrac{\sinh^2 a}{\cosh^2 a} = \dfrac{\cosh^2 a - \sinh^2 a}{\cosh^2 a} = \dfrac{1}{\cosh^2 a}$, using (2). The other relation is proved similarly. ∎

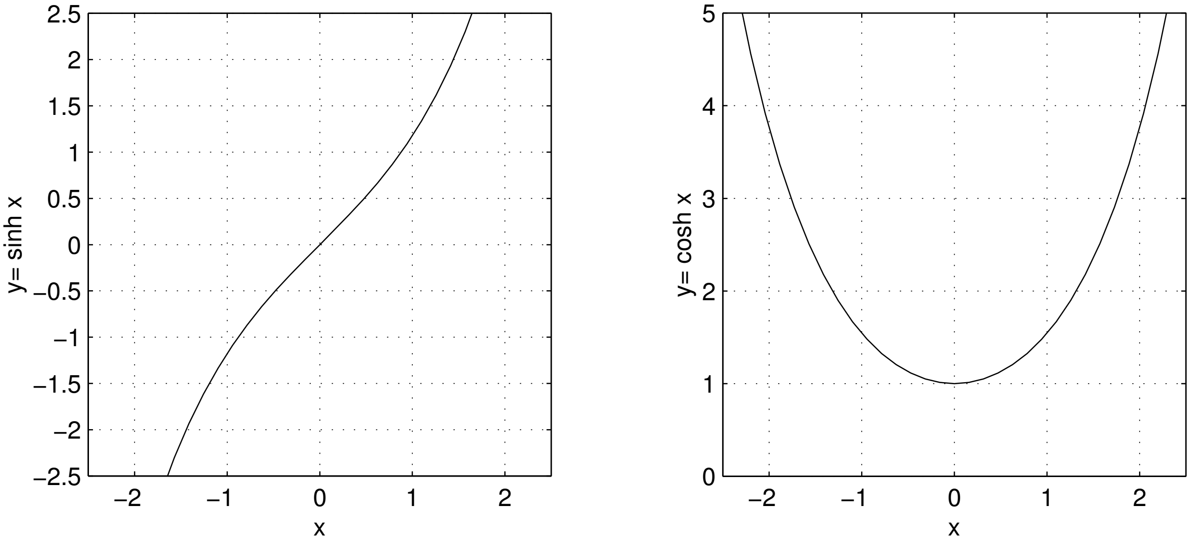


Figure 2.1: Plots of sinh x and cosh x

11. $\tanh' x = \text{sech}^2 x$, $\coth' x = -\text{cosech}^2 x$.

Proof. $\tanh' x = \dfrac{d}{dx}\left(\dfrac{\sinh x}{\cosh x}\right) = \dfrac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \text{sech}^2 x$, using (5) and (2). The other equation is proved similarly. ∎

12. Let arcsinh be the inverse function of sinh. Then $\text{arcsinh}\, x$ is that number y such that $\sinh y = x$, ie,

$$\boxed{\sinh(\text{arcsinh}\, x) = x}$$

In fact $\text{arcsinh}\, x = \log(x + \sqrt{1 + x^2})$ and $\text{arcsinh}' x = \frac{1}{\sqrt{1+x^2}}$.

13. In a similar way, let arccosh be the inverse function of cosh. Then $\text{arccosh}\, x$ is that positive number y such that $\cosh y = x$, ie,

$$\boxed{\cosh(\text{arccosh}\, x) = x}.$$

In fact $\text{arccosh}\, x = \log(x + \sqrt{x^2 - 1})$ and $\text{arccosh}' x = \frac{1}{\sqrt{x^2-1}}$.

2.5 Basic principle of integration

The problem, given a function $f(x)$ and an interval $a \le x \le b$ in the real number line, is to compute the integral

$$I = \int_a^b f(x)dx.$$

Because of the consequence of the Fundamental Theorem of Calculus described in Section 2.2, there are two steps:

(a) find (somehow!) a function $F(x)$ such that $F'(x) = f(x)$ (ie find an indefinite integral of $f(x)$);

(b) then $I = F(b) - F(a)$.

Of course, the hard part is step (a), and for this there are no holds barred—if you can find an $F(x)$ that works, you've solved the problem!

Remember that the indefinite integral of $f(x)$ is not unique: if $F(x)$ is an indefinite integral then so is $F(x) + c$ for any real number c.

Example 2.5. Let

$$I = \int_1^3 e^{2x} dx.$$

Note that $F(x) = \frac{1}{2}e^{2x}$ will do, because this happens to satisfy $F'(x) = e^{2x}$. So will $F(x) = \frac{1}{2}e^{2x} + c$ for any constant c, as we've just mentioned. However, c is irrelevant here, since it cancels in step (b):

$$I = F(3) - F(1) = \frac{1}{2}\left(e^6 - e^2\right).$$

We write an indefinite integral as $\int f(x)dx$.

2.6 Methods of integration

2.6.1 Some useful indefinite integrals

Here are some useful indefinite integrals. To check them, all you need to do is to notice that in each case the derivative of the right-hand side is the function inside the integral sign.

$$\begin{aligned}
\int x^a\, dx &= \frac{x^{a+1}}{a+1} + c, \qquad a \neq -1 \\
\int x^{-1}\, dx &= \ln|x| + c \\
\int \sin x\, dx &= -\cos x + c \\
\int \sec^2 x\, dx &= \tan x + c \\
\int \sinh x\, dx &= \cosh x + c \\
\int \frac{dx}{a^2 + x^2} &= \frac{1}{a}\arctan\frac{x}{a} + c \\
\int \frac{dx}{1 - x^2} &= \text{arctanh } x + c \\
\int \frac{dx}{\sqrt{1 + x^2}} &= \text{arcsinh } x + c \\
\int \frac{dx}{\sqrt{1 - x^2}} &= \text{arcsin } x + c
\end{aligned}$$

This list is not meant to be complete (shortage of memory and of paper!).

A word about the first integral in this list. Of course, it works because the derivative of x^a is ax^{a-1}. When $a = 1, 2, 3, \ldots$ is a whole number this can be checked using the

Binomial Theorem. But more generally (for example, if a is irrational, like $a = \sqrt{2}$ or $a = \pi$), we define x^a as follows:

$$x^a = e^{a \ln x}.$$

We can differentiate this using the chain rule:

$$\begin{aligned} \frac{d}{dx}(x^a) &= \frac{a}{x} e^{a \ln x} \\ &= \frac{a}{x} x^a \\ &= a x^{a-1}. \end{aligned}$$

2.6.2 Integration by Parts

Recall the *product rule* for differentiation:

$$\frac{d}{dx}\left(u(x)v(x)\right) = \frac{du}{dx} v(x) + u(x) \frac{dv}{dx}.$$

This says that $u(x)v(x)$ is an indefinite integral of the right-hand side, or in other words:

$$uv = \int \left(\frac{du}{dx} v + u \frac{dv}{dx} \right) dx.$$

Rearranging this gives the rule for *integration by parts*:

$$\boxed{\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx}$$

Example 2.6. Consider

$$I_1 = \int x e^x dx.$$

Take $u(x) = x$ and $v'(x) = e^x$. Then $u'(x) = 1$ and $v(x) = e^x$, and we can find I_1 by substituting for u and v in the rule for integration by parts.

$$I_1 = xe^x - \int e^x dx = xe^x - e^x + c.$$

This can be checked by differentiating the right hand side.

Example 2.7. Find

$$I_2 = \int x^2 e^x dx.$$

Take $u(x) = x^2$ and $v'(x) = e^x$ Then $u'(x) = 2x$ and $v(x) = e^x$, and we get:

$$\begin{aligned} I_2 &= x^2 e^x - \int 2x e^x dx \\ &= x^2 e^x - 2I_1, \end{aligned}$$

where I_1 is the integral from the previous example. Using that example we conclude:

$$I_2 = \left(x^2 - 2x + 2\right) e^x + c.$$

Again, you could check that this is correct by differentiating it!

Remark How would you do *all* the integrals

$$I_n := \int x^n e^x dx, \qquad n = 1, 2, 3, 4, \ldots?$$

Well, by exactly the same reasoning as in the last example, we find that

$$I_n = x^n e^x - nI_{n-1}.$$

You can use this to find I_3, I_4, etc up to I_n for any n (although you wouldn't want n to be too big!). This is an example of a *recurrence relation*.

Sometimes you have to integrate by parts twice in order to evaluate an integral:

Example 2.8. Let

$$I = \int e^{2x} \cos 3x dx.$$

Integrate by parts with $u(x) = \cos 3x$ and $v'(x) = e^{2x}$. Then $v(x) = \frac{1}{2}e^{2x}$, so we get

$$I = \frac{1}{2}e^{2x} \cos 3x + \frac{3}{2} \int e^{2x} \sin 3x dx.$$

Now integrate the right-hand integral by parts, to get

$$\begin{aligned} I &= \frac{1}{2}e^{2x} \cos 3x + \frac{3}{4}e^{2x} \sin 3x - \frac{9}{4} \int e^{2x} \cos 3x dx \\ &= \frac{1}{2}e^{2x} \cos 3x + \frac{3}{4}e^{2x} \sin 3x - \frac{9}{4} I. \end{aligned}$$

Now solve for I, to conclude:

$$I = \frac{1}{13}e^{2x} (2 \cos 3x + 3 \sin 3x) + c.$$

Again, you could differentiate this to make sure it works.

Example 2.9. Here's a cunning example:

$$I = \int \ln x \ dx.$$

There is a trick here: integrate by parts with $u(x) = \ln x$ and $v'(x) = 1$. Then $v(x) = x$ so we get

$$\begin{aligned} I &= x \ln x - \int \frac{1}{x} x dx \\ &= x \ln x - x + c. \end{aligned}$$

It's quite quick to differentiate this to check that it works!

For definite integrals, just put the limits on the previous formula:

$$\boxed{\int_a^b u \frac{dv}{dx} dx = \left[uv\right]_a^b - \int_a^b v \frac{du}{dx} dx}$$

where (as usual when computing definite integrals) $[uv]_a^b = u(b)v(b) - u(a)v(a)$. Sometimes, as the next example shows, you can save a lot of time by working directly with definite integrals.

Example 2.10. Find

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} dt \ , \qquad n = 1, 2, \ldots$$

This is similar to the problem of computing the indefinite integrals I_n in example 2.7 above. It turns out to be an easier calculation, because the limits of the definite integral have been chosen in a helpful way. However, the upper $(+\infty)$ limit does need extra care. In general, an integral over an infinite range such as $\int_a^\infty f(t)\,dt$ is defined as the limiting value of $\int_a^b f(t)\,dt$ as b 'tends to infinity'; i.e., as b becomes bigger and bigger. Later in the course we'll see in more detail how to calculate these sorts of limiting values; for now, you might have to take some of the discussion on trust.

First we'll take the case $n = 1$:

$$\Gamma(1) = \int_0^\infty e^{-t} dt = \left[-e^{-t}\right]_0^\infty = 0 - (-1) = 1\,.$$

(The zero in the second-last formula follows since the limiting value of e^{-t} as t gets very large is zero.)

Next suppose that $n > 1$. The formula for $\Gamma(n)$ can be integrated by parts, taking $u(t) = t^{n-1}$ and $v'(t) = e^{-t}$. Then $u'(t) = (n{-}1)t^{n-2}$ and $v(t) = -e^{-t}$, and so

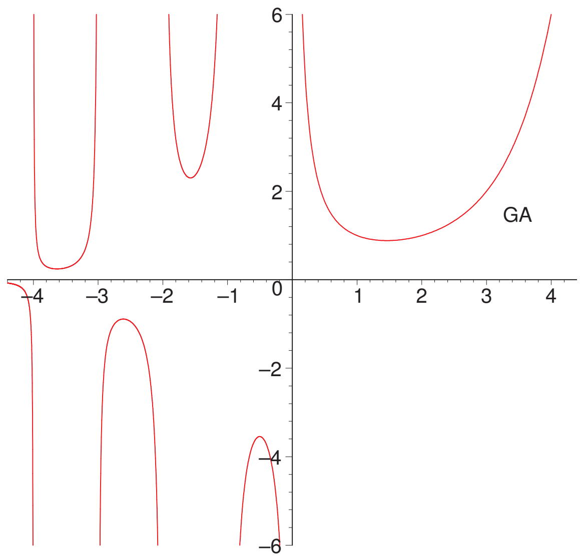
$$\begin{aligned}\Gamma(n) &= \left[-t^{n-1}e^{-t}\right]_0^\infty + \int_0^\infty (n{-}1)t^{n-2}e^{-t}dt \\ &= 0 - 0 + (n{-}1)\int_0^\infty t^{n-2}e^{-t}dt \\ &= (n{-}1)\Gamma(n{-}1)\,.\end{aligned}$$

(The zeroes on the middle line are because the limiting value of $t^{n-1}e^{-t}$ as t becomes large is zero, as, for $n > 1$, is the value of this function at $t = 0$.)

Armed with this formula we can calculate $\Gamma(2) = 1.\Gamma(1) = 1$, $\Gamma(3) = 2.\Gamma(2) = 2$, $\Gamma(4) = 3.\Gamma(3) = 6$ and so on; the general result is easily seen to be $\Gamma(n) = (n{-}1)!$ (as practice, you could try to write out a full proof of this using induction).

A neat feature of this example is that, unlike the 'traditional' definition of the factorial, the integral formula for $\Gamma(n)$ makes sense when n is not an integer – it works fine if n is replaced by any positive real number x, thereby defining the **gamma function** $\Gamma(x)$.

It's even possible to define the gamma function for negative values of x, by using the formula $\Gamma(x) = (x-1)\Gamma(x-1)$, but this goes well beyond the material of this course. Just for interest, the function is plotted on the right – you can check that at positive integers it has the expected values. The gamma function has many remarkable properties – for example, it can be shown that $\Gamma(1/2) = \sqrt{\pi}$, and that $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$.



2.6.3 Substitution

This method corresponds to the Chain Rule for differentiation.

The idea here is to solve an integral by making a substitution of u for a suitable function of x, in order to simplify the integral.

This is done as follows.

$$\text{Let} \quad \begin{cases} u = u(x) \\ du = u'(x)dx \end{cases} \qquad \text{and} \qquad \begin{cases} c = u(a) \\ d = u(b) \end{cases}$$

Then

$$\boxed{\int_{x=a}^{b} f(u(x))u'(x)dx = \int_{u=c}^{d} f(u)du.}$$

In general, a good choice of substitution is the best trick for evaluating an integral, and is probably the first thing to look for.

Example 2.11. Let

$$I = \int_0^{\pi/2} e^{\sin x} \cos x \; dx.$$

The correct substitution to make here is $u = \sin x$, $du = \cos x \; dx$. This gives

$$\begin{aligned} I &= \int_{u=0}^{1} e^u du \\ &= [e^u]_{u=0}^1 \\ &= e - 1. \end{aligned}$$

Example 2.12. Let

$$I = \int_0^1 15x^2\sqrt{5x^3+4} \; dx.$$

What you need to notice here is that $15x^2$ is exactly the derivative of $5x^3+4$. Hence the substitution to make is $u = 5x^3+4$, $du = 15x^2dx$. Then the integral becomes

$$\begin{aligned} I &= \int_{u=4}^{9} \sqrt{u}du \\ &= \left[\frac{2}{3}u^{3/2}\right]_4^9 \\ &= \frac{38}{3}. \end{aligned}$$

This method works in the same way for indefinite integrals. We play exactly the same game to find an indefinite integral, but without having to worry about the limits of the integral. However, don't forget to express the answer as a function of x rather than the substituted variable u.

Example 2.13. Let

$$I = \int \frac{(\ln x)^2}{x} dx.$$

Noticing that $1/x$ is the derivative of $\ln x$ it is natural to make a substitution $u = \ln x$, $du = dx/x$. Then

$$\begin{aligned} I &= \int u^2 du \\ &= \tfrac{1}{3}u^3 + c \\ &= \tfrac{1}{3}(\ln x)^3 + c. \end{aligned}$$

Here is a trigonometric example.

Example 2.14. Let $I = \int \tan x \; dx$.

Here the integrand is $\sin x / \cos x$, where the top is (essentially) the derivative of the bottom. So make a substitution $u = \cos x$, $du = -\sin x \; dx$. Then

$$\begin{aligned} I &= -\int \frac{du}{u} \\ &= \ln(1/u) + c \\ &= \ln(\sec x) + c. \end{aligned}$$

Example 2.15. Let $I = \displaystyle\int \frac{dx}{(3x+4)^2}$.

Here, we make a substitution $u = 3x + 4$, so that $du = 3dx$. So the integral becomes $I = \dfrac{1}{3}\displaystyle\int \frac{du}{u^2} = -\frac{1}{3u}$. Thus $I = -\dfrac{1}{3(3x+4)}$.

Example 2.16. Let $I = \displaystyle\int \frac{dx}{x^2 + 4x + 7}$.

Here, completing the square in the denominator, we get $I = \displaystyle\int \frac{dx}{(x+2)^2 + 3}$. Now substitute $u = x + 2$, $du = dx$. Then

$$I = \int \frac{du}{u^2 + 3} = \int \frac{du}{u^2 + (\sqrt{3})^2} = \frac{1}{\sqrt{3}} \arctan\left(\frac{x+2}{\sqrt{3}}\right).$$

Finally a different sort of example. We make a substitution in 'the opposite direction'.

Example 2.17. Let

$$I = \int_{2/\sqrt{3}}^{\sqrt{2}} \frac{dx}{x\sqrt{x^2 - 1}}.$$

We will tidy this up using the identity

$$\sec^2 u = 1 + \tan^2 u.$$

Make a substitution $x = \sec u$, $dx = \sec u \; \tan u \; du$. Then the integral reads

$$I = \int_a^b \frac{\sec u \tan u}{\sec u \sqrt{\tan^2 u}} du = \int_a^b du.$$

Very neat! What are the limits a, b? Well, $\sec a = 2/\sqrt{3}$ and $\sec b = \sqrt{2}$. So $a = \pi/6$ and $b = \pi/4$. Hence

$$I = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}.$$

2.6.4 Partial fractions

These are useful when finding integrals of functions which are a quotient of polynomials. So, if $p(x), q(x)$ are polynomials we'll see how to integrate $p(x)/q(x)$ using partial fractions.

'Partial fractions' means separating a fraction whose denominator is a product of polynomial factors into a sum of fractions with *those factors* as their denominators.

If the degree of $p(x)$ is **strictly less** than the degree of $q(x)$ then $p(x)/q(x)$ can be written as a sum of functions of the form

$$\frac{A}{(\ell x+m)^r}, \qquad \frac{Bx+C}{(ax^2+bx+c)^s}$$

where $(\ell x+m)^r, (ax^2+bx+c)^s$ are divisors of $q(x)$ with ax^2+bx+c having no real roots, that is $b^2-4ac<0$. The expression so obtained is called the **partial fraction expansion** of $p(x)/q(x)$.

Example 2.18. $\dfrac{x+1}{(x-1)(x+2)} = \dfrac{A}{x-1}+\dfrac{B}{x+2}$. (We have to find A and B)

Example 2.19. $\dfrac{x^2+x-3}{(2x-1)(x+4)(x-7)} = \dfrac{A}{2x-1}+\dfrac{B}{x+4}+\dfrac{C}{x-7}$.

Example 2.20. $\dfrac{x^2-7x+2}{(2x+5)(x^2-2x+5)} = \dfrac{A}{2x+5}+\dfrac{Bx+C}{x^2-2x+5}$.

Example 2.21. $\dfrac{2x+1}{(3x-2)^2(x+4)} = \dfrac{A}{3x-2}+\dfrac{B}{(3x-2)^2}+\dfrac{C}{x+4}$.

Example 2.22. $\dfrac{x+1}{(x^2+x+1)^2(x-3)^2} = \dfrac{Ax+B}{x^2+x+1}+\dfrac{Cx+D}{(x^2+x+1)^2}+\dfrac{E}{x-3}+\dfrac{F}{(x-3)^2}$.

The constants could be found by clearing fractions and equating constants and appropriate powers of x on both sides (but don't try it for the above examples, some of the numbers could be ugh!).

Example 2.23. $\dfrac{x+2}{(x-1)(x-2)} = \dfrac{A}{x-1}+\dfrac{B}{x-2}$.

Clearing fractions, we get

$$x+2 = A(x-2)+B(x-1),$$

so, equating constants and coeffs of x on both sides we get

$$2 = -2A-B \quad \text{and} \quad 1 = A+B.$$

These equations are easily solved to give $A=-3$ and $B=4$.

The above method always works, but the equations are sometimes a nuisance to solve. For instance, in Example (2.21) you'd get 3 equations in 3 unknowns, while in Example (2.22) you'd get 6 equations in 6 unknowns!

However, in the case where the denominator $q(x)$ has some non-repeated linear factors there is a simpler method (called the *cover-up* rule) of determining the corresponding constants.

The idea is that if the factor $(ax+b)$ (but not raised to a power) appears in the denominator of the LHS then you 'cover-up' that factor on the LHS and evaluate what you can still see at the value of x which makes that factor equal to 0 (ie put $x=-b/a$).

Example (2.23) **(again)** To find A, cover up the factor $(x-1)$ on the LHS and evaluate what you can still see $\left(\text{ie } \dfrac{x+2}{x-2}\right)$ at $x=1$. This gives $A = \dfrac{3}{-1} = -3$, in agreement with above. In a similar way you can see that $B = 4$.

This method is really labour-saving when there are more unknowns involved.

Example 2.24. $\dfrac{2x^2 - x}{(x+1)(x-2)(x-1)} = \dfrac{A}{x+1} + \dfrac{B}{x-2} + \dfrac{C}{x-1}$. Here, the cover-up rule quickly gives $A = 1/2$, $B = 2$ and $C = -1/2$. Try doing this by the original method - although it's not hard, I hope it will take you longer than the cover-up method!

We can also partially apply this technique to more general situations. For instance in the following example, we can use the cover-up rule to find the constant on top of $(x-1)$

Example 2.25. Using the cover-up rule, $\dfrac{5x^2+4}{(x^2+x+1)(x-1)} = \dfrac{3}{x-1} + \dfrac{Bx+C}{x^2+x+1}$. If you then clear the denominators and compare constants you'll get $C = -1$, while comparing coeffs of x^2 gives $B = 2$. This is much easier than the original method which would mean your solving 3 equations in 3 unknowns.

Using the partial fraction expansion, the integral $\displaystyle\int \frac{p(x)}{q(x)} dx$ may thus be written as a sum of integrals, each of a standard form. If $\deg p(x) \geq \deg q(x)$ then we divide $p(x)$ by $q(x)$ to get $p(x) = g(x)q(x) + r(x)$, where $\deg r(x) < \deg q(x)$. Then

$$\frac{p(x)}{q(x)} = g(x) + \frac{r(x)}{q(x)}$$

where $g(x)$ is a polynomial and $r(x)/q(x)$ is of the form discussed above.

Before we see how this method of integration works, here are two integrals which we will often need:

$$\boxed{\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + c, \qquad \int \frac{dx}{x-a} = \ln|x-a| + c.}$$

Now let's do some integrals using partial fractions.

Example 2.26. $\displaystyle\int \frac{dx}{x^2-x-2}$. We have

$$\frac{1}{x^2-x-2} = \frac{1}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

and $A = 1/3$, $B = -1/3$. Thus

$$\int \frac{dx}{x^2-x-2} = \frac{1}{3}\int \frac{dx}{x-2} - \frac{1}{3}\int \frac{dx}{x+1} = \frac{1}{3}\ln\left|\frac{x-2}{x+1}\right| + c.$$

Example 2.27. Let

$$I = \int \frac{(x+3)dx}{x^2-3x+2}.$$

Note that the denominator factorises, so the integrand is:

$$\frac{x+3}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2},$$

for suitable constants A and B. To find them, either use the cover-up rule or multiply up to get rid of denominators and compare the numerators on each side. You get two equations in the two unknowns A, B. Solving, we get $A = -4$, $B = 5$. So the integral is

$$\begin{aligned} I &= 5\int \frac{dx}{x-2} - 4\int \frac{dx}{x-1} \\ &= 5\ln|x-2| - 4\ln|x-1| + c. \end{aligned}$$

Example 2.28. Let

$$I = \int \frac{(x+3)dx}{(x-1)(x^2+2)}.$$

This time the integrand is

$$\frac{x+3}{(x-1)(x^2+2)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+2},$$

for some constants A, B, C. This time, after clearing the denominators, the numerator of each side is quadratic (though on the left the coefficient of x^2 happens to be zero!). We therefore have three equations in three unknowns A, B, C which we can solve (but we could use the cover-up rule to find A). We get $A = 4/3$, $B = -4/3$ and $C = -1/3$. So the integral is

$$\begin{aligned} I &= \frac{1}{3}\int \frac{4dx}{x-1} - \frac{1}{3}\int \frac{(4x+1)dx}{x^2+2} \\ &= \frac{4}{3}\ln|x-1| - \frac{2}{3}\ln(x^2+2) - \frac{1}{3\sqrt{2}}\arctan\frac{x}{\sqrt{2}} + c. \end{aligned}$$

The next example illustrates what to do when there are repeated factors.

Example 2.29. Let

$$I = \int \frac{(6x+7)dx}{(x+2)^2}.$$

The correct form of partial fraction expansion is

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2},$$

so that

$$6x+7 = A(x+2) + B.$$

Equating coefficients of x on both sides gives $A = 6$, and then equating constants gives $B = -5$. Thus the integrand is

$$\frac{6x+7}{(x+2)^2} = \frac{6}{x+2} - \frac{5}{(x+2)^2}$$

Making a substitution $u = x+2$, $du = dx$ now gives

$$I = 6\ln|x+2| + \frac{5}{x+2} + c.$$

Example 2.30. $\displaystyle\int \frac{dx}{x^4-1}$. Observe that $x^4 - 1 = (x-1)(x+1)(x^2+1)$ so that

$$\frac{1}{x^4-1} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

and, by the cover-up rule, $A = 1/4$ and $B = -1/4$. Then, clearing fractions,

$$4 = (x+1)(x^2+1) - (x-1)(x^2+1) + 4(Cx+D)(x^2-1)$$

so that $4 = 2(x^2+1) + 4(Cx+D)(x^2-1)$. Comparing cofficients of x^3 gives $C = 0$, while comparing constants gives $D = -1/2$. Thus

$$\frac{1}{x^4-1} = \frac{1}{4}\left(\frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{x^2+1}\right).$$

Hence

$$\begin{aligned}\int \frac{dx}{x^4-1} &= \frac{1}{4}\int \frac{dx}{x-1} - \frac{1}{4}\int \frac{dx}{x+1} - \frac{1}{2}\int \frac{dx}{x^2+1} \\ &= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \frac{1}{2}\text{arctan x} + c \\ &= \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\text{arctan x} + c.\end{aligned}$$

Example 2.31. Let

$$I = \int \frac{-2x+4}{(x^2+1)(x-1)^2}dx.$$

We first note that the partial fractions form of the integrand is

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2},$$

for some constants A, B, C, D.

In fact, a slight extention of the cover-up rule enables us to find D. Cover-up $(x-1)^2$ on LHS and evaluate at $x = 1$ what you can still see. You'll get $D = 1$. Now, getting rid of denominators and comparing numerators gives

$$0x^3 + 0x^2 - 2x + 4 = (Ax+B)(x-1)^2 + C(x^2+1)(x-1) + (x^2+1).$$

Comparing coefficients gives, respectively:

$$\begin{aligned} x^3: \quad 0 &= A + C \\ x: \quad -2 &= A - 2B + C \\ \text{const}: \quad 4 &= B + 1 - C. \end{aligned}$$

Solving these equations gives

$$A = 2, \quad B = 1, \quad C = -2.$$

Hence the integral is

$$\begin{aligned} I &= \int \frac{(2x+1)dx}{x^2+1} - 2\int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2} \\ &= \ln(x^2+1) + \arctan x - 2\ln|x-1| - \frac{1}{x-1} + c. \end{aligned}$$

2.6.5 Powers of trigonometric functions

Odd powers of $\sin x$ or $\cos x$ are relatively easy, and can be dealt with by the method of the next example.

Example 2.32. Let

$$I = \int \cos^5 x \; dx.$$

Make a substitution $u = \sin x$, $du = \cos x \; dx$, and use the fact that $\cos^2 x = 1 - \sin^2 x$. Then

$$\begin{aligned} I &= \int \cos^4 x \cos x dx = \int (1 - \sin^2 x)^2 \cos x dx \\ &= \int (1 - u^2)^2 du \\ &= \int (1 - 2u^2 + u^4) du \\ &= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + c \\ &= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + c. \end{aligned}$$

Any product of sines and cosines, at least one of which is an odd power, can be done using the above method.

Example 2.33. Let

$$I = \int_0^{\pi/2} \cos^2 x \sin^3 x \; dx.$$

Here, $\sin x$ occurs to an odd power, so substitute $u = \cos x$, $du = -\sin x \; dx$. Then

$$\begin{aligned} I &= -\int_{u=1}^{0} u^2(1 - u^2) du \\ &= \int_0^1 (u^2 - u^4) du \\ &= \left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right]_0^1 \\ &= \frac{2}{15}. \end{aligned}$$

What about even powers? For these the method of the last two examples doesn't work. Instead, we can make use of the identities

$$\cos 2x = \begin{cases} 2\cos^2 x - 1 \\ 1 - 2\sin^2 x \end{cases}$$

to reduce from powers to multiple angles.

Example 2.34. Let

$$I = \int \cos^4 x \ dx.$$

The integrand can be rewritten

$$\begin{aligned} \cos^4 x &= \left(\frac{\cos 2x + 1}{2}\right)^2 \\ &= \frac{1}{4}\cos^2 2x + \frac{1}{2}\cos 2x + \frac{1}{4}. \end{aligned}$$

Similarly

$$\cos^2 2x = \frac{\cos 4x + 1}{2},$$

and hence

$$\cos^4 x = \frac{1}{8}\cos 4x + \frac{1}{2}\cos 2x + \frac{3}{8}.$$

So the integral is

$$\begin{aligned} I &= \frac{1}{8}\int (\cos 4x + 4\cos 2x + 3)dx \\ &= \frac{1}{32}\sin 4x + \frac{1}{4}\sin 2x + \frac{3}{8}x + c. \end{aligned}$$

Note that if you differentiate this to check that it's right, you will have a little work to do to get back to $\cos^4 x$.

Later on we might find a more efficient way to integrate even powers of this sort using complex numbers.

2.7 Line integrals (in 2 and 3 dimensions)

Let C be a curve in the plane, parametrised by $\mathbf{r}(t) = (x(t), y(t))$, and let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables (a "scalar" function; its values are scalar). The **line integral** of f along the curve C from $\mathbf{r}(t_0) = (x(t_0), y(t_0))$ to $\mathbf{r}(t_1) = (x(t_1), y(t_1))$ is given by

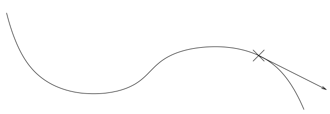
$$\int_{t_0}^{t_1} f(\mathbf{r}(t))|\mathbf{r}'(t)|dt = \int_{t_0}^{t_1} f(x(t), y(t))\sqrt{(x'(t))^2 + (y'(t))^2}dt.$$

2.7.1 Arclength

If C is a curve in the plane, parametrised by $\mathbf{r}(t) = (x(t), y(t))$, then the **arclength** of the curve from $(x(t_0), y(t_0))$ to $(x(t_1), y(t_1))$ is given by

$$\int_{t_0}^{t_1} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

WHY?[Not part of course] $\mathbf{r}'(t) = (x(t), y(t))$ so that $|\mathbf{r}'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}$ and the length of the curve is $\int_{t_0}^{t_1} |\mathbf{r}'(t)|dt$.



Example 2.35. Find the length of the curve C parametrised by $\mathbf{r}(t) = (\cos t, \sin t)$ between $t = 0$ and $t = 2\pi$. Clearly this is a circle of unit radius so its length is 2π.

Example 2.36. Find the length of the curve C parametrised by $\mathbf{r}(t) = (\cos^2 t, \sin^2 t)$ between $t = 0$ and $t = \pi/2$.

Here $x(t) = \cos^2 t$ so that $x'(t) = -2\cos t \sin t$, while $y(t) = \sin^2 t$ so that $y'(t) = 2\sin t \cos t$. So

$$\sqrt{(x'(t))^2 + (y'(t))^2} = \sqrt{8}\cos t \sin t = \sqrt{2}\sin 2t$$

So the required length of C is

$$3\int_0^{\pi/2} \sqrt{2}\sin 2t dt = -\sqrt{2}\frac{\cos 2t}{2}\Big|_0^{\frac{\pi}{2}} = \sqrt{2}\,.$$

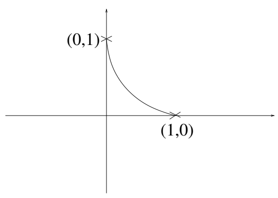
In fact this is the length of the staight line from (0,1) to (1,0) (Pythagoras).

Example 2.37. Find the length of the curve C parametrised by $\mathbf{r}(t) = (\cos^3 t, \sin^3 t)$ between $t = 0$ and $t = \pi/2$.

Here, $x(t) = \cos^3 t$ so that $x'(t) = -3\cos^2 t \sin t$, while $y(t) = \sin^3 t$ so that $y'(t) = 3\sin^2 t \cos t$. So

$$\sqrt{(x'(t))^2 + (y'(t))^2} = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t} = 3\sin t \cos t\sqrt{\cos^2 t + \sin^2 t}$$

$$= 3\sin t \cos t.$$

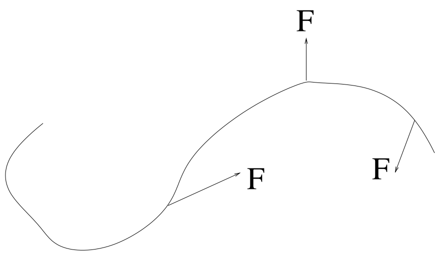


So the required length of C is

$$3\int_0^{\pi/2} \sin t \cos t \, dt = 3/2.$$

2.7.2 Work done by a force

If C is the curve in the plane parametrised by $\mathbf{r}(t) = (x(t), y(t))$ if $\mathbf{F}(x,y) = (f(x,y), g(x,y))$ (a *vector field* or *vector-valued function*, i.e., a function associating a vector to each point (x,y)) then the **work done** by the force $\mathbf{F}$ as you move along $\mathbf{r}(t)$ from $(x(t_0), y(t_0))$ to $(x(t_1), y(t_1))$ is given by



$$\int_{t_0}^{t_1} (f(x(t), y(t))x'(t) + g(x(t), y(t))y'(t))\, dt. \quad (*)$$

You encounter this very often in physics *e.g.* when you consider electrostatic energy of a particle of charge q in electric field $\mathbf{E}$. You get $-q\int \mathbf{E}\cdot d\mathbf{r}$

WHY?[Not part of course] The work done by the force $\mathbf{F}(x,y)$ is $\displaystyle\int_{t_0}^{t_1} \mathbf{F}(\mathbf{r}(t)).\frac{d\mathbf{r}}{dt}\, dt$, where the dot in the integral denotes the scalar product. Write this out in components to get (*).

Example 2.38. Find the above integral when $\mathbf{F}(x,y) = (x+y, x-y)$ and $\mathbf{r}(t) = (1+t^2, 2t)$, $t_0 = -1$, $t_1 = 1$.

Here, $f(x,y) = x+y$ and $g(x,y) = x-y$, while $x(t) = 1+t^2$ and $y(t) = 2t$. Thus $f(x(t), y(t)) = 1+t^2+2t = (1+t)^2$, while $g(x(t), y(t)) = 1+t^2-2t = (1-t)^2$.

So the required integral is

$$\int_{-1}^{1} \{(1+t)^2 2t + (1-t)^2 2\} dt.$$

If you expand the brackets and then integrate you should get the answer 8.

Note: The equation $(*)$ is often written as

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_C (f(x,y)dx + g(x,y)dy).$$

or, in 3 dimensions, as

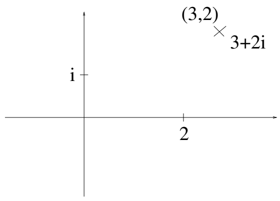
$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_C (f(x,y,z)dx + g(x,y,z)dy + h(x,y,z)dz).$$

Chapter 3

Complex numbers

3.1 What are complex numbers?

A *complex number* is simply a point in the (x, y)-plane,
but rather than writing (x, y) for the point we write $x + iy$.
So, for example, we write $3 + i2$ or $3 + 2i$ rather than $(3, 2)$,
2 rather than $(2, 0)$, and i rather than $(0, 1)$.



Just like we can add and multiply points of the real line, we would like to be able to add and multiply points of the complex plane. Addition of complex numbers is, as you'd expect,

$$(x_1 + iy_1) + (x_2 + iy_2) = x_1 + x_2 + i(y_1 + y_2),$$

so, for example, $(2 + i5) + (3 + i7) = 5 + i12$. The reason for the notation involving i described above is that we multiply complex numbers by doing the natural thing but putting $i^2 = -1$. So, for example,

$$(3 + i4)(2 + i5) = 6 - 20 + i(8 + 15) = -14 + i23.$$

Summarising: The set of complex numbers is called the complex (or *Argand*) plane,

$$\mathbf{C} = \{x + iy \mid x, y \text{ are real numbers}\}, \qquad \text{where } i^2 = -1.$$

The horizontal axis (ie the x-axis) is called the *real axis* and the vertical axis (ie the y-axis) is called the *imaginary axis*.

Complex numbers are at least as relevant to the real world as real numbers. The real numbers are 1-dimensional, but complex numbers are 2-dimensional.

See the 'links' tab on the course webpage for more on other aspects of complex numbers, including sites where you can read about the Riemann hypothesis.

Note: Complex number cannot be ordered, *i.e.* you **cannot** say whether z_1 is larger or smaller than z_2!.

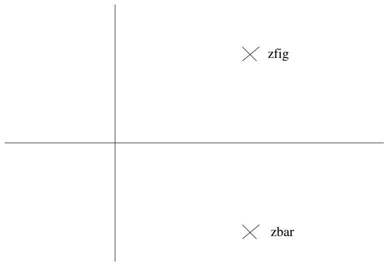
3.2 Conjugate and modulus

Definition 3.1. Let $z = x + iy$. We write Re $z = x$, the *real part* of z, and Im $z = y$, its *imaginary part*.

So, if $z = 3 - i4$ then Re $z = 3$ and Im $z = -4$.

Definition 3.2. The *complex conjugate* of z is:

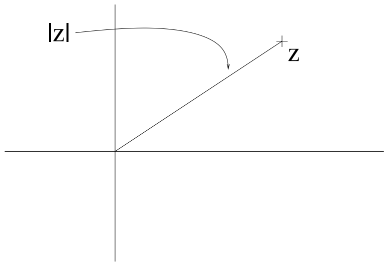
$$\boxed{\overline{z} = x - iy}$$



So the complex conjugate of z is obtained by reflecting z in the real axis. Note that Re $z = \frac{1}{2}(z + \overline{z})$ and Im $z = \frac{1}{2i}(z - \overline{z})$.

Definition 3.3. The modulus of z is:

$$\boxed{|z| = \sqrt{x^2 + y^2}}$$



This *nonnegative real* number is the distance of z from the origin.

Note that $|z|^2 = z\overline{z} = x^2 + y^2$.

As well as adding and multiplying as described above, we can also *divide* by (nonzero) complex numbers by using the fact that $z\overline{z}$ is real.

In fact :

$$\frac{w}{z} = \frac{w\overline{z}}{z\overline{z}} = \frac{w\overline{z}}{|z|^2}.$$

So, to evaluate a quotient of complex numbers, multiply top and bottom by the conjugate of the bottom.

Example 3.1. Find the real and imaginary parts of the complex number

$$z = \frac{2-i}{2+i3}.$$

$$\frac{2-i}{2+i3} = \frac{(2-i)(2-i3)}{(2+i3)(2-i3)} = \frac{4-3+i(-6-2)}{4+9} = \frac{1}{13} - i\frac{8}{13}.$$

Hence Re $z = 1/13$ and Im $z = -8/13$.

Example 3.2. Find the real and imaginary parts of the complex number

$$z = \frac{1}{1+i} + \frac{2-i}{3-i}.$$

You could, of course, write each fraction in real and imaginary parts using the method of the preceding example, and then just add. Here is an alternative method which might be quicker. First simplify:

$$\begin{aligned} z &= \frac{3-i+(2-i)(1+i)}{(1+i)(3-i)} \\ &= \frac{6}{4+2i} \\ &= \frac{3}{2+i}. \end{aligned}$$

Now multiply top and bottom by the conjugate of the denominator:

$$\begin{aligned} z &= \frac{3}{2+i}\,\frac{2-i}{2-i} \\ &= \frac{6-3i}{5}. \end{aligned}$$

Hence Re $z = 6/5$ and Im $z = -3/5$. We note that the modulus is:

$$|z| = \sqrt{\left(\frac{6}{5}\right)^2 + \left(\frac{3}{5}\right)^2} = \frac{3}{\sqrt{5}}.$$

3.2.1 Further properties

For any complex numbers z, w we have

(1) $$\boxed{\overline{zw} = \overline{z}\,\overline{w}} \qquad \boxed{\overline{z+w} = \overline{z} + \overline{w}}$$

(2) $$\boxed{|zw| = |z|\,|w|}$$

Proof of (1). Suppose $z = x + iy$ and $w = u + iv$. Then

$$zw = (xu - yv) + i(uy + xv),$$

and so by definition

$$\overline{zw} = (xu - yv) - i(uy + xv).$$

On the other hand,

$$\begin{aligned} \overline{z}\,\overline{w} &= (x - iy)(u - iv) \\ &= (xu - yv) + i(-xv - yu), \end{aligned}$$

which is the same. ∎

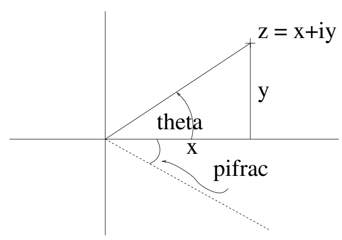
Proof of (2). No need to use real and imaginary parts here:

$$\begin{aligned} |zw|^2 &= (zw)\overline{(zw)} \\ &= zw\overline{z}\,\overline{w} \qquad \text{using (1)}, \\ &= z\overline{z}\,w\overline{w} = |z|^2|w|^2. \end{aligned}$$

Now take the square root of both sides and we're done. ∎

3.3 Polar representation of complex numbers

Definition 3.4. Let $z = x+iy \in \mathbf{C}$. We define the *argument* of z, $\arg(z)$, to be the angle θ between the x-axis and the line through (0,0) and z. The angle is measured anticlockwise from the x-axis. Negative angles are measured clockwise. Note that θ, $\theta + 2\pi$, $\theta + 4\pi$ etc all correspond to the *same* complex number z – the argument is only defined modulo 2π.



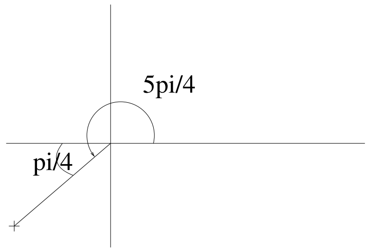
Trigonometry shows that

$$\boxed{\tan\theta = y/x}$$

but **beware:** care is needed since this equation doesn't completely determine θ: if we replace x and y by $-x$ and $-y$, then the value of y/x is unchanged, but the angle θ changes by π. As a result, plugging $\theta = \arctan(y/x)$ into your calculator might *not* give you the right value of θ – you'll need to draw a picture to decide whether θ is $\arctan(y/x)$ or $\arctan(y/x) + \pi$ (which might also be written as $\arctan(y/x) - \pi$). (Most often, once you have drawn the picture you'll be able to decide what $\arg(z)$ is without using your calculator at all, especially if you can remember the useful triangles from section 1.2.2)

Example 3.3.
Find the argument θ of $z = -1 - i$.

Solution:
Since $\tan\theta = y/x = 1$, we see that $\theta = \pi/4$ or $5\pi/4$. A picture shows you that $\theta = 5\pi/4$.

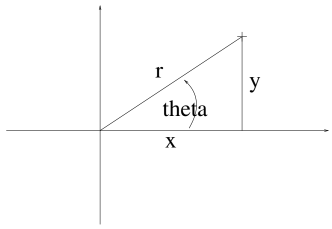


Note that if we write $r = |z|$ and $\theta = \arg z$ then the number $z \in \mathbf{C}$ is determined by its modulus r and argument θ, called the *polar coordinates* of the complex number:

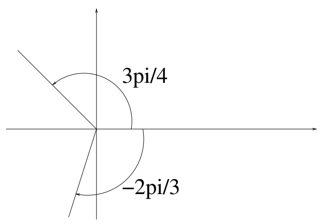
$$\boxed{\operatorname{Re} z = x = r\cos\theta, \qquad \operatorname{Im} z = y = r\sin\theta.}$$

or, in other words,

$$\boxed{z = r(\cos\theta + i\sin\theta).}$$



Remark. It is a common convention to take $-\pi < \arg z \le \pi$, and it is clear geometrically that this is always possible. However, as we'll see later, it is sometimes important to remember that adding any integer multiple of 2π to the argument of a complex number leaves it unchanged.

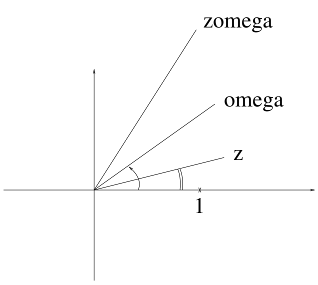


Addition in $\mathbf{C}$ is geometrically clear but, at the moment, multiplication is not. We'll get a geometrical picture of complex multiplication using polar coordinates.

Suppose $z = r(\cos\theta + i\sin\theta)$ and $w = s(\cos\psi + i\sin\psi)$. Then multiplying out gives

$$\begin{aligned} zw &= rs\left((\cos\theta\cos\psi - \sin\theta\sin\psi) + i(\cos\theta\sin\psi + \sin\theta\cos\psi)\right) \\ &= rs\left(\cos(\theta+\psi) + i\sin(\theta+\psi)\right). \end{aligned}$$

This says that $|zw| = |z|\,|w|$ (which we already knew) and $\arg zw = \arg z + \arg w$ (after adding a multiple of 2π to bring it in the range $(-\pi, \pi]$, if necessary).



In other words: **to multiply two complex numbers together you should multiply their moduli and add their arguments**, ie

$$\boxed{|zw| = |z|\,|w|, \quad \arg zw = \arg z + \arg w}$$

Example 3.4. Let $w = i$. This has modulus $|w| = 1$ and argument $\arg w = \pi/2$. So multiplication by i has the effect of rotating anticlockwise through angle $\pi/2$.

Example 3.5. Let $z = 1 + i$, and $w = \sqrt{3} + i$. Then $|z| = \sqrt{2}$, $|w| = \sqrt{3+1} = 2$. Also, $\arg z = \pi/4$, $\arg w = \pi/6$. Then $zw = (1+i)(\sqrt{3}+i) = (\sqrt{3}-1) + i(\sqrt{3}+1)$. So

$$|zw| = \sqrt{3 - 2\sqrt{3} + 1 + 3 + 2\sqrt{3} + 1} = \sqrt{8} = \sqrt{4}\sqrt{2} = 2\sqrt{2} = |z||w|.$$

If $\theta = \arg\,(zw)$, then $\tan\theta = \dfrac{\sqrt{3}+1}{\sqrt{3}-1}$. Using a calculator, $\theta = \dfrac{5\pi}{12} = \arg\, z + \arg\, w$.

Another consequence is:

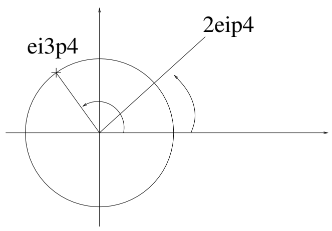
Theorem 3.1 (de Moivre)**.** For any positive whole number n,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Exercise: Prove this by induction.

Definition 3.5. For any real number $\theta \in \mathbf{R}$ we write

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$



This is a complex number on the unit circle (i.e. its modulus is one).

Then any complex number may be written

$$z = re^{i\theta},$$

where r is the modulus of z and θ is an argument.

Some properties:

1. $(e^{i\theta})^n = e^{in\theta}$. (This is what de Moivre says.)
2. $e^{i\theta}\, e^{i\psi} = e^{i(\theta+\psi)}$. (Since multiplication adds arguments.)
3. $e^{i0} = 1$. (Since $\cos\, 0 = 1$ and $\sin\, 0 = 0$.)

Remark. In other words, $e^{i\theta}$ behaves just like the exponential function of a *real* variable—hence our choice of notation.

Notice that if we replace θ by $-\theta$ we get

$$e^{-i\theta} = \cos\theta - i\sin\theta.$$

Adding and subtracting $e^{i\theta}$, respectively, this yields:

$$\boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.}$$

This should remind you of hyperbolic sine and cosine. Its first applications are to deriving trigonometric identities and evaluating trigonometric integrals.

Example 3.6. Let's derive the identity $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$.

$$\begin{aligned}\cos^2\theta &= \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^2 \\ &= \frac{1}{4}\left(e^{2i\theta}+e^{-2i\theta}+2\right) \\ &= \frac{\cos 2\theta + 1}{2}.\end{aligned}$$

Example 3.7. Express $\sin 3\theta$ as a polynomial in $\sin\theta$.

Well, $\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$, so $\sin 3\theta$ will be equal to the imaginary part of the RHS. But,

$$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta\, i\,\sin\theta + 3\cos\theta\, i^2\,\sin^2\theta + i^3\sin^3\theta.$$

So

$$\begin{aligned}\sin 3\theta &= 3\cos^2\theta\sin\theta - \sin^3\theta \\ &= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta \\ &= -4\sin^3\theta + 3\sin\theta.\end{aligned}$$

Example 3.8. Here's an integral.

$$I = \int_0^{\pi/2} \cos^6\theta\, d\theta.$$

Just as in the previous example we can expand the power (using the binomial theorem) and rearrange to pick out cosines of multiple angles:

$$\begin{aligned}\cos^6\theta &= \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^6 \\ &= \frac{1}{64}\left(e^{6i\theta}+6e^{4i\theta}+15e^{2i\theta}+20+15e^{-2i\theta}+6e^{-4i\theta}+e^{-6i\theta}\right) \\ &= \frac{1}{64}\left(2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20\right).\end{aligned}$$

Now we can integrate:

$$\begin{aligned}I &= \frac{1}{64}\left[\frac{1}{3}\sin 6\theta + 3\sin 4\theta + 15\sin 2\theta + 20\theta\right]_0^{\pi/2} \\ &= \frac{5\pi}{32}.\end{aligned}$$

There's a moral here. Real problems have real solutions—but the quickest way to get to them is often via the complex numbers.

3.4 Functions of a complex variable

We are now going to extend the familiar functions of a real variable to handle a *complex variable* $z = x + iy$.

3.4.1 The exponential function

We have already defined the exponential function for a *real* variable, and for a *pure imaginary* variable (motivated by de Moivre's Theorem). We now define:

$$\boxed{e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)}$$

We can read off from this its real and imaginary parts, Re $e^z = e^x \cos y$, Im $e^z = e^x \sin y$, and the modulus and argument

$$|e^z| = e^x, \qquad \arg e^z = y.$$

Example 3.9. Find the modulus of $e^{(1+i)(2-i)}$.

Well, $(1+i)(2-i) = 3+i$, so $|e^{(1+i)(2-i)}| = |e^{(3+i)}| = e^3$.

Try drawing a picture of how horizontal and vertical lines in the z-plane map under $z \mapsto e^z$, and compare with the animations of e^z on the course webpage.

Some properties:

1. Note that
$$e^{z+2\pi i} = e^z.$$
In other words, the exponential function is *periodic*, with imaginary period $2\pi i$.

2. For any complex numbers z, w we have
$$e^{z+w} = e^z e^w.$$

3. The complex conjugate is
$$\overline{(e^z)} = e^{\bar{z}}.$$

3.4.2 Trigonometric (circular) and hyperbolic functions

These will all be defined using the exponential function. We define

$$\boxed{\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.}$$

This makes sense because we have just defined e^z (and therefore e^{iz}), and it agrees with the usual expression in the case when z is real. It is an exercise to verify (using the properties of the complex exponential function) that sine and cosine have all the usual properties of the real case: periodicity with period 2π, addition formulae and so on.

Example 3.10. Show that $\sin(z+w) = \sin z \cos w + \cos z \sin w$.

$$\begin{aligned}
\text{RHS} &= \tfrac{1}{2i}(e^{iz} - e^{-iz})\tfrac{1}{2}(e^{iw} + e^{-iw}) + \tfrac{1}{2}(e^{iz} + e^{-iz})\tfrac{1}{2i}(e^{iw} - e^{-iw}) \\
&= \tfrac{1}{4i}(e^{i(z+w)} + e^{i(z-w)} - e^{i(w-z)} - e^{-i(w+z)}) + (\text{similar terms}) \\
&= \tfrac{1}{2i}(e^{i(z+w)} - e^{-i(z+w)}) = \sin(z+w) = \text{ LHS}.
\end{aligned}$$

While we're about it we should also define:

$$\boxed{\cosh z = \frac{e^z + e^{-z}}{2}, \qquad \sinh z = \frac{e^z - e^{-z}}{2}.}$$

These have exactly the same addition formulae, and so on, as for the real case. Unlike the real case, sinh and cosh are now periodic, with period $2\pi i$.

In fact, you can see that for any complex number z,

$$\begin{cases} \cos iz = \cosh z \\ \sin iz = i \sinh z, \end{cases} \qquad \begin{cases} \cosh iz = \cos z \\ \sinh iz = i \sin z. \end{cases}$$

In other words, the hyperbolic functions are obtained from the circular functions *by a* $90°$ *rotation of the complex plane*, and vice versa.

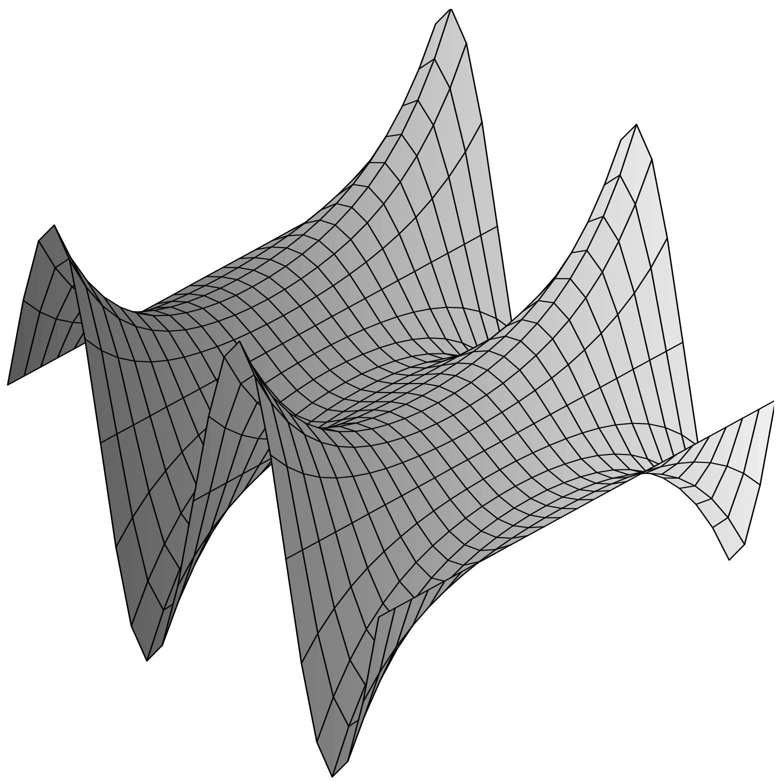
Let's calculate the real and imaginary parts of the function $\sin z$. (Cosine and the hyperbolic functions are entirely similar.)

$$\begin{aligned} \sin z &= \sin(x+iy) \\ &= \sin x \cos(iy) + \cos x \sin(iy) \\ &= \sin x \cosh y + i \cos x \sinh y. \end{aligned}$$

This says that for $z = x + iy$,

$$\text{Re } \sin z = \sin x \cosh y, \qquad \text{Im } \sin z = \cos x \sinh y.$$

Can you visualise the graphs of these two (real) functions over the complex plane? Here is the real part, in the range $0 < x < 4\pi$ and $-3 < y < 3$. Notice that the profile in the real direction is that of $\sin x$, and in the imaginary direction is that of $\cosh y$.



What about the modulus? This is given by:

$$\begin{aligned} |\sin z|^2 &= (\sin x \cosh y)^2 + (\cos x \sinh y)^2 \\ &= \sin^2 x (1 + \sinh^2 y) + \cos^2 x \sinh^2 y \\ &= \sin^2 x + (\sin^2 x + \cos^2 x) \sinh^2 y \\ &= \sin^2 x + \sinh^2 y. \end{aligned}$$

Hence
$$|\sin z| = \sqrt{\sin^2 x + \sinh^2 y}.$$
Note that, as the picture above also suggests, $|\sin z| \to \infty$ as $z \to \infty$ in the imaginary direction.

3.4.3 The derivative of $e^{i\theta}$

Recall the definition:
$$e^{i\theta} = \cos\theta + i\sin\theta$$
Taking the derivative with respect to θ,
$$\frac{d}{d\theta}\, e^{i\theta} = -\sin\theta + i\cos\theta = i(\cos\theta + i\sin\theta) = ie^{i\theta}$$
and so
$$\boxed{\frac{d}{d\theta}\, e^{i\theta} = ie^{i\theta}}$$
By the chain rule, this implies $\frac{d}{d\theta}\, e^{i\lambda\theta} = i\lambda e^{i\lambda\theta}$, and, differentiating again, $\frac{d^2}{d\theta^2}\, e^{i\lambda\theta} = -\lambda^2 e^{i\lambda\theta}$. All of this means that the function $e^{i\lambda\theta}$ solves
$$\frac{d^2}{d\theta^2} f + \lambda^2 f = 0\,,$$
an equation which crops up all the time in physics.

3.5 Equations in a complex variable

3.5.1 Transcendental equations

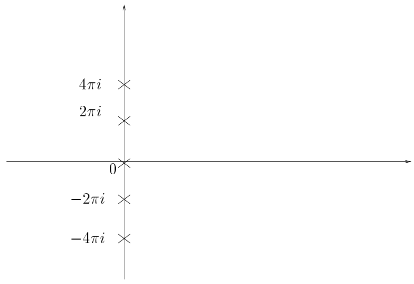
A good way to get to know these complex functions is to try solving some equations involving them, and I'll give some examples. The first example is fundamental, and will be used repeatedly in what follows.

Example 3.11. The Basic Example. Find all complex solutions of the equation
$$e^z = 1.$$
Of course, the only *real* solution is $z = 0$. But the complex plane is a bigger place. And to start with, periodicity (i.e. $e^{z+2\pi i} = e^z$) already tells us that there must be infinitely many solutions
$$(\star) \qquad z = 2\pi i m, \quad \text{for any whole number } m.$$
So: are there any other solutions? If $z = x + iy$ then (by definition of the exponential function) we have to solve
$$1 = e^x \cos y + ie^x \sin y.$$
Equating real and imaginary parts, this gives two equations
$$\begin{cases} e^x \cos y = 1 \\ e^x \sin y = 0. \end{cases}$$

Now x is real, so $e^x > 0$ is real and positive. Therefore the second equation says that $\sin y = 0$, and hence (since y is real) $y = n\pi$ for some whole number n. But $\cos n\pi = (-1)^n$, so the first equation says

$$(-1)^n e^x = 1.$$

This forces n to be even, $n = 2m$ say, and $x = 0$. Hence $z = 2m\pi i$, and we conclude that *all* the solutions are those given by ($\star$).



Example 3.12. Find all complex solutions of the equation

$$\sinh z = 0.$$

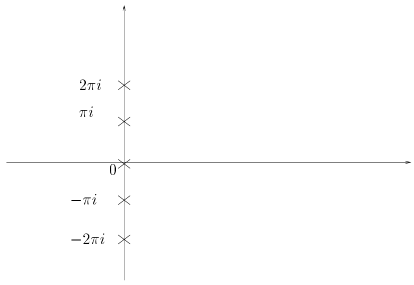
Again, in the confines of the real world you will only see one solution $z = 0$—but don't be satisfied with that. We have to solve

$$\frac{e^z - e^{-z}}{2} = 0,$$

or, multiplying through by $2e^z$,

$$e^{2z} = 1.$$

Example 3.11 tells us that the solutions of this are precisely:



$$2z = 2\pi i m, \quad \text{for any whole number } m,$$

or in other words, $z = m\pi i$ for any whole number m.

Example 3.13. Solve the equation

$$\cos z = 0.$$

This is very similar to the previous example. We have to solve

$$\frac{e^{iz} + e^{-iz}}{2} = 0,$$

or equivalently,

$$e^{2iz} = -1.$$

As in the last example, we would like to use the result of Example 3.11, but in this case we can't do so directly. To get the equation into the right form, we write $-1 = e^{\pi i}$. Then the equation says $e^{2iz} = e^{\pi i}$, or

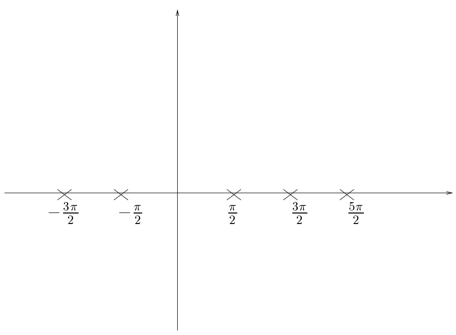
$$e^{2iz - \pi i} = 1,$$

and now we can use Example 3.11:

$$2iz - \pi i = 2m\pi i, \quad \text{for any whole number } m.$$

Tidying this up, it says that the general solution is

$$z = \pi(m + \frac{1}{2}) \quad \text{for any whole number } m.$$



In other words, the solutions of $\cos z = 0$ are exactly the real solutions you already know, and no others.

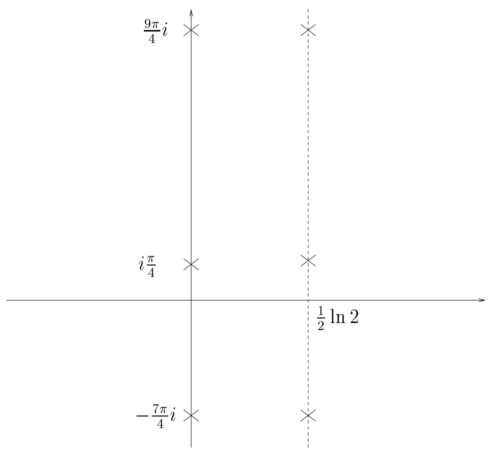
Example 3.14. Find all complex solutions of the equation

$$e^z = 1 + i.$$

We want to use a similar method to the previous example - so we first write the the right-hand side in polar form. The equation then becomes

$$e^z = \sqrt{2}e^{i\pi/4}.$$

We really want this in the form $e^{??} = 1$, and to do this we write



$$\sqrt{2} = 2^{1/2} = e^{\frac{1}{2}\ln 2},$$

so that the original equation reads

$$e^z = e^{\frac{1}{2}\ln 2}\ e^{i\frac{\pi}{4}} = e^{\frac{1}{2}\ln 2 + i\frac{\pi}{4}}.$$

Dividing by the right-hand side, we get

$$e^{z - \frac{1}{2}\ln 2 - i\frac{\pi}{4}} = 1.$$

We can now use Example 3.11. This says that $z - \frac{1}{2}\ln 2 - i\frac{\pi}{4} = 2m\pi i$ or:

$$z = \frac{1}{2}\ln 2 + i\pi\left(2m + \frac{1}{4}\right), \quad \text{for any whole number } m.$$

3.5.2 Algebraic equations

The examples above, involving the exponential function and so on, are examples of *transcendental equations*, and typically they have infinitely many solutions. *Algebraic equations* involve just polynomials, and have only finitely many solutions.

Here is the general method for solving equations of the form

$$\boxed{z^n = a,}$$

where a is some (real or) complex number.

First write a in polar form as $a = re^{i\theta}$, but, remembering the periodicity of e^z we have that $a = re^{i(\theta + 2\pi m)}$ for $m = 0, 1, 2, \ldots$. The equation now becomes

$$z^n = re^{i(\theta+2\pi m)} \qquad \text{for } m = 0, 1, 2, \ldots ,$$

so the solutions are

$$z = \sqrt[n]{r}\ e^{i\theta/n},\ \sqrt[n]{r}\ e^{i\frac{\theta+2\pi}{n}}\ \sqrt[n]{r}\ e^{i\frac{\theta+4\pi}{n}},\ \sqrt[n]{r}\ e^{i\frac{\theta+6\pi}{n}}, \ldots.$$

Notice, though, that after you've written down the first n solutions then the periodicity of e tells you that you are repeating solutions you've already written down.

An example should (as usual) make this rather more clear.

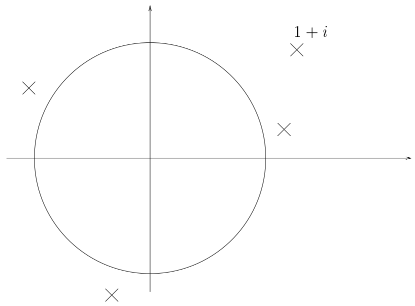
Example 3.15. Solve the equation

$$z^3 = 1 + i.$$

In this case $n = 3$ and $a = 1 + i = \sqrt{2}\ e^{i\left(\frac{\pi}{4}+2\pi m\right)}$, $m = 0, 1, 2, \ldots$.
Hence the solutions are

$$z = 2^{1/6}e^{i\pi/12}, \quad 2^{1/6}e^{i3\pi/4}, \quad 2^{1/6}e^{i17\pi/12},$$

since the next solution you'd write down would be for $m = 3$ which would give $z = 2^{1/6}e^{i25\pi/12} = 2^{1/6}e^{i\pi/12}$.
Here is a picture of the solutions.



Complex roots of unity

The complex n-th roots of unity are the solutions of the equation

$$\boxed{z^n = 1.}$$

To solve this, we write the right hand side in polar form, $1 = e^{i2\pi m}$, $m = 0, 1, 2, \ldots$, so the equation becomes

$$z^n = e^{i2\pi m}, \quad m = 0, 1, 2, \ldots.$$

The solutions are

$$z = 1 (= e^{i0}), e^{i\frac{2\pi}{n}}, e^{i\frac{4\pi}{n}}, \ldots e^{i\frac{2(n-1)\pi}{n}}.$$

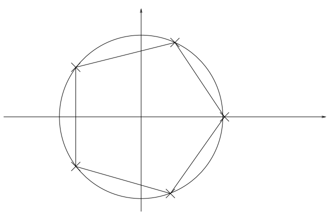
Note that the solutions are just the powers of $e^{i\frac{2\pi}{n}}$. So, if we put $\omega = e^{i\frac{2\pi}{n}}$ then the solutions are

$$z = 1, \omega, \omega^2, \omega^3, \ldots \omega^{n-1}.$$

For example, taking $n = 5$ we find that the fifth roots of unity (ie the solutions to $z^5 = 1$) are

$$z = 1, \omega, \omega^2, \omega^3, \omega^4,$$

where $\omega = e^{i\frac{2\pi}{5}}$. Here is a picture of these solutions in the complex plane. They form a regular pentagon with one vertex at $z = 1$.



This happens in general for complex n-th roots of unity, we get n points, all on the unit circle forming a regular n-sided polygon, one of whose vertices is at $z = 1$.

Example 3.16. Find all complex solutions of the equation

$$z^6 - 2z^3 + 2 = 0.$$

As you've already noticed (yes?), this is a quadratic in z^3 so we can solve for z^3 using the quadratic formula to get

$$z^3 = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i.$$

(If you prefer, you can complete the square:

$$(z^3 - 1)^2 + 1 = 0,$$

so that $z^3 - 1 = \pm i$, ie $z^3 = 1 \pm i$, as before.)

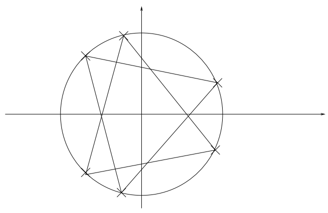
So, we now need to solve the two equations

$$z^3 = 1 + i \quad \text{and } z^3 = 1 - i.$$

The first of these two equations have been solved, by some coincidence, in Example 3.15; so let's consider the second one. Here we can take $a = 1 - i = \sqrt{2}e^{-\pi i/4}$, and by the same method as in Example 3.15, we find three solutions

$$z = 2^{1/6}e^{-i\pi/12}, \quad 2^{1/6}e^{7i\pi/12}, \quad 2^{1/6}e^{5i\pi/4}.$$

So the six solutions of the original equation are the vertices of two equilateral triangles on the circle of radius $2^{1/6}$ centred at the origin.



Remark Notice that in Example 3.16 the set of solutions is symmetric in the real axis. This is because if

$$a_0z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n = 0$$

is any equation with *real* coefficients $a_0, a_1, \ldots, a_n$, and z is a solution, then the complex conjugate $\overline{z}$ is also a solution. (To see this, just conjugate the whole equation.) If the coefficients are not all real (as in Example 3.15) then this is no longer true.

Finally, we quote one of the most important facts about complex numbers.

3.5.3 The Fundamental Theorem of Algebra

Theorem 3.2 (The fundamental theorem of algebra)**.** Every polynomial equation

$$a_0z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n = 0$$

with (real or) complex coefficients $a_0, a_1, \ldots, a_n \in \mathbf{C}$ has a (real or) complex solution z.

Remark Once one root, say z_0, is known, the polynomial can be divided by $(z - z_0)$ and the theorem used again to find another root, and so on. Hence the theorem implies that, with multiplicities, every polynomial of degree n has n complex roots. As a quick check, notice that examples 3.15 and 3.16 above involved polynomials of degrees 3 and 6, and had 3 and 6 complex roots respectively.

Chapter 4

Analysis of real numbers and real-valued functions

4.1 Various types of real number

Whole numbers (or **integers**). $\ldots -3, -2, -1, 0, 1, 2, 3, \ldots$. Although the sum, difference and product of two integers is an integer, the quotient of two integers is not usually an integer.
Rational numbers. These are real numbers of the form m/n where m and n are integers with $n \neq 0$. So, for example, 2/5, -17/9, 12/8 are all rational numbers. Note that the sum, difference, product and quotient of two rational numbers is a rational number (except that you can't divide by zero!).
Irrational numbers. These are real numbers that are not rational. So, for example, π, e, $\sqrt{2}$ are all irrational numbers. If you carry out algebraic operations on irrational numbers the answer could be rational or it could be irrational.
Exercise Assuming that $\sqrt{3}$ and $\sqrt{5}$ are irrational (which they are - see below), show that $\sqrt{3} + \sqrt{5}$ is irrational.

However, it is easy to find an example of two irrational numbers whose sum is rational; it is also easy to find an example of two irrational numbers whose product is rational.

Question. What happens if you add (or multiply) a rational with an irrational?

Exercise If a and b are rational with $a \neq 0$ and $b > 0$, and if $(a + \sqrt{b})^2$ is rational, prove that $\sqrt{b}$ is rational.

IMPORTANT: Infinity (ie ∞) is NOT a real number!

Here are a couple of facts about **integers** which will be useful.
(a) If n is an even integer then n^2 is also even. (Why this is true?)
(b) If n is an odd integer then n^2 is also odd. (Why this is true?)

Theorem 4.1. $\sqrt{2}$ is irrational.

Proof. We **assume** that there is a rational number whose square is equal to 2 and obtain a contradiction. Cancelling common factors from the numerator and the denominator, our assumption implies that there exist integers m and n with $n \neq 0$, and with no common factors, such that

$$\left(\frac{m}{n}\right)^2 = 2.$$

Then

$$m^2 = 2n^2, \tag{4.1}$$

so that m^2 is even. So, by result **(b)** from the previous page, we see that m itself is even. Hence $m = 2r$ for some integer r. Then, from (4.1), $4r^2 = 2n^2$ so that $2r^2 = n^2$. Thus n^2 and hence n is even. Thus 2 divides both m and n which contradicts the statement that m and n have no common factors. We have seen that our initial assumption leads to a contradiction, so our initial assumption (the existence of a rational number whose square is equal to 2) is false. This proves the theorem. ∎

Challenge Try a similar thing for $\sqrt{3}$, ${}^3\sqrt{2}$ etc.

In fact: **If n is an integer then $\sqrt{n}$ is irrational unless n is a perfect square.**
Question: How are the rational and irrational numbers distributed on the real line?

4.2 Limits of functions of a real variable

Let f be a real valued function, such that $f(x)$ is defined for all x near a point $a \in \mathbb{R}$, but not necessarily defined at the point a itself. If the value of $f(x)$ approaches a real number l as x approaches a from both sides, then we say that l is the *limit* of the function at a and write

$$\lim_{x \to a} f(x) = l.$$

Example 4.1. 1. $\lim_{x \to 2} x^2 = 4$.

2. $\lim_{x \to 0} 1/(1 + x) = 1$.

3. $\lim_{x \to 0} \cos x = 1$.

This gives the first method for evaluating limits.
Method 1. If $f(x)$ is continuous at $x = a$ then $\lim_{\mathbf{x} \to \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a})$.

Most decent functions are continuous whenever they are defined. For example, polynomials, trig functions, rational functions (ie quotients of polynomials), hyperbolic functions etc all fall into this category.

Example 4.2. Let

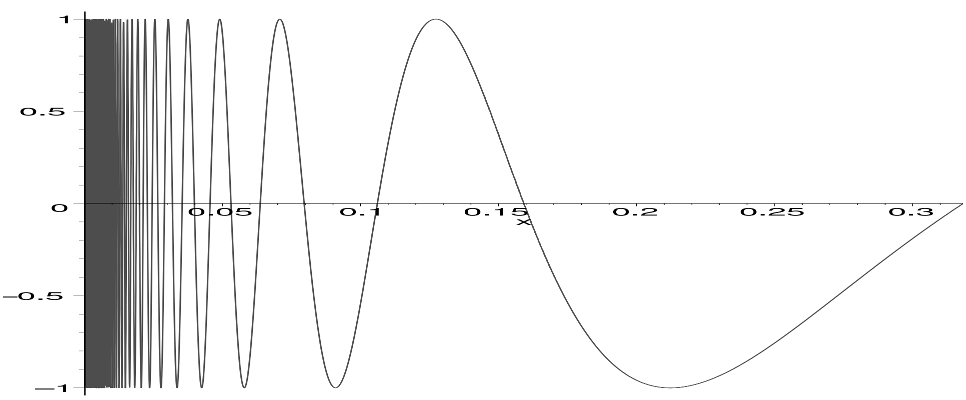
$$f(x) = \begin{cases} 1 + x & \text{if } x \geq 0, \\ -1 + x & \text{if } x < 0, \end{cases}$$

and let $a = 0$. In this case $f(x)$ tends to $+1$ as x approaches 0 from above and tends to a different value -1 as x approaches 0 from below. So in this case a limit does not exist.

Example 4.3. Let

$$h(x) = \sin \frac{1}{x} \quad \text{for } x \neq 0.$$

This has no limit as $x \to 0$. Here is what the function looks like over $(0, 1/\pi)$:



If a function $f(x)$, instead of approaching a real number, becomes arbitrarily large as $x \to a$, then we write

$$\lim_{x\to a} f(x) = \infty.$$

Example 4.4. $\lim_{x\to 0} 1/x^2 = \infty$. Note that this is an example in which the function is not defined *at* the point a – but it is defined for all x *near* the point a.

Finally, instead of a point of the real number line, a might be $\pm\infty$. For example,

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

Here are a couple more examples.

Example 4.5. Find $\lim_{x\to 1} \frac{x^2+x-2}{x^2+2x-3}$. Mmm - get 0/0 at the limit point.

Example 4.6. Find $\lim_{x\to 0} \frac{\sin x}{x}$. Once again we get 0/0 at the limit point.

Question. Why do we care about limits like the above?
Answer. Well, we do think differentiation is useful, don't we, and we do remember that the derivative f' of a function f at $x = a$ is given by

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

We note that this is always 0/0 at the limit point.

Example 4.7. Let $f(x) = \sin x$. Find $f'(0)$. Well,

$$\lim_{h\to 0} \frac{\sin h - \sin 0}{h} = \lim_{h\to 0} \frac{\sin h}{h} =?$$

Example 4.8. Let $f(x) = x^2$. Find $f'(3)$.

$$\lim_{h\to 0} \frac{(3+h)^2 - 3^2}{h} = \lim_{h\to 0} \frac{6h+h^2}{h} = \lim_{h\to 0}(6+h) = 6.$$

In fact, the method of this above example gives the clue how to do Example 4.5.

Method 2. If, when you put in the limiting value of x you get 0/0, then try to cancel the (same) factor from the top and bottom that gives these zeros. Hopefully, what's left will then be continuous at the limit point, so can simply be evaluated at the limit point (Method 1).

So, here is how to do Example 4.5.

$$\lim_{x\to 1} \frac{x^2+x-2}{x^2+2x-3} = \lim_{x\to 1} \frac{(x-1)(x+2)}{(x-1)(x+3)} = \lim_{x\to 1} \frac{(x+2)}{(x+3)} = \frac{3}{4}.$$

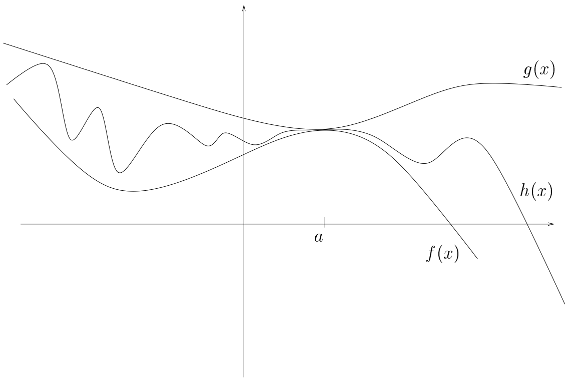
We'll now move on to another useful method, motivated by the following example.

Example 4.9. Find $\lim_{x\to 0} x\sin(1/x)$. Since $x \to 0$ and $\sin(1/x)$ is trapped between ± 1, the x should kill the $\sin(1/x)$ and the limit should be zero.

Method 3. (Pinching Theorem). Suppose that $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \lim_{\mathbf{x}\to\mathbf{a}} \mathbf{g}(\mathbf{x}) = \boldsymbol{l}$ **and that**

$$\mathbf{f}(\mathbf{x}) \le \mathbf{h}(\mathbf{x}) \le \mathbf{g}(\mathbf{x})$$

for all x near a (but not necessarily at $x = a$, where the functions don't even need to be defined). Then $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{h}(\mathbf{x}) = \boldsymbol{l}$.



So, going back to Example 4.9

$$-|x| \le x\sin(1/x) \le |x|,$$

and $\lim_{x\to 0} -|x| = \lim_{x\to 0} |x| = 0$. So, by the Pinching theorem, $x\sin(1/x)$ has a limit as $x \to 0$ and, in fact,

$$\lim_{x\to 0} x\sin(1/x) = 0.$$

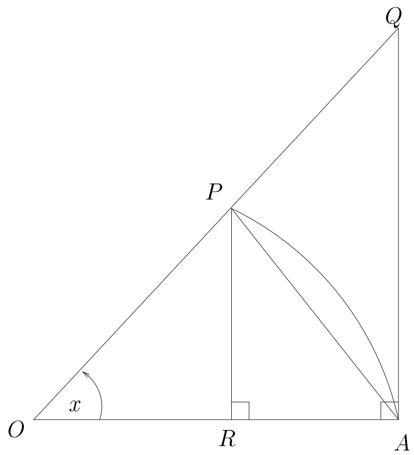
(For a picture see Example 4.18 below.)

Example 4.10. Find $\lim_{x\to\infty} \frac{\cos x}{x}$.

Here, $-\dfrac{1}{x} \le \dfrac{\cos x}{x} \le \dfrac{1}{x}$, so, by the Pinching theorem, the required limit is 0.

We still can't do Example 4.6, but here is a geometrical way to evaluate $\lim_{x\to 0} \frac{\sin x}{x}$ which uses the Pinching theorem.

Consider the following diagram, where the curve AP is an arc of the circle centre O, radius 1:



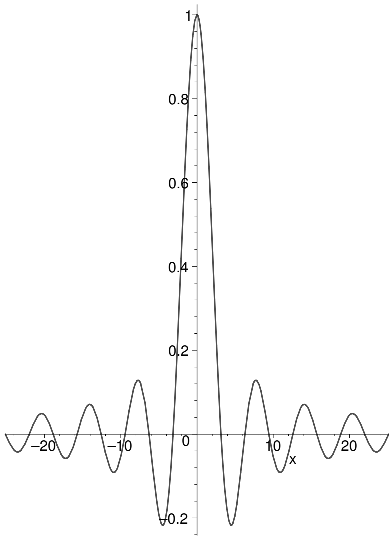
For $0 < x < \pi/2$, trigonometry shows that length $QA = \tan x$ and length $PR = \sin x$. Thus area $\Delta OAP = \frac{1}{2}\sin x$, area sector $OAP = \frac{1}{2}x$ and area $\Delta OAQ = \frac{1}{2}\tan x$. So, for $0 < x < \pi/2$, we have that $\sin x < x < \tan x$. Dividing by $\sin x$ we get $1 < \dfrac{x}{\sin x} < \dfrac{1}{\cos x}$, so taking reciprocals gives

$$1 > \frac{\sin x}{x} > \cos x, \quad 0 < x < \pi/2.$$

However, 1, $\frac{\sin x}{x}$ and $\cos x$ are all even functions so the above inequality is true for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $x \neq 0$. But $\cos x$ is continuous, so $\lim_{x\to 0}\cos x = 1$. It now follows from the Pinching theorem that

$$\boxed{\lim_{\mathbf{x}\to\mathbf{0}} \frac{\sin \mathbf{x}}{\mathbf{x}} = \mathbf{1}}$$

The function $\frac{\sin x}{x}$ is even, and its graph looks like (between $\pm 8\pi$):



(You could use the Pinching theorem to show that $\lim_{x\to\infty}\frac{\sin x}{x} = 0$.)

Method 4. Use $\lim_{\mathbf{x}\to\mathbf{0}}\frac{\sin\mathbf{x}}{\mathbf{x}} = \mathbf{1}$ to find other limits of a similar nature.

Example 4.11.

$$\lim_{x\to 0}\frac{\sin 3x}{x} = \lim_{x\to 0} 3\frac{\sin 3x}{3x} = \lim_{y\to 0} 3\frac{\sin y}{y} = 3.$$

A complicated limit can often be broken down into several easier ones using the following method.

Method 5 (The Calculus of Limits Theorem). Suppose that $\lim_{\mathbf{x}\to\mathbf{a}}\mathbf{f}(\mathbf{x}) = \boldsymbol{l}$ and $\lim_{\mathbf{x}\to\mathbf{a}}\mathbf{g}(\mathbf{x}) = \boldsymbol{m}$ with $\boldsymbol{l}$ and $\boldsymbol{m}$ finite. Then:

(i) $\lim_{\mathbf{x}\to\mathbf{a}}(\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})) = \boldsymbol{l} + \boldsymbol{m}$;

(ii) $\lim_{\mathbf{x}\to\mathbf{a}}\mathbf{f}(\mathbf{x})\mathbf{g}(\mathbf{x}) = \boldsymbol{lm}$;

(iii) $\lim_{\mathbf{x}\to\mathbf{a}}\mathbf{f}(\mathbf{x})/\mathbf{g}(\mathbf{x}) = \boldsymbol{l}/\boldsymbol{m}$ **provided that $\boldsymbol{m} \neq 0$.**

We'll use methods 4 and 5 to find the following limits.

Example 4.12. Find $\lim_{x\to 0}\frac{\tan 3x}{x}$.

Here, $\lim_{x\to 0}\frac{\tan 3x}{x} = \lim_{x\to 0}\frac{\sin 3x}{x}\frac{1}{\cos 3x}$. We know from Example 4.11 that $\lim_{x\to 0}\frac{\sin 3x}{x} = 3$ and continuity of $\cos 3x$ at $x = 0$ shows that $\lim_{x\to 0}\frac{1}{\cos 3x} = 1$. So, by COLT, the required limit is 3.1=3.

Example 4.13. Find $\lim_{x\to 0}\frac{1-\cos 2x}{x^2}$.

Both top and bottom tend to 0 as $x \to 0$. Start with the double angle formula:

$$\cos 2x = 1 - 2\sin^2 x.$$

Using this identity and the limit in Method 4, we find:

$$\lim_{x\to 0}\frac{1-\cos 2x}{x^2} = \lim_{x\to 0} 2\left(\frac{\sin x}{x}\right)^2 = 2.1^2 = 2.$$

Example 4.14. Find $\lim\limits_{x\to\infty} \dfrac{2x^2 - 7x + 2}{3x^2 + 4}$.

The x^2 term on the top and bottom makes them both go to infinity as $x \to \infty$. To get over this we divide top and bottom by this nasty term so that all limits become finite. Here goes:

$$\lim_{x\to\infty} \frac{2x^2 - 7x + 2}{3x^2 + 4} = \lim_{x\to\infty} \frac{2 - (7/x) + (2/x^2)}{3 + (4/x^2)} = \frac{2 - 0 + 0}{3 + 0} = \frac{2}{3}.$$

Example 4.15. Find $\lim\limits_{x\to\infty} \dfrac{2x}{x + \cos x}$.

Here, it is the x term on the top and bottom which makes them both go to infinity as $x \to \infty$, so we divide top and bottom by this. We get

$$\lim_{x\to\infty} \frac{2x}{x + \cos x} = \lim_{x\to\infty} \frac{2}{1 + \frac{\cos x}{x}} = \frac{2}{1 + 0} = 2, \quad \text{using Example 4.10.}$$

4.3 Continuous functions

A function $f\colon I \to \mathbb{R}$ is *continuous* at a point a in its domain $I = (x_0, x_1)$ (or $I = [x_0, x_1]$) or $I = \mathbb{R}$ etc.) if

$$\lim_{x\to a} f(x) = f(a).$$

Example 4.16. The function f given by

$$f(x) = \begin{cases} 1 + x & \text{if } x \geq 0, \\ -1 + x & \text{if } x < 0, \end{cases}$$

is defined at the origin but is not continuous there, by Example 4.2. ∎

Example 4.17. The function f given by

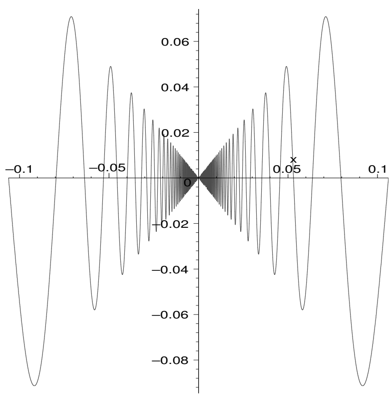
$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is also discontinuous at the origin because of Example 4.3.

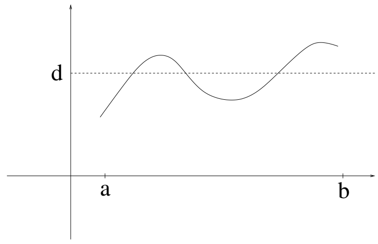
Example 4.18. The function f given by

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

on the other hand, is continuous at the origin, by Example 4.9. Here is the graph (plotted between $\pm 1/3\pi$):



Theorem 4.2 (Intermediate value theorem, or IVT)**.** If f is continuous between a and b ($f\colon [a,b] \to \mathbb{R}$ is continous) and if d lies between $f(a)$ and $f(b)$ then $f(c) = d$ for at least one number c between a and b. (So, a continuous function takes all values between its starting value and its end value.)



Example 4.19. Show that $x^8 - 9x^2 + 6 = 0$ has at least one solution x between $x = 1$ and $x = 2$.

To see this we note that $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^8 - 9x^2 + 6$ is continuous. Since 0 lies between $f(1) = -2$ and $f(2) = 226$ we see that $f(c) = 0$ for some c between $x = 1$ and $x = 2$. This is the required solution.

4.4 Differentiable functions

A function f is *differentiable* at a point $a \in \mathbb{R}$ if f is defined near a and if there exists a limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Letting $h = x - a$, we can also write this as

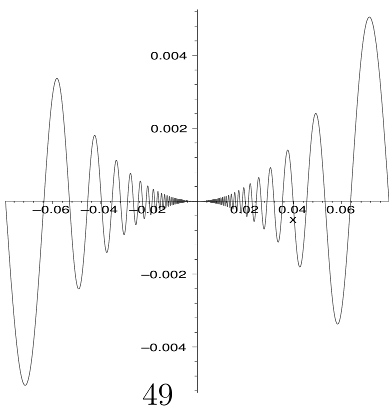
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Example 4.20.

1. The function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|$ is continuous at $x = 0$, but *not* differentiable there.

2. The function $f\colon \mathbb{R} \to \mathbb{R}$ given by
$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$
also fails to be differentiable at $x = 0$, by Example 4.3.

3. The function $f\colon \mathbb{R} \to \mathbb{R}$ given by
$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$
on the other hand, *is* differentiable at the origin, with derivative
$$f'(0) = \lim_{x \to 0} \frac{x^2 \sin(1/x)}{x} = \lim_{x \to 0} x \sin(1/x) = 0,$$
(by Example 4.9). Here is a plot of this function between $\pm 1/4\pi$:



Remark. If a function f is differentiable at a point a then it is automatically continuous there too. This is because for the limit of $(f(x) - f(a))/(x - a)$ as $x \to a$ to exist, we must have $(f(x) - f(a)) \to 0$ as $x \to a$, or in other words $\lim_{x \to a} f(x) = f(a)$.

One can show that all the usual functions f given by $f(x) = x^n$, $f(x) = e^x$, and the trigonometric (circular) and hyperbolic functions are differentiable (and hence continuous) at all points at which they are defined.

Example 4.21. Let's compute the derivative of sin. We have to consider

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \left(\frac{\cos h - 1}{h}\right) \sin x + \left(\frac{\sin h}{h}\right) \cos x.\end{aligned}$$

You could use Method 4 to show that $(\cos h - 1)/h \to 0$ as $h \to 0$ (use $\cos h = 1 - 2\sin^2(h/2)$), and we know that $(\sin h)/h \to 1$ as $h \to 0$. Hence, by COLT, the limit as $h \to 0$ is

$$f'(x) = \cos x.$$

4.5 Three important theorems

Definition. Let $f(x)$ be defined in some open interval (a, b), and let $c \in (a, b)$. We say that f has a *local maximum* at c if there exists a number $\delta > 0$ such that $f(x) \le f(c)$ for all $x \in (c - \delta, c + \delta)$. There is a similar definition for local minimum.

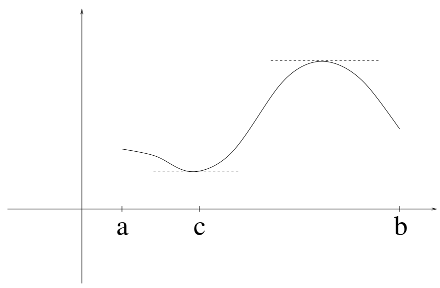
Theorem 4.3 (Theorem.)**.** If $f(x)$ is differentiable at $x = c$, and has a local maximum or minimum at c, then $f'(c) = 0$.

Proof. Suppose that f has a local maximum at c (the proof for local minimum is similar). So there exists $\delta > 0$ such that $f(c + h) \le f(c)$ provided $|h| < \delta$. Consider the function $R(h) = [f(c + h) - f(c)]/h$. When $h > 0$, we have $R(h) \le 0$, and when $h < 0$, we have $R(h) \ge 0$. But since f is differentiable at c, $\lim_{h \to 0} R(h)$ exists and is $f'(c)$. If we take a sequence of values $\{h_n\}$ with $h_n < 0$ and $h_n \to 0$ as $n \to \infty$, we get $\lim_{n \to \infty} R(h_n) = f'(c) \ge 0$; and if we take a sequence of values $\{h_n\}$ with $h_n > 0$, tending to 0, then we get $\lim_{n \to \infty} R(h_n) = f'(c) \le 0$. Thus $f'(c) = 0$.

In each of the following theorems we assume that **$\mathbf{f, g}$ are continuous functions defined and differentiable for all $\mathbf{x}$ between $\mathbf{x = a}$ and $\mathbf{x = b}$, where $\mathbf{a < b}$.**

Theorem 4.4 (Rolle's theorem)**.** Suppose that $f(a) = f(b)$. Then there is at least one point c between $x = a$ and $x = b$ for which

$$f'(c) = 0.$$



If we apply Rolle's theorem to the function h given by

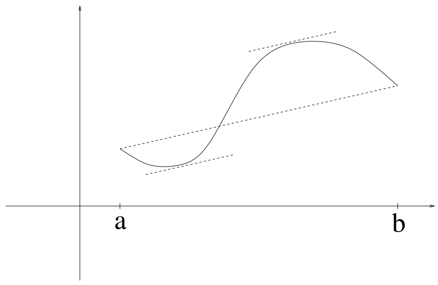
$$h(x) = f(x) - \left(\frac{f(b) - f(a)}{b - a}\right)(x - a)$$

then we get the following generalisation:

Theorem 4.5 (Mean value theorem, or MVT)**.** There is at least one point c between $x = a$ and $x = b$ for which

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

(ie the derivative takes on its mean value at some point c between $x = a$ and $x = b$). ∎



This theorem is important for its consequences, several of which you already know. For example, a function f is **increasing** between $x = a$ and $x = b$ if $f(x_2) > f(x_1)$ whenever $x_2 > x_1$ lie between $x = a$ and $x = b$. It is a consequence of MVT that if $f'(x) > 0$ between $x = a$ and $x = b$ then f is increasing between $x = a$ and $x = b$. (Similar remarks hold for **decreasing**).

Example 4.22. Let $f(x) = x^3 + 3x^2 - 9x + 6$. Show that f is decreasing between $x = -3$ and $x = 1$.

Well, $f'(x) = 3x^2 + 6x - 9 = 3(x + 3)(x - 1)$ which is less than zero between $x = -3$ and $x = 1$. So $f(x)$ is decreasing between $x = -3$ and $x = 1$. we also see that $f'(x) > 0$ for $x > 1$ and for $x < -3$, so that $f(x)$ is increasing on both these intervals.

Example 4.23. If $x > 1$, show that $\ln x > 1 - \dfrac{1}{x}$.

Let $f(x) = \ln x - (1 - \frac{1}{x})$. Then $f(1) = 0$ and $f'(x) = \dfrac{1}{x} - \dfrac{1}{x^2} > 0$ for $x > 1$. The MVT shows that $f(x)$ is increasing for $x > 1$, so that $f(x) > f(1) = 0$ for $x > 1$. Thus $\ln x > 1 - \dfrac{1}{x}$ for $x > 1$.

4.6 l'Hôpital's Rule

This is a ubiquitous tool for computing limits of quotients in which numerator and denominator both $\to 0$. A function f is called *continuously differentiable* at a point x_0 if it is differentiable (in a neighbourhood of x_0) and its derivative f' is continuous at x_0. L'Hôpital's Rule is proved by using the Mean Value Theorem.

Theorem 4.6 (l'Hopital's rule)**.** Suppose that f and g are continuously differentiable at a point a, that $f(a) = g(a) = 0$ and that $f'(x)/g'(x)$ has a limit as $x \to a$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Note: it is important that the limit of $f'(x)/g'(x)$ as $x \to a$ exists – if not, l'Hôpital's rule can't be used.

Example 4.24. Find

$$\lim_{x\to 0} \frac{e^{2x} - 1}{x}.$$

Here, $f(x) = e^{2x} - 1$ and $g(x) = x$ satisfy the requirements of l'Hôpital at $a = 0$, so the limit is equal to

$$\lim_{x\to 0} \frac{2e^{2x}}{1} = 2.$$

Example 4.25. Find

$$\lim_{x \to 1} \frac{\sin \pi x}{\ln x}.$$

Both top and bottom go to 0 as $x \to 1$, so by l'Hôpital the limit is

$$\lim_{x \to 1} \frac{\pi \cos \pi x}{(1/x)} = -\pi.$$

∎

Example 4.26. Find

$$\lim_{x \to 0} \frac{\cos x - 1}{x^2}$$

For this case we can apply l'Hôpital twice:

$$\begin{aligned} \lim_{x \to 0} \frac{\cos x - 1}{x^2} &= \lim_{x \to 0} \frac{-\sin x}{2x} \\ &= \lim_{x \to 0} \frac{-\cos x}{2} \\ &= -\tfrac{1}{2}. \end{aligned}$$

Example 4.27. Find the limit

$$E = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x.$$

Let $f(x) = x \ln\left(1 + \frac{1}{x}\right)$. Then $E = \lim_{x \to \infty} e^{f(x)}$, and since e^x defines a continuous function this is the same as e^L where $L = \lim_{x \to \infty} f(x)$. But this is

$$\begin{aligned} \lim_{x \to \infty} f(x) &= \lim_{u \to 0} \frac{\ln(1+u)}{u} \qquad \text{where } u = \frac{1}{x}, \\ &= \lim_{u \to 0} \frac{1}{1+u} \qquad \text{by l'Hôpital's Rule,} \\ &= 1. \end{aligned}$$

Hence $E = e^1 = e$.

Letting x run through the natural numbers, this shows that e is the limit of the sequence:

$$2, \quad \left(\frac{3}{2}\right)^2, \quad \left(\frac{4}{3}\right)^3, \quad \left(\frac{5}{4}\right)^4, \quad \left(\frac{6}{5}\right)^5, \quad \ldots$$

Try this on a calculator!

Some A level revision. Here's a reminder for the use of differentiation to find maximum and minimum values of a function.

Example 4.28. Find the max and min values taken by $f(x) = 2x^3 - 3x^2 - 12x + 7$ between $x = 0$ and $x = 3$.

Solution: The max/min values are taken at the end points of the interval or at points inside the interval where $f'(x) = 0$. Here, $f'(x) = 6x^2 - 6x - 12 = 6(x-2)(x+1)$, so that the max/min values are taken at either $x = 0$ or $x = 2$ or $x = 3$. It is quick to check that $f(0) = 7$, $f(2) = -13$ and $f(3) = -2$. So max value taken between $x = 0$ and $x = 3$ is 7, while the min value is -13.

Rubric on the Collection Exam for SMA in January:

Time allowed: 45 minutes. Answer all questions. Electronic calculators may **not** *be used.*

Note:

There will be 8 questions altogether. All will be worth an equal amount of credit.

That's all folks! And finally... I hope you all have a very happy Christmas.