Analysis — Problems 2013/14

1. Find all those values of x for which $\dfrac{3x+4}{2} \le \dfrac{6-x}{4}$.

2. Find all those values of x for which $x^2 - x < 2$.

3. Find all those values of x for which $\dfrac{-3}{x-4} \le x$.

4. Find all those values of x for which $\dfrac{3}{x-4} < -x$.

5. Find all real values of x such that $|x^2 + x - 4| = 2$.

6. For all real x show that $|8x - 9| < 7x - 6$ if and only if $|x - 2| < 1$.

7. For all real x show that $|2x + 1| < 3x$ if and only if $x > 1$.

8. Find all values of x for which $|2x + 5| > 4$.

9. Find all values of x for which $|2x + 1| \le |3x - 6|$.

10. * Show that the following statements about the real numbers x and y are equivalent:
(a) $x \ge y$; (b) For every $\epsilon > 0$, $x > y - \epsilon$; (c) For every $\epsilon > 0$, $x + \epsilon > y$.

11. Calculate $\lim_{n\to\infty} x_n$ in each of the following cases (or show that no limit exists).
(a) $x_n = \cos(n^2)/\sqrt{n^2+n}$ (b) $x_n = (3n+1)^2(4n^4+1)^{-1/2}$
(c) $x_n = [(1+2n)/(2n)]^n$ (d) $x_n = [(2n+1)/(n+1)]^{2n}$ *[Hint: show* $\frac{2n+1}{n+1} \ge \frac{3}{2}$*]*
(e) $x_n = (n^2 + \log n)/\sqrt{2n^3 - 1}$ (f) $x_n = (n^5 + \log n)^{2/n}$
(g) $x_n = n^2[1/(n+1) - 1/(n-1)]$.

12. Calculate $\lim_{n\to\infty} x_n$ in each of the following cases.
(a) $x_n = (2n+1)^2/\sqrt{n^4+1}$ (b) $x_n = n(\sqrt{1+n^2} - n)$ (c) $x_n = \log n - \log(n+1)$
(d) $x_n = (n^2 + \mathrm{e}^{-n})/(\log n + 5n^3)$ (e) $x_n = (n!)^2/[(n-2)!(n+2)!]$ (f) $x_n = n!\, n^{-n}$
(g) $x_n = 2^n/n!$ (h) $x_n = n\sin(\pi/n)$ *[Use* $\frac{\sin\theta}{\theta} \to 1$ *as* $\theta \to 0$*]* (i) $x_n = (1+n^2)^{1/n}$
(j) $x_n = (n+3)!/(n!\, n^3)$ (k) $x_n = n^2[n^{-1} - (n+1)^{-1}]$.

13. Let x_n be as in (d) of the previous question. Given $\epsilon > 0$, show that the distance from x_n to its limit is less than ϵ if $n > 2/(5\epsilon)$.

14. Calculate $\lim_{n\to\infty} x_n$ in each of the following cases (or show that no limit exists).
(a) $x_n = (n + \log n^2)/\sqrt{n^2+2}$ (b) $x_n = \sqrt{n}\,(n + \mathrm{e}^{-n})^{-1}\sin(\mathrm{e}^n)$

15. Find the limit of each of the following sequences as $n \to \infty$, or show that no limit exists. (a) $x_n = (n^2 + e^n)^{1/n}$ (b) $x_n = \sqrt{n}\left(\sqrt{n+1} - \sqrt{n-1}\right)$ (c) $x_n = \left(\frac{n-1}{n+1}\right)^n$

16. Compute $\lim_{n\to\infty} x_n$ for the following: (a) $x_n = (n^2+n)^{1/n}$ (b) $x_n = n\left(\sqrt{n+1} - \sqrt{n}\right)^2$

17. If $\{x_n\}$ is a sequence such that $x_n \to L$ as $n \to \infty$, and $x_n < 0$ for all n, prove that $L \leq 0$. Is it necessarily true that $L < 0$?

18. Compute $\lim_{n\to\infty} \left[(n+1)^2 - (n-1)^2\right] / \left(n + \sqrt{n}\right)$, or show that the limit does not exist.

19. Find $\lim_{n\to\infty}$ for each of the following sequences, or show that no limit exists.
(a) $x_n = (n^2+2)^{1/n}$ (b) $x_n = [(n+2)/(n+1)]^{2n}$ (c) $x_n = n\left(\sqrt{n^2+1} - \sqrt{n^2-1}\right)$

20. Compute $\lim_{n\to\infty} \left[(n+1)^2 - n^2\right] / \left(n + \log n\right)$, or show that the limit does not exist.

21. Find the limit of each of the following sequences as $n \to \infty$, or show that the limit does not exist.
(a) $x_n = \left(\frac{n}{n+1}\right)^n$ (b) $x_n = \left(\frac{1}{n+1} - \frac{1}{n-1}\right) / \sin\left(\frac{1}{n^2}\right)$ (c) $x_n = \left(\frac{3n+2}{2n+1}\right)^n$

22. Prove that limits are unique: i.e. if $x_n \to l$ and $x_n \to k$ as $n \to \infty$, then $l = k$.

23. Show that if $\{x_n\}$ is a sequence with $x_n \leq b$ for all n, and $\lim_{n\to\infty} x_n = L$, then $L \leq b$.

24. * If $\{x_n\}$ is a bounded sequence (ie there is a number K such that $|x_n| \leq K$ for all n), and if $y_n \to 0$ as $n \to \infty$, prove that $x_n y_n \to 0$ as $n \to \infty$.

25. Prove that one of the following statements is true and that the other is false.
(a) If $x_n \to 1$ as $n \to \infty$, then $(x_n)^n \to 1$ as $n \to \infty$.
(b) * If $0 < r < 1$ and $x_n \to r$ as $n \to \infty$, then $(x_n)^n \to 0$ as $n \to \infty$.

26. ** The sequence $\{x_n\}$ is defined recursively by $x_1 = 10$ and $x_{n+1} = \sqrt{6 + x_n}$. Find $\lim_{n\to\infty} x_n$. *[Hint: first find the fixed points of the iteration.]*

27. Compute $\lim_{n\to\infty}(t + 1/n)^n$ for each positive real value of t for which this limit exists.

28. Determine for which real values of x the sequence $\{n^{-1}x^n\}$ tends to a limit as $n \to \infty$.

29. Compute $\lim_{n\to\infty} x_n$, or show that the limit does not exist, for each of the following.
(a) $x_n = (p^n + q^n)^{1/n}$, with $p \geq q \geq 0$
(b) $x_n = (1 + 2 + \ldots + n)/n^2$ *[Hint: sum this arithmetic series]*

(c) $x_n = n \int_0^n \mathrm{e}^{-nx}\, dx$ *[Hint: do the integral]*
(d) $x_n = (1 + 1/n)^{n^2}$ *[Hint: log]*.

30. Determine the sup and inf, where they exist, of the following sets:
(a) $S = \{x : |2x - 1| < 11\}$ (b) $S = \{x + |x - 1| : x \in \mathbf{R}\}$.

31. Compute the sup and the inf of the function $f(x) = e^x/(1 + e^x)$, where $x \in \mathbf{R}$.

32. Compute the sup and the inf of the function $f(x) = x/(1 + |x|)$, where $x \in \mathbf{R}$.

33. Determine the least upper bound and the greatest lower bound of each of the following sets (where n and m range over the positive integers):
(a) $X = \{(n^2 - n)/(n^2 + 1)\}$ (b) $Y = \{(2m + n)/(m + 3n)\}$.

34. Determine the supremum of $g(x) = (x \cos^2 x - 2)/(2x \cos^2 x + 2)$ for $x \geq 0$.

35. Determine the least upper bound and the greatest lower bound of each of the following functions (where x ranges over the real numbers):
(a) $f(x) = (1 + x^2 \cos x)/(2 + x^2)$ (b) $g(x) = x^2 \exp(-x^2)$.

36. If g is bounded above, and $f(x) < g(x)$ for all x, prove that $\sup f \leq \sup g$. Is it necessarily true that $\sup f < \sup g$?

37. Compute the supremum and the infimum of the function
$f(x) = x^2/(1 + x^2)$ on $\mathbf{R}$.

38. Compute the supremum and the infimum of the function
$f(x) = \sqrt{x}/(2 + x)$ for $x > 0$.

39. Compute the supremum and the infimum of the function
$f(x) = x/(x^2 + 1)$ for $x > 0$.

40. Determine whether or not the series $\Sigma_{n=1}^{\infty}(2 + n)/\sqrt{4n^4 - 1}$ converges.

41. Determine whether or not the series $\Sigma_{n=1}^{\infty}\sqrt{n}/(n^3 + 1)$ converges.

42. Determine whether or not the series $\sum_{n=1}^{\infty} \sin(2^n)/2^n$ converges.

43. Determine whether or not the series $\sum_{n=1}^{\infty}(n - 3)(2 + 9n^6)^{-1/2}$ converges.

44. Use the comparison test to decide whether or not $\sum_{n=1}^{\infty} x_n$ converges in each of the following cases. (You may assume that $\sum_{n=1}^{\infty} n^{-\alpha}$ converges iff $\alpha > 1$.)
(a) $x_n = n/\sqrt{1+n^6}$ (b) $x_n = 1/(n+\sqrt{n})$ (c) $x_n = (3-n\sqrt{n})/n^6$
(d) $x_n = n!\,n^2/(n+3)!$ (e) $x_n = n^2 \exp(-\sqrt{n})$ (f) $x_n = (n\cos n)/(n^3+\log n)$
(g) $x_n = n^{-1}\sin(n^{-1})$ *[Use $\sin\theta < \theta$ for $\theta > 0$]* (h) $x_n = n^{-2}(\log n)^4$
(i) $x_n = \sqrt{1+n^2} - n$.

45. For which values of α do the following series converge?
(a) $\sum_{n=1}^{\infty}(n^2+1)^{-\alpha}\log(1+\frac{1}{n})$ (b) $\sum_{n=1}^{\infty} n^{\alpha}\left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$.

46. Show that the series $(x_1 - x_2) + (x_2 - x_3) + (x_3 - x_4) + \ldots$ converges if and only if the sequence $\{x_n\}$ tends to a limit as $n \to \infty$.

47. If $\sum_{n=1}^{\infty} x_n$ converges to s, and $y_n = (x_n + x_{n+1})/2$ for all n, does $\sum_{n=1}^{\infty} y_n$ converge, and if so to what?

48. Given that $\sum_{n=1}^{\infty} x_n$ converges, and $\sum_{n=1}^{\infty} y_n$ converges absolutely, prove that $\sum_{n=1}^{\infty} x_n y_n$ converges absolutely. If we knew only that $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ converged, would it follow that $\sum_{n=1}^{\infty} x_n y_n$ converged as well?

49. Determine whether or not each of the following series converges.
(a) $\sum_{n=3}^{\infty}\tan(\pi/n)\cos(n\pi)$ (b) $\sum_{n=2}^{\infty} n^{-1}(\log n)^{-3}$ (c) $\sum_{n=1}^{\infty}(2n)!\,5^{-n}(n!)^{-2}$

50. Determine whether or not each of the following series converges.

$$(a)\ \sum_{n=1}^{\infty}\frac{1}{(-1)^n\sqrt{n}\tanh n} \qquad (b)\ \sum_{n=1}^{\infty}\frac{2^n(2n)!}{9^n(n!)^2} \qquad (c)\ \sum_{n=1}^{\infty}\frac{n-1}{(n^2+2)(n^2+1)^{1/4}}.$$

51. Discuss whether or not $\sum_{n=1}^{\infty} x_n$ converges in each of the following cases.
(a) $x_n = (n!)^2/(2n)!$ (b) $x_n = 1/[(n+1)\log(n+1)]$
(c) $x_n = (\cos\pi n)/(n\log(n+1))$.

52. For what values of α does the series $\sum_{n=1}^{\infty} x_n$ converge, in each of the following cases? *[Be careful to investigate all real values of α. In each case except (c), use the ratio test first, and then deal with the remaining values of α separately.]*
(a) $x_n = \alpha^n n^{\alpha}$ (b) $x_n = \alpha^{n-1}/(n3^n)$ (c) $x_n = n^{-1}(\log(n+1))^{-\alpha}$ (d) $x_n = n!\alpha^n$
(e) $x_n = n\alpha^n/(2^n(3n-1))$. *[For (c): first compare with $y_n = (n+1)^{-1}(\log(n+1))^{-\alpha}$].*

53. Find values of z for which the series $\sum a_n(z-z_0)^n$ converges in the following cases:
(a) $z_0 = 0$, $a_n = 1/n!$; (b) $z_0 = 1$, $a_n = 1/(n-1)!$, $n > 1$; (c) $z_0 = 0$, $a_n = c^n$;
(d) $z_0 = 0$, $a_n = n$; (e) $z_0 = 0$, $a_n = n!$.