Analysis.

These notes are not meant to be complete

Contents

1 Numbers and Inequalities.

Numbers. Below is a hierarchy of number sets. The first three "exist in Nature", in the sense that measurements yield answers that belong to $\mathbb{Q}$; the others are "invented".

- The *Natural Numbers* or positive integers $\mathbb{N} = \{1, 2, 3, \ldots\}$. Sometimes 0 is also added to *Natural Numbers*.

- The *Integers* $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

- The *Rational Numbers* $\mathbb{Q} = \{p/q \,:\, p, q \in \mathbb{Z},\ q \neq 0\}$.

- The *Algebraic Numbers* are the solutions x of polynomial equations $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$, where the a_j are integers. For example, $\sqrt{2}$ is algebraic (being a solution of $x^2 - 2 = 0$), but not rational.

- The *Real Numbers* $\mathbb{R}$ will not be defined here; but a crucial defining property is that $\mathbb{R}$ is *complete*: it has no gaps. Real numbers that are not algebraic are called *transcendental*; examples of transcendental numbers are e and π. Fact: almost all real numbers are transcendental.

- The set of *Complex Numbers* $\mathbb{C} = \{x + \mathrm{i}y \,:\, x, y \in \mathbb{R},\ \mathrm{i}^2 = -1\}$ is also complete.

Rules for Inequalities. The real numbers are *ordered*, and the order relations $>$, $<$, $\geq$ and $\leq$ satisfy properties such as the following.

- If $x < y$ and $y < z$, then $x < z$ (transitivity).

- If $x < y$ and $a < b$, then $a + x < b + y$.

- If $x < y$ and $c > 0$, then $cx < cy$.

- If $x < y$ and $c < 0$, then $cx > cy$.

- If $0 < x < y$, then $x^{-1} > y^{-1}$.

Note Complex numbers are NOT ordered and so for them we cannot use the concept of inequality.

Examples.

1. Find all $x \in \mathbb{R}$ such that $-3(4-x) \leq 12$. One solution:
$-3(4-x) \leq 12 \Leftrightarrow 4-x \geq -4 \Leftrightarrow x \leq 8$.

2. Solve $\dfrac{x+2}{3} < \dfrac{5-2x}{4}$. Solution: the original inequality is equivalent to
$4x+8 < 15-6x \Leftrightarrow 10x < 7 \Leftrightarrow x < 7/10$.

3. Solve $x^2-4x+3 > 0$. Solution: the original inequality is equivalent to
$(x-3)(x-1) > 0 \Leftrightarrow x > 3$ or $x < 1$.

4. Solve $\dfrac{3}{x-2} \leq x$. Solution: bring to a common denominator. The original inequality is equivalent to $\dfrac{(x-3)(x+1)}{x-2} \geq 0 \Leftrightarrow x \geq 3$ or $-1 \leq x < 2$.

Inequalities involving Absolute Value.

Note: As Absolute values of complex numbers are real nonnegative numbers such inequalities can be used for both complex and real numbers.

- $|x+y| \leq |x|+|y|$ (the Triangle Inequality).
- $||x|-|y|| \leq |x-y|$ (a variant of the Δ inequality).
- $|x| < c \Leftrightarrow x^2 < c^2 \Leftrightarrow -c < x < c$.

Examples.

1. Solve $|3x-4| \leq 2$. Solution:
$|3x-4| \leq 2 \Leftrightarrow -2 \leq 3x-4 \leq 2 \Leftrightarrow 2/3 \leq x \leq 2$.

2. Solve $|2x+3| > 5$. Solution:
$|2x+3| > 5 \Leftrightarrow 2x+3 > 5$ or $2x+3 < -5 \Leftrightarrow x > 1$ or $x < -4$.

3. Solve $|x+2| \leq |2x-1|$. Solution: the original inequality is equivalent to
$(x+2)^2 \leq (2x-1)^2 \Leftrightarrow 3x^2-8x-3 \geq 0 \Leftrightarrow (3x+1)(x-3) \geq 0 \Leftrightarrow x \geq 3$ or $x \leq -1/3$.

2 Sequences and Limits.

Definition. A *real sequence* is a function from $\mathbb{N}$ to $\mathbb{R}$. The usual notation for a sequence is $\{x_n\}_{n=1}^{\infty} = \{x_1, x_2, x_3, \ldots\}$, with each of x_1, x_2 etc being a real number.

Examples.

1. If $x_n = 6$ for all $n \in \mathbb{N}$, then we get the *constant sequence* $\{6, 6, 6, 6, \ldots\}$.
2. If $x_n = (n-1)/n$ for all $n \in \mathbb{N}$, then we get the sequence $\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\}$.
3. If $x_n = (-1)^{n+1}$ for all $n \in \mathbb{N}$, then we get the sequence $\{1, -1, 1, -1, 1, \ldots\}$.
4. If $x_n = n^2$ for all $n \in \mathbb{N}$, then we get the sequence $\{1, 4, 9, 16, 25, \ldots\}$.

Remark. Of these four sequences, the first two have *limits* as $n \to \infty$, whereas the other two do not. The idea is that x_n tends to the limit L as n tends to infinity if x_n gets close to L as n gets big; but we have to quantify 'close' and 'big'. So a more accurate definition involves saying: "If someone tells you what 'close' means, then you can tell them what 'big' means". Closeness is often measured by the Greek letter ε, pronounced 'epsilon'. Here is the definition.

Definition. Let $\{x_n\}_{n=1}^{\infty}$ be a real sequence. Then we say that $x_n \to L$ as $n \to \infty$ (also written $\lim_{n\to\infty} = L$) if: given $\varepsilon > 0$, there exists a number N such that $|x_n - L| < \varepsilon$ whenever $n \geq N$.

Examples.

1. If $x_n = 6$, then clearly $x_n \to 6$ as $n \to \infty$: in fact, given $\varepsilon > 0$, we have $|x_n - 6| < \varepsilon$ for all n. (In other words, N could be anything.)
2. If $x_n = (n-1)/n$, then I claim that $x_n \to 1$ as $n \to \infty$. Proof: given $\varepsilon > 0$, take N to be any number greater than $1/\varepsilon$; then $1/N < \varepsilon$ and so $|x_n - 1| = 1/n < \varepsilon$ for all $n \geq N$. (Note that N depends on ε, as is usually the case.)
3. If $x_n = (-1)^{n+1}$, then I claim that there is no limit L. Proof: suppose there were such an L. Take a smallish ε, say $\varepsilon = 1$, and find the corresponding N. Then, taking $n \geq N$ to be even gives $|L - 1| < 1$, while taking $n \geq N$ to be odd gives $|L + 1| < 1$. The first inequality implies that $L > 0$, while the second implies $L < 0$, and these cannot both be true; so no such L exists.
4. If $x_n = n^2$, then there is no limit as $n \to \infty$ (similar proof). In this case, we could say that $x_n \to \infty$ as $n \to \infty$; but that doesn't count as a limit (which has to be finite).

Remark. If $x_n \to L$, then L has to be a fixed number, not depending on n. In other words, a statement such as $1/(n+1) \to 1/n$ is not allowed.

Theorem (*Calculus of Limits Theorem*). If $x_n \to L$ and $y_n \to K$ as $n \to \infty$, and if A and B are constants, then

(i) $Ax_n + By_n \to AL + BK$ as $n \to \infty$;

(ii) $x_n y_n \to LK$ as $n \to \infty$;

(iii) $x_n/y_n \to L/K$ as $n \to \infty$, provided K and all the y_n are non-zero.

Sample Proof. We prove (i) with $A = B = 1$. Let $\varepsilon > 0$ be given. Find P such that $|x_n - L| < \varepsilon/2$ for all $n \geq P$, and find Q such that $|y_n - K| < \varepsilon/2$ for all $n \geq Q$. Take $N = \max(P, Q)$. Then for $n \geq N$, we have

$$\begin{aligned} |(x_n + y_n) - (L + K)| &= |(x_n - L) + (y_n - K)| \\ &\leq |x_n - L| + |y_n - K| \quad \text{by } \Delta \text{ inequality} \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon; \end{aligned}$$

and so we have shown that $x_n + y_n \to L + K$ as $n \to \infty$.

Theorem. If $x_n \to L$ as $n \to \infty$, and if the function $f(x)$ is continuous at $x = L$, then $f(x_n) \to f(L)$ as $n \to \infty$.
Remark. We will prove this later in the course, after we have defined what 'continuous' means. For the time being, just use the fact that the following are all continuous: polynomials $p(x)$; ratios $p(x)/q(x)$ of polynomials away from the zeros of $q(x)$; $\sin(x)$; $\exp(x)$; and $\log(x)$ for $x > 0$. So, for example, if we know that $x_n \to L$ as $n \to \infty$, we can deduce that $\exp(x_n) \to \mathrm{e}^L$ as $n \to \infty$.

Theorem. (*Squeezing Theorem*). If $|x_n| \leq y_n$ for all n, and $y_n \to 0$ as $n \to \infty$, then $x_n \to 0$ as $n \to \infty$.
Proof. Let $\varepsilon > 0$ be given, and find N such that $|y_n| < \varepsilon$ for all $n \geq N$. Then $|x_n - 0| = |x_n| \leq y_n = |y_n| < \varepsilon$ for all $n \geq N$.

Facts (*proved later*). "Exponentials beat powers, and powers beat logs." In other words,

$$n^p/\mathrm{e}^n \to 0 \quad \text{and} \quad n^{-q} \log n \to 0 \quad \text{as } n \to \infty,$$

for all $p \in \mathbb{R}$ and for all $q > 0$. Note that log means log to base e (in other words, log means the same as ln).

Examples.

1. Compute $\lim_{n\to\infty} x_n$, where $x_n = n\sqrt{3n^2-2}/\sqrt{1+8n^4}$. Solution:
$$\begin{aligned} x_n &= \frac{\sqrt{3-2/n^2}}{\sqrt{1/n^4+8}} \\ &\to \frac{\sqrt{3-0}}{\sqrt{0+8}} = \sqrt{\frac{3}{8}} \quad \text{as } n\to\infty. \end{aligned}$$

2. Compute $\lim_{n\to\infty} x_n$, where $x_n = (n+\sin n)/\sqrt{4n^2+1}$. Solution: First note that $|n^{-1}\sin n| \le n^{-1} \to 0$ as $n\to\infty$, so $n^{-1}\sin n \to 0$ as $n\to\infty$ by squeezing. Now
$$\begin{aligned} x_n &= (1+n^{-1}\sin n)/\sqrt{4+1/n^2} \\ &\to (1+0)/\sqrt{4+0} = 1/2 \quad \text{as } n\to\infty. \end{aligned}$$

3. Compute $\lim_{n\to\infty} x_n$, where $x_n = (n^2+n^3\mathrm{e}^{-n})/\left[\log(2^n)+\log(n^8)\right]^2$. Solution:
$$\begin{aligned} x_n &= (1+n\mathrm{e}^{-n})/\left[\log 2 + 8n^{-1}\log n\right]^2 \\ &\to (1+0)/(\log 2+0)^2 = (\log 2)^{-2} \quad \text{as } n\to\infty. \end{aligned}$$

4. Compute $\lim_{n\to\infty} x_n$, where $x_n = n^2\, n!/(n+2)!$. Solution:
$$x_n = \frac{n^2}{(n+2)(n+1)} \to 1 \quad \text{as } n\to\infty.$$

5. Compute $\lim_{n\to\infty} x_n$, where $x_n = \sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right)$. Solution: The trick here is to use $A-B = (A-B)(A+B)/(A+B) = (A^2-B^2)/(A+B)$, where $A=\sqrt{n+1}$ and $B=\sqrt{n}$. We get
$$x_n = \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \to \frac{1}{2} \quad \text{as } n\to\infty.$$

6. Compute $\lim_{n\to\infty} x_n$, where $x_n = t^{1/n}$ with $t>0$ fixed. Solution: $\log x_n = n^{-1}\log t \to 0$ as $n\to\infty$, and applying the continuous function exp to this gives $x_n \to 1$ as $n\to\infty$. Exercise: try it on your calculator, by entering a positive number and then pressing the $\sqrt{}$ key repeatedly.

7. Compute $\lim_{n\to\infty} x_n$, where $x_n = n^{2/n}$. Solution: $\log x_n = 2n^{-1}\log n \to 0$ as $n\to\infty$, and applying the continuous function exp to this gives $x_n\to 1$ as $n\to\infty$.

8. Compute $\lim_{n\to\infty} x_n$, where $x_n = n^{-1}\log(3^n+n^3)$. Solution:
$$\begin{aligned} x_n &= n^{-1}\log\left[3^n(1+n^3\,3^{-n})\right] \\ &= \log 3 + n^{-1}\log(1+n^3\,3^{-n}) \\ &\to \log 3 + 0\times\log(1+0) = \log 3 \quad \text{as } n\to\infty. \end{aligned}$$

Theorem. If $x_n > 0$ for all n, and $x_n \to L$ as $n \to \infty$, then $L \geq 0$.
Proof. Suppose, on the contrary, that $L < 0$. Put $\varepsilon = -L/2$, and find a positive integer N such that $|x_n - L| < \varepsilon$ for all $n \geq N$. Now

$$|x_N - L| < \varepsilon \Rightarrow L/2 < x_N - L < -L/2 \Rightarrow x_N < L/2 < 0,$$

which contradicts the requirement that all the x_n are positive. So we must have $L \geq 0$.

Example. If t is a fixed number with $|t| < 1$, then $t^n \to 0$ as $n \to \infty$.
Proof. Let $\varepsilon > 0$ be given. Note that $n^{-1} \log \varepsilon \to 0$ as $n \to \infty$, and exponentiating this gives $\varepsilon^{1/n} \to 1$ as $n \to \infty$. So we can find an integer N such that $|t| < \varepsilon^{1/N}$, and hence $|t^N| < \varepsilon$. Finally, $|t^{N+1}| = |t||t^N| < 1 \times \varepsilon = \varepsilon$ and so forth: by induction, it follows that $|t^n| < \varepsilon$ for all $n \geq N$. Exercise: try it on your calculator, by entering a suitable number (such as 0.999) and then pressing the x^2 key repeatedly.

Remark. It follows that if $|t| > 1$, then t^n has no limit as $n \to \infty$. For suppose that $t^n \to L$, and also since $|1/t| < 1$ we have $1/t^n \to 0$ by the previous result, so $t^n/t^n \to L \times 0 = 0$ which is clearly false. Carrying this further, we see that if $x_n > t^n$ for some fixed $t > 1$, and for all n, then $\{x_n\}$ has no limit (this is like a complement of the Squeezing Theorem).

Example. If c is a fixed number, then $(1 + c/n)^n \to \mathrm{e}^c$ as $n \to \infty$.
Proof. We show that $n \log(1+c/n) \to c$, which is equivalent. And we take log to be defined by $\log t = \int_1^t x^{-1}\, dx$. For $x \in [1, 1 + c/n]$, we have

$$\frac{n}{c+n} \leq \frac{1}{x} \leq 1 \Rightarrow \frac{cn}{c+n} \leq n \log\left(1 + \frac{c}{n}\right) \leq c;$$

and since $cn/(c + n) \to c$ as $n \to \infty$, the desired result follows (by squeezing).

Example. $x_n = n^{-q} \log n \to 0$ as $n \to \infty$, provided $q > 0$.
Proof. It is sufficient to prove it for $0 < q < 1$, since the $q > 1$ case then follows by squeezing. For simplicity, I'll take $q = 1/2$; the other cases you can do as an exercise (the argument is similar). We have

$$0 \leq x_n = \frac{1}{\sqrt{n}} \int_1^n \frac{dx}{x} < \frac{1}{n^{1/2}} \int_1^n \frac{dx}{x^{3/4}} = 4(n^{-1/4} - n^{-1/2}) \to 0$$

as $n \to \infty$. So $x_n \to 0$ by squeezing.

Example. $x_n = n^p \mathrm{e}^{-n} \to 0$ as $n \to \infty$, for any fixed $p \in \mathbb{R}$.
Proof. Start with $n^{p/n} \to 1$ as $n \to \infty$ (same proof as $p = 2$ case, which was done above). Therefore

$$\frac{n^{p/n}}{\mathrm{e}} \to \frac{1}{\mathrm{e}} < \frac{1}{2}.$$

Taking the n-th power of this, we see that for n large enough, we have

$$0 < \frac{n^p}{\mathrm{e}^n} < \frac{1}{2^n}.$$

Since $2^{-n} \to 0$ as $n \to \infty$, the required result follows by squeezing.

3 Sup, Inf and Completeness.

Definition. Let X be a set of real numbers. A number k is the *maximum* of X, written $k = \max(X)$, if $k \in X$ and $x \leq k$ for all $x \in X$. Similarly one defines $\min(X)$.

Examples.

1. $X = \{1, 2, 3, 7\}$ has $\min(X) = 1$ and $\max(X) = 7$.
2. $X = \{1/n : n = 1, 2, 3, \ldots\}$ has $\max(X) = 1$, but $\min(X)$ does not exist.
3. $X = [0, \infty)$ has $\min(X) = 0$, but $\max(X)$ does not exist.

Definition. A set X of real numbers is *bounded above* if there exists a number K such that $x \leq K$ for all $x \in X$; then K is called an *upper bound* for X. Similarly: bounded below, lower bound. A set is *bounded* if it is bounded both above and below.

Examples.

1. $X = \{1/n : n = 1, 2, 3, \ldots\}$ is bounded above (the numbers 1, 1.7, π are some upper bounds), and also bounded below (the numbers 0, -11, $-17/9$ are some lower bounds). So X is bounded.
2. $X = [0, \infty)$ is bounded below, but not bounded above.

Remark. Note that if X is bounded above, then it has many (in fact infinitely many) upper bounds. But one of them is special, namely the smallest one.

Definition. Let X be bounded above. A number k is called a *least upper bound* (or *supremum* or *sup*) of X if

(i) k is an upper bound for X; and

(ii) no number less than k is an upper bound for X.

We then write $k = \sup(X)$. Similarly, if X is bounded below, we can define a *greatest lower bound* (or *infimum* or *inf*) of X, written $\inf(X)$. Another (equivalent) way of saying (ii) is

(ii') either $k \in X$ or there is a sequence $\{x_n\}$ of elements of X such that $x_n \to k$ as $n \to \infty$.

Examples.

1. $X = \{1/n : n = 1, 2, 3, \ldots\}$ has $\inf(X) = 0$ and $\sup(X) = 1$.

2. $X = (0, \infty)$ has $\inf(X) = 0$, but $\sup(X)$ does not exist.

3. $X = \{n/(1 + n^2) : n = 1, 2, 3, \ldots\}$. Here $X = \{1/2, 2/5, 3/10, \ldots\}$, so we guess that $\inf(X) = 0$ and $\sup(X) = 1/2$. Write $x_n = n/(1 + n^2)$.
   Proof of $\inf(X) = 0$: (i) $x_n > 0$ for all $n \in \mathbb{N}$; (ii) $x_n \to 0$ as $n \to \infty$.
   Proof of $\sup(X) = 1/2$: (i) We want $x_n \leq 1/2 \Leftrightarrow (n - 1)^2 \geq 0$ clearly true;
   (ii) $x_1 = 1/2$.

4. $X = \{mn/(1 + m^2 + n^2) : m, n \in \mathbb{N}\}$. Looking at some values in the 2-dimensional array of X motivates the guess that $\inf(X) = 0$ and $\sup(X) = 1/2$.
   Write $x_{mn} = mn/(1 + m^2 + n^2)$.
   Proof of $\inf(X) = 0$: (i) $x_{mn} > 0$ for all $m, n \in \mathbb{N}$; (ii) $x_{1n} \to 0$ as $n \to \infty$.
   Proof of $\sup(X) = 1/2$: (i) We want $x_{mn} \leq 1/2 \Leftrightarrow 1 + (n - m)^2 \geq 0$ clearly true;
   (ii) $x_{nn} \to 1/2$ as $n \to \infty$.

5. $X = \{x_n = (n^2 - 4n + 4)/(1 + 2n^2) : n = 1, 2, 3, \ldots\}$. Here $X = \{1/3, 0, 1/19, \ldots\}$, and $x_n \to 1/2$ as $n \to \infty$; so we guess that $\inf(X) = 0$ and $\sup(X) = 1/2$.
   Proof of $\inf(X) = 0$: (i) $x_n \geq 0$ for all $n \in \mathbb{N}$; (ii) $x_2 = 0$.
   Proof of $\sup(X) = 1/2$: (i) We want $x_n \leq 1/2 \Leftrightarrow 8n - 7 \geq 0$ clearly true;
   (ii) $x_n \to 1/2$ as $n \to \infty$.

Axiom *(the Completeness Axiom for $\mathbb{R}$).* Every non-empty set of real numbers which is bounded above has a supremum. [This is either an axiom or a theorem, depending on how the real numbers are constructed: the point is that it is a defining feature of $\mathbb{R}$.]

Definition. Let X be a set, and $f : X \to \mathbb{R}$ a real-valued function on X. Let $f(X)$ denote the set $f(X) = \{f(x) : x \in X\}$. We say that f is *bounded above* if the set $f(X)$ is bounded above. Similarly *bounded below* and *bounded*. And $\sup(f(X))$ is written $\sup_{x \in X} f(x)$, or simply $\sup(f)$ when it is clear what X is.

Examples.

1. $f(x) = x^2$ on $X = \mathbb{R}$ has $\inf(f) = 0$, but $\sup(f)$ does not exist.

2. $f(x) = x^2 \cos x/(1 + x^2)$ for $x > 0$. Note that $f(x) = g(x) \cos x$, where $g(x) = x^2/(1 + x^2)$. Since $0 \leq g(x) < 1$ and $-1 \leq \cos x \leq 1$, we guess that $\inf(X) = -1$ and $\sup(X) = 1$.
   Proof of $\sup(X) = 1$: (i) $g(x) < 1$ and $\cos x \leq 1$ imply $f(x) < 1$;
   (ii) $f(2n\pi) = g(2n\pi) \to 1$ as $n \to \infty$ (where $n \in \mathbb{N}$).

3. $f(x) = -x^{-2} + x^{-1} - 1$ for $x > 1$. Note that $f(1) = -1$, and $f(x) \to -1$ as $x \to \infty$. Also, f is differentiable, and $df/dx = (2 - x)/x^3$ is zero only at $x = 2$; and $f(2) = -3/4$. So clearly $\inf(X) = -1$ and $\sup(X) = -3/4$.
   Proof of $\sup(X) = -3/4$: (i) We want $f(x) \leq -3/4 \Leftrightarrow (x - 2)^2 \geq 0$ which is true for all x; (ii) $f(2) = -3/4$.
   Proof of $\inf(X) = -1$: (i) $f(x) \geq -1 \Leftrightarrow x \geq 1$; (ii) $f(x) \to -1$ as $x \to \infty$.

Remark. In the last example, we have the fact from calculus (more on this later in the course), that if $f(x)$ is differentiable at $x = c$, and $f(x)$ has a local maximum or a local minimum at $x = c$, then $df(x)/dx = 0$ at $x = c$.

Theorem. Let $f, g : X \to \mathbb{R}$ be two functions, with f bounded above and g bounded. Then $f + g$ is bounded above, and $\sup(f) + \inf(g) \leq \sup(f + g) \leq \sup(f) + \sup(g)$.

Proof. From $f(x) \leq \sup(f)$ and $g(x) \leq \sup(g)$ we get $f(x) + g(x) \leq \sup(f) + \sup(g)$ for all $x \in X$. Thus $f + g$ is bounded above, and $\sup(f) + \sup(g)$ is an upper bound; hence $\sup(f + g) \leq \sup(f) + \sup(g)$.
For the other part, begin with $f(x) + g(x) \leq \sup(f + g)$, which leads to
$f(x) \leq \sup(f + g) - g(x) \leq \sup(f + g) - \inf(g)$, for all $x \in X$.
Hence $\sup(f) \leq \sup(f + g) - \inf(g)$, which gives the other required inequality.

Exercise. Write out the corresponding theorem with sup and inf interchanged, and with $\leq$ and $\geq$ interchanged.

4 Sequences Revisited, and Bolzano-Weierstrass.

Definition. A sequence $\{x_n\}_{n=1}^{\infty}$ is *increasing* if $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$, and *decreasing* if $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$.

Theorem. If x_n are real and $\{x_n\}_{n=1}^{\infty}$ is increasing and bounded above, then $x_n \to L$ as $n \to \infty$, where $L = \sup(\{x_n\})$.

Proof. By completeness, $L = \sup(\{x_n\})$ exists. Let $\varepsilon > 0$ be given. By definition of sup, there is an integer N such that $x_N > L - \varepsilon$. Since the sequence is increasing, it follows that $x_n > L - \varepsilon$ for all $n \geq N$. Thus $-\varepsilon < x_n - L < 0 < \varepsilon \Rightarrow |x_n - L| < \varepsilon$ for all $n \geq N$.

Remark. Similarly, a decreasing sequence which is bounded below tends to a limit.

Theorem. If $x_n \to L$ as $n \to \infty$, then the set $S = \{x_n\}_{n=1}^{\infty}$ is bounded.

Proof. Choose $\varepsilon = 1$, and find $N \in \mathbb{N}$ such that $|x_n - L| < 1$ for all $n \geq N$. Then $x_n < L + 1$ for all $n \geq N$, and so $K = \max(L + 1, x_1, x_2, \ldots, x_{N-1})$ is an upper bound for S. Similarly you can write down a lower bound.

Remark. The converse of this is not, of course, true. But the Bolzano-Weierstrass theorem provides a partial converse: it says that *every bounded sequence contains a subsequence which tends to a limit*. This is a rather general result — it does not just apply to sets of real numbers — but it makes essential use of completeness. The Bolzano-Weierstrass theorem for real numbers follows immediately from the previous theorem, together with the Lemma below.

Definition. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence. A *subsequence* of $\{x_n\}$ is a sequence $\{x_{n_i}\}_{i=1}^{\infty}$, where $n_1 < n_2 < n_3 < \ldots$.

Example. Let $x_n = (-1)^n(1 - 1/n)$, so the sequence is $S = \{0, 1/2, -2/3, 3/4, -4/5, \ldots\}$. Clearly this sequence does not have a limit. But the subsequence defined by $\{n_i\}_{i=1}^{\infty} = \{2, 4, 6, 8, \ldots\}$ is $\widehat{S} = \{1/2, 3/4, 5/6, \ldots\}$, and this does have a limit, namely 1.

Lemma. Every sequence of real numbers contains a subsequence which is either increasing or decreasing.

Proof. Given a sequence $\{x_n\}_{n=1}^{\infty}$, we say that n_0 is a *peak index* if $x_n \leq x_{n_0}$ for all $n \geq n_0$. There are two possibilities.

- Case 1: there are infinitely many peak indices $n_1 < n_2 < n_3 < \ldots$. Then by definition $x_{n_2} \leq x_{n_1}$ (since n_1 is peak) etc; so $\{x_{n_i}\}_{i=1}^{\infty}$ is a decreasing subsequence.
- Case 2: there are finitely many peak indices. Let $n_1 \in \mathbb{N}$ be larger than all the peak indices: so n_1 is not peak, and so there is an n_2 such that $x_{n_2} > x_{n_1}$. But then n_2 is not peak, so we can continue in this way, and we get an increasing subsequence $\{x_{n_i}\}_{i=1}^{\infty}$.

5 Infinite Series.

Definition. Let $\{x_n\}$ be a real sequence. Define a new sequence $\{S_k\}_{k=1}^{\infty}$ (the sequence of *partial sums*) by $S_k = \sum_{n=1}^{k} x_n$. If $S_k \to L$ as $k \to \infty$, we say that the series $\sum_{n=1}^{\infty} x_n$ *converges* to L. If S_k has no limit as $k \to \infty$, we say that the series $\sum_{n=1}^{\infty} x_n$ *diverges*.

Examples.

1. If $x_n = 1$ for all n, then $S_k = k$, and so $\sum_{n=1}^{\infty} x_n$ diverges.
2. Take $x_n = \dfrac{1}{n(n+1)}$. Then using $\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$, we get $S_k = 1 - \dfrac{1}{k+1}$, and so $\sum_{n=1}^{\infty} x_n$ converges to 1.
3. Take $x_n = t^n$, where t is a fixed real number. This time, begin the sum at $n = 0$, ie take $S_k = \sum_{n=0}^{k} x_n = 1 + t + t^2 + \ldots + t^k$. It's an elementary result that $S_k = (1 - t^{k+1})/(1-t)$, and so we conclude that $\sum_{n=0}^{\infty} x_n$ converges to $1/(1-t)$ if $|t| < 1$, but diverges if $|t| \geq 1$. This is the *Geometric Series.*

Theorem *(Linearity).* If $\sum_{n=1}^{\infty} x_n$ converges to S, and $\sum_{n=1}^{\infty} y_n$ converges to T, and if A and B are real numbers, then $\sum_{n=1}^{\infty}(Ax_n + By_n)$ converges to $AS + BT$.

Proof. Apply the corresponding theorem for sequences to the partial sums.

Theorem. If $\sum_{n=1}^{\infty} x_n$ converges, then $x_n \to 0$ as $n \to \infty$.

Proof. Suppose that $\sum_{n=1}^{\infty} x_n$ converges to S, ie the partial sums S_k tend to S as $k \to \infty$. Then also $S_{k-1} \to S$ as $k \to \infty$. But $x_n = S_n - S_{n-1}$, so if we let $n \to \infty$ and use linearity, we get $x_n \to S - S = 0$ as $n \to \infty$.

Remark *(Very Important).* If $x_n \to 0$, it does *not* follow that $\sum_{n=1}^{\infty} x_n$ converges. The following example illustrates this.

Example. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. This divergent series is called the *Harmonic Series.*

Proof. Consider the partial sums S_1, S_2, S_4, S_8 etc. We have

$$\begin{aligned} S_1 &= 1 \\ S_2 &= S_1 + \frac{1}{2} = \frac{3}{2} \\ S_4 &= S_2 + \frac{1}{3} + \frac{1}{4} > \frac{3}{2} + \frac{1}{4} + \frac{1}{4} = 2 \end{aligned}$$

and it is easy to prove by induction that

$$S_{2^n} \geq \frac{1}{2}n + 1.$$

So this subsequence is unbounded and has no limit, and therefore the sequence $\{S_k\}_{k=1}^{\infty}$ has no limit.

Remark. It is clear from this that the harmonic series diverges logarithmically; in fact, we have $S_k - \log k \to \gamma$ as $k \to \infty$, where $\gamma = 0.5772156649\ldots$ is *Euler's constant.*

Theorem *(Comparison Test).* Suppose that $0 \le x_n \le y_n$ for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} y_n$ converges to T, then $\sum_{n=1}^{\infty} x_n$ converges to $S \le T$. Equivalently, if $\sum_{n=1}^{\infty} x_n$ diverges, then so does $\sum_{n=1}^{\infty} y_n$.

Proof. Write $S_k = \sum_{n=1}^{k} x_n$ and $T_k = \sum_{n=1}^{k} y_n$. Note that $S_k \le T_k$, and also that $T_k \le T$ (since $\{T_k\}$ is an increasing sequence which tends to T as $k \to \infty$). So $\{S_k\}$ is an increasing sequence which is bounded above by T; hence by completeness it has a limit as $k \to \infty$.

Examples. Investigate $\sum_{n=1}^{\infty} x_n$ in each of the following cases.

1. $x_n = \dfrac{\sqrt{n^2+1}}{n^2}$. Here $x_n \ge \dfrac{1}{n}$, and so $\sum x_n$ diverges by comparison with the harmonic series.

2. $x_n = \dfrac{1}{n^2}$. Here $0 < x_{n+1} = \dfrac{1}{(n+1)^2} \le \dfrac{1}{n(n+1)} = y_n$, and $\sum y_n$ converges (shown earlier), so $\sum x_{n+1}$ converges by comparison, so $\sum x_n$ converges as well (just add x_1). [*Remark:* in fact, it converges to $\pi^2/6$.]

3. $x_n = \dfrac{(n^2+5n+7)(n^2+3n+1)}{(n^3+8n^2+6)^2}$. Note that $n^2 x_n \to 1$ as $n \to \infty$. So there is a number K such that $n^2 x_n \le K$ for all n. Hence $0 < x_n \le K/n^2$, and the series $\sum x_n$ converges by comparison.

4. *Claim:* $\sum_{n=1}^{\infty} n^{-p}$ converges if and only if $p > 1$. We already know this (by comparison) for $p \ge 2$ and for $p \le 1$. A proof for $1 < p < 2$ will be given later.

5. $x_n = \dfrac{n}{\sqrt{n^7+2}}$. Here $0 < x_n < n^{-5/2}$, and $\sum n^{-5/2}$ converges, so $\sum x_n$ converges by comparison.

6. $x_n = \dfrac{n+3}{\sqrt{2n^3-1}}$. Here $x_n > 1/\sqrt{2n}$, and $\sum 1/\sqrt{2n}$ diverges, so $\sum x_n$ diverges by comparison.

7. $x_n = n^8/\mathrm{e}^n$. Note that $n^2 x_n \to 0$ as $n \to \infty$, so there exists a number K such that $n^2 x_n \le K$ for all n. Hence $0 < x_n \le K/n^2$, and the series $\sum x_n$ converges by comparison.

8. $x_n = \log(n^2)/n^2$. Note that $n^{3/2} x_n \to 0$ as $n \to \infty$, so there exists a number K such that $n^{3/2} x_n \le K$ for all n. Hence $0 < x_n \le K n^{-3/2}$, and the series $\sum x_n$ converges by comparison.

Definition. If $\sum_{n=1}^{\infty} |x_n|$ converges, then we say that $\sum_{n=1}^{\infty} x_n$ *converges absolutely*. If $\sum_{n=1}^{\infty} x_n$ converges but $\sum_{n=1}^{\infty} |x_n|$ diverges, then we say that $\sum_{n=1}^{\infty} x_n$ *converges conditionally*.

Theorem *(Absolute Convergence Theorem).* If $\sum_{n=1}^{\infty} x_n$ converges absolutely, then it converges.

Proof. We are given that $\sum_{n=1}^{\infty} |x_n|$ converges, and hence so does $\sum 2|x_n|$. Put $y_n = |x_n| - x_n$: then $0 \leq y_n \leq 2|x_n|$ and so $\sum y_n$ converges by comparison. Finally, $x_n = |x_n| - y_n$, so $\sum x_n$ converges by linearity.

Example. The series $\sum_{n=1}^{\infty} (-1)^{n+1}/n^2 = 1 - 1/4 + 1/9 - 1/16 + \ldots$ converges (since we already know that $\sum_{n=1}^{\infty} 1/n^2$ does).

Theorem *(Ratio Test).* Let $\{x_n\}$ is a sequence of non-zero numbers such that $|x_{n+1}/x_n| \to L$ as $n \to \infty$. Then:

(i) $\sum_{n=0}^{\infty} x_n$ converges absolutely if $L < 1$;

(ii) $\sum_{n=0}^{\infty} x_n$ diverges if $L > 1$;

(iii) if $L = 1$ we cannot conclude anything.

Proof. The prototype of this is the Geometric Series, and we shall compare with this series. First, suppose that $L < 1$. Choose a number M such that $L < M < 1$. Then there exists an integer N such that $|x_{n+1}/x_n| \leq M$ for all $n \geq N$. Now observe that

$$\begin{aligned} |x_{N+1}| &\leq M|x_N| \\ |x_{N+2}| &\leq M|x_{N+1}| \leq M^2|x_N| \end{aligned}$$

and so forth: we see that $|x_{N+k}| \leq M^k |x_N|$ for all $k \in \mathbb{N}$. But the geometric series $\sum_{k=0}^{\infty} |x_N| M^k$ converges (since $M < 1$); and hence so does $\sum_{n=N}^{\infty} |x_n|$, by comparison. Now add $x_1, \ldots, x_{N-1}$ to deduce that $\sum_{n=0}^{\infty} x_n$ converges absolutely.
Next, suppose that $L > 1$. Then there exists an integer N such that $|x_{n+1}/x_n| > 1$ for all $n \geq N$. So $|x_N| < |x_{N+1}| < \ldots$, from which it follows that x_n cannot tend to zero as $n \to \infty$. Thus $\sum x_n$ diverges.
Finally, consider the two examples $x_n = 1$ and $x_n = 1/n^2$. In each case, we have $L = 1$; but in the first case we have divergence, whereas in the second case we have convergence.

Examples. Investigate $\sum_{n=0}^{\infty} x_n$ in each of the following cases.

1. $x_n = t^n/n!$ where t is a constant. Here $|x_{n+1}/x_n| = |t|/(n+1) \to 0$ as $n \to \infty$. So $\sum x_n$ converges for all values of t.

2. $x_n = nt^n$, where t is a constant. Here $|x_{n+1}/x_n| = |t|(n+1)/n \to |t|$ as $n \to \infty$. So $\sum x_n$ converges for $|t| < 1$, and diverges for $|t| > 1$. The $|t| = 1$ cases can be done separately: for $t = \pm 1$, the series clearly diverges.

3. $x_n = n!\, t^n$, where t is a constant. Here $|x_{n+1}/x_n| = |t|(n+1)$ which tends to infinity as $n \to \infty$ (unless $t = 0$). So $\sum x_n$ converges only if $t = 0$.

4. $x_n = t^n/(n^2\, 3^n)$, where t is a constant. Here $|x_{n+1}/x_n| = |t|(n+1)^2/(3n^2) \to |t|/3$ as $n \to \infty$. So $\sum x_n$ converges for $|t| < 3$, and diverges for $|t| > 3$. The $|t| = 3$ cases can be done separately: for $t = \pm 3$, the series converges absolutely.

Theorem *(Integral Test).* Suppose $f(x)$ is a positive decreasing function on $[1, \infty)$. Write $x_n = f(n)$ for $n \in \mathbb{N}$, and $F(m) = \int_1^m f(t)\, dt$ for $m \in \mathbb{N}$. Then $\sum_{n=1}^{\infty} x_n$ converges if and only if $F(m)$ tends to a limit as $m \to \infty$.

Proof. Write $I_n = \int_n^{n+1} f(t)\, dt$. Note that $F(m)$ is the partial sum for the series $\sum_{n=1}^{\infty} I_n$, so the theorem claims that the two series $\sum x_n$ and $\sum I_n$ behave in the same way. Now we have $x_{n+1} \le I_n \le x_n$ (since f is decreasing), and so we can play the comparison test both ways to see that this is indeed the case.

Examples. Investigate $\sum_{n=0}^{\infty} x_n$ in each of the following cases.

1. $x_n = 1/n^p$ where $p \in \mathbb{R}$ is a constant. Define f by $f(x) = 1/x^p$. Note f is decreasing (at least for $p \ge 0$: for $p < 0$ we have divergence anyway). If $p = 1$, we get $F(m) = \log m$ which has no limit as $m \to \infty$, and so the series diverges (we already knew this: it's the harmonic series). If $p \ne 1$, we get $F(m) = (m^{1-p} - 1)/(1-p)$ which has a limit (as $m \to \infty$) if and only if $p > 1$. The conclusion, from the Integral Test, is that $\sum n^{-p}$ converges if and only if $p > 1$.

2. $x_n = 1/(n \log n)$ for $n \ge 2$. Define f by $f(x) = 1/(x \log x)$ for $x \ge 2$. Note f is decreasing. Then $F(m) = \log\log m - \log\log 2$, which has no limit as $m \to \infty$, and so the series $\sum_{n=2}^{\infty} x_n$ diverges.

Definition. An *alternating series* is one whose terms alternate in sign.

Theorem *(Alternating Sign Test).* Suppose $\{y_n\}$ is a decreasing sequence of positive numbers such that $y_n \to 0$ as $n \to \infty$. Then the alternating series $\sum(-1)^{n+1} y_n = y_1 - y_2 + y_3 - \ldots$ converges.

Proof. First, consider the odd partial sums S_1, S_3, S_5, ... Note that $S_3 = S_1 - y_2 + y_3 \le S_1$ etc, so this is a decreasing sequence. Also, $S_{2n+1} = (y_1 - y_2) + (y_3 - y_4) + \ldots + (y_{2n-1} - y_{2n}) + y_{2n+1}$, and each term of this is ≥ 0, so $\{S_{2n+1}\}$ is bounded below by zero. Thus S_{2n+1} tends to a limit L as $n \to \infty$.
Next, consider the even partial sums S_2, S_4, S_6, ... Here we have $S_4 = S_2 + y_3 - y_4 \ge S_2$ etc, so the sequence is increasing. Also, $S_{2n} = y_1 - (y_2 - y_3) - \ldots - (y_{2n-2} - y_{2n-1}) - y_{2n} < y_1$,

so $\{S_{2n}\}$ is bounded above by y_1. Thus S_{2n} tends to a limit K as $n \to \infty$.
Finally, we have $y_{2n} = S_{2n} - S_{2n-1}$, and taking the $n \to \infty$ limit of this gives $0 = K - L$.
So $K = L$, and $S_k \to L$ as $k \to \infty$.

Examples. Investigate $\sum_{n=1}^{\infty} x_n$ in each of the following cases.

1. $x_n = (-1)^n/n$. Since $\{1/n\}$ is a decreasing sequence which tends to zero, the Alternating Sign Test says that the series $\sum x_n$ converges. [In fact, it converges to $-\log 2$.] This is therefore an example of a conditionally-convergent series.

2. $x_n = \cos(\pi n)\sin(\pi/n)$. Note that $\cos(\pi n) = (-1)^n$, and that $\{\sin(\pi/n)\}$ is a decreasing sequence which tends to zero. So by the Alternating Sign Test, the series $\sum x_n$ converges.

Extra tests for the convergence of the series.

- **Quotient test**

  If $u_n \geq 0$ and $v_n > 0$ and if
  $$\lim \frac{u_n}{v_n} = A$$
  and $A \neq 0$ and is finite then either $\sum u_n$ and $\sum v_n$ both converge or both diverge.

  Examples

  1. $\sum_n \frac{n}{4n^3-3}$ converges as $n^2 \frac{n}{4n^3-3} \to \frac{1}{4}$
  2. $\sum_n \frac{\log n}{\sqrt{n^2+1}}$ diverges as $\sum \frac{\log n}{n}$ diverges (shown before)

- n^{th} **root test**

  The series $\sum_n x_n$ converges if $|x_n|^{\frac{1}{n}} \to L < 1$

  The series $\sum_n x_n$ diverges if $|x_n|^{\frac{1}{n}} \to L > 1$

  Example:

  Consider $1 + 2r + r^2 + 2r^3 + r^4 + 2r^5 + \cdots$

  The ratio test cannot be used as the series has no limit (odd, even terms etc)

  But look at $|x_n|^{\frac{1}{n}}$ this is

  $|r|$ for n even and $|r|2^{\frac{1}{n}}$ for n odd
  $$\text{so} \qquad \lim_{n\to\infty} |x_n|^{\frac{1}{n}} = |r|$$
  and we see that the series converges for $|r| < 1$.

- **Raabe's test**

  Let
$$\lim_{n\to\infty} n\left(1 - \frac{|x_{n+1}|}{|x_n|}\right) = L$$
  Then series $\sum x_n$ converges absolutely if $L > 1$ and diverges or converges conditionally if $L < 1$.

- **Gauss's test**

  If
$$\left|\frac{x_{n+1}}{x_n}\right| = 1 - \frac{L}{n} + \frac{c_n}{n^2} \quad \text{and} \quad |c_n| < P$$
  where P is a constant then the series $(\sum_n x_n)$ converges absolutely if $L > 1$ and diverges (or converges conditionally) if $L \leq 1$.

- **Leibnitz's test**

  Take $x_1, x_2, \cdots$ a sequence of real monotonic numbers decreasing to zero and

  $x_1 \geq x_2 \geq x_3 \geq ...$ and $\lim_{m\to\infty} x_m = 0$ then the series
$$x_1 - x_2 + x_3 - x_4 + \cdots$$
  converges. Moreover the remainder R_n (after the n^{th} term) satisfies
$$|R_n| \leq x_{n+1}$$
  .

6 Complex Sequences and Series.

Complex Numbers. A complex number has the form $z = x + \mathrm{i}y$, where $\mathrm{i}^2 = -1$, $x = \mathrm{Re}(z)$ is the real part of z, and $y = \mathrm{Im}(z)$ is the imaginary part of z. The complex conjugate of z is $\bar{z} = x - \mathrm{i}y$, and its modulus $|z|$ is defined by $|z|^2 = z\bar{z} = x^2 + y^2$. The polar form of z is $z = r\mathrm{e}^{\mathrm{i}\theta}$, where $r = |z|$. Note that $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$, and that $|\mathrm{e}^{\mathrm{i}\theta}| = 1$.

Definition. Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. Then we say that $z_n \to c \in \mathbb{C}$ as $n \to \infty$ if $|z_n - c| \to 0$ as $n \to \infty$. [Note that $\{|z_n - c|\}$ is a *real* sequence, so we already know what this means.]

Theorem. Write $z_n = x_n + \mathrm{i}y_n$ and $c = a + \mathrm{i}b$, where x_n, y_n, a and b are real. Then $z_n \to c$ if and only if both $x_n \to a$ and $y_n \to b$ as $n \to \infty$.
Proof. Note that $0 \le |z_n - c| = \sqrt{(x_n - a)^2 + (y_n - b)^2} \le |x_n - a| + |y_n - b|$. So if $x_n \to a$ and $y_n \to b$, then $z_n \to c$ by squeezing.
Conversely, we also have $|x_n - a| \le |z_n - c|$ and $|y_n - b| \le |z_n - c|$. So if $z_n \to c$, then both $x_n \to a$ and $y_n \to b$ by squeezing.

Examples.

1. $z_n = 1/(\mathrm{i} + n)$.
   Method 1. $z_n = n^{-1}/(1 + \mathrm{i}/n) \to 0/(1+0) = 0$ as $n \to \infty$.
   Method 2. $|z_n - 0| = 1/\sqrt{n^2+1} < 1/n \to 0$ as $n \to \infty$, so $z_n \to 0$ by squeezing.
   Method 3. The real and imaginary parts of z_n are $x_n = n/(n^2+1)$ and $y_n = -1/(n^2+1)$, and both of these tend to 0 as $n \to \infty$.

2. $z_n = \dfrac{\sqrt{n^2+\mathrm{i}}}{n+2\mathrm{i}} \exp\left(\dfrac{\mathrm{i}\pi n}{\sqrt{n^2+1}+\sqrt{n^2-1}}\right)$.
   By Calculus of Limits, this tends to $1.\exp(\mathrm{i}\pi/2) = \mathrm{i}$ as $n \to \infty$.

3. $z_n = \dfrac{\sqrt{n^3+1}}{n^2+2\mathrm{i}} \exp(\mathrm{i}n^2)$. Here we can use squeezing:
   $|z_n|^2 = (n^3+1)/(n^4+4) \to 0$ as $n \to \infty$, so $z_n \to 0$ as $n \to \infty$.

Remark. In the second example, we should say what is meant by $\sqrt{c}$, where $c \in \mathbb{C}$. The equation $z^2 = c$ has two solutions, of the form $\pm\sqrt{c}$; but which one is $\sqrt{c}$? Our convention is that $\sqrt{c}$ is the root which has positive real part. This fails only when c is a negative real number, which does not occur for the examples encountered in this course.

Definition. We say that the complex series $\sum_{n=1}^{\infty} z_n$ *converges* to S if the partial sums $S_k = \sum_{n=1}^{k} z_n$ tend to S as $k \to \infty$.

Theorem. Write $z_n = x_n + \mathrm{i}y_n$, where x_n and y_n are real. Then $\sum_{n=1}^{\infty} z_n$ converges if and only if both $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ converge.

Proof. Apply the previous theorem to the partial sums of $\sum z_n$, $\sum x_n$ and $\sum y_n$.

Definition. We say that the complex series $\sum_{n=1}^{\infty} z_n$ *converges absolutely* if the real series $\sum_{n=1}^{\infty} |z_n|$ converges.

Theorem *(Absolute Convergence Theorem).* If $\sum_{n=1}^{\infty} z_n$ converges absolutely, then $\sum_{n=1}^{\infty} z_n$ converges.

Proof. Note that $0 \leq |x_n| \leq |z_n|$ and $0 \leq |y_n| \leq |z_n|$. So if $\sum |z_n|$ converges, then so do $\sum |x_n|$ and $\sum |y_n|$, by comparison. Thus $\sum x_n$ and $\sum y_n$ converge, by the real Absolute Convergence Theorem. And finally this implies that $\sum z_n$ converges, by the previous theorem.

Theorem *(Vanishing Theorem).* If $\sum_{n=1}^{\infty} z_n$ converges, then $z_n \to 0$ as $n \to \infty$.

Proof. Same as the real case.

Example (the Geometric Series). The series $\sum_{n=0}^{\infty} c^n$ converges to $1/(1-c)$ if $|c| < 1$, and diverges if $|c| \geq 1$. To see this, use the explicit formula for the partial sum, exactly as in the real case.

Remark. The Ratio Test works exactly as in the real case (and its proof is the same). Suppose $|z_{n+1}/z_n| \to L$ as $n \to \infty$.

- If $L < 1$, then $\sum z_n$ converges absolutely.
- If $L > 1$, then $\sum z_n$ diverges.

Examples.

1. $\sum z_n$, where $z_n = 1/(1 + \mathrm{i}n^2)$. Here $|z_n| = 1/\sqrt{1+n^4} < 1/n^2$, so the series is absolutely convergent, by comparison with the convergent series $\sum n^{-2}$.

2. $\sum z_n$, where $z_n = 1/(1 + \mathrm{i}\sqrt{n})$. Here $-\mathrm{Im}(z_n) = \sqrt{n}/(1+n) > 1/(2\sqrt{n})$, and $\sum (2\sqrt{n})^{-1}$ diverges, so $\sum \mathrm{Im}(z_n)$ diverges by comparison, so $\sum z_n$ diverges.

3. $\sum z_n$, where $z_n = (n + \mathrm{i})/(2^n + \mathrm{i})$. Here $|z_{n+1}/z_n| \to 1/2$ as $n \to \infty$, so the series converges by the Ratio Test.

7 Power Series, Radius of Convergence, and Taylor.

Definition. A *power series* in $z \in \mathbb{C}$ is a series of the form $\sum_{n=0}^{\infty} a_n z^n$, where the a_n are complex constants. Given a power series in z, a real number $R \geq 0$ such that the series converges for $|z| < R$ and diverges for $|z| > R$ is called its *radius of convergence*. [Two special cases are: $R = 0$, where the series converges only for $z = 0$; and R infinite, where the series converges for all $z \in \mathbb{C}$.]

Remark. If $|a_{n+1}/a_n| \to L$ as $n \to \infty$, then the Ratio Test tells us that the radius of convergence is $R = 1/L$. [If $L = 0$, then R is infinite; if $|a_{n+1}/a_n| \to \infty$ as $n \to \infty$, then $R = 0$.] This fact is often used in calculations; as we shall see later, however, the radius of convergence exists even when $|a_{n+1}/a_n|$ does not tend to a limit as $n \to \infty$.

Examples.

1. $a_n = 2^n/n$. Here $|a_{n+1}/a_n| = 2n/(n+1) \to 2$ as $n \to \infty$. So the series $\sum a_n z^n$ has radius of convergence $R = 1/2$.

2. $a_n = 1/n!$. Here $|a_{n+1}/a_n| = 1/(n+1) \to 0$ as $n \to \infty$. So the series $\sum a_n z^n$ has infinite radius of convergence .

3. $a_n = n^n$. Here $|a_{n+1}/a_n| > n+1$, and hence it tends to infinity as $n \to \infty$. So the series $\sum a_n z^n$ has radius of convergence $R = 0$.

4. $a_n = n^n/n!$. Here $|a_{n+1}/a_n| = [(n+1)/n]^n \to \mathrm{e}$ as $n \to \infty$. So the series $\sum a_n z^n$ has radius of convergence $R = \mathrm{e}^{-1}$.

Lemma. Suppose that $\sum_{n=0}^{\infty} a_n c^n$ converges, for some $c \neq 0$. Then $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for all z such that $|z| < |c|$.

Proof. Since $\sum a_n c^n$ converges, the Vanishing Theorem implies that $a_n c^n \to 0$ as $n \to \infty$. So there exists a positive real number M such that $|a_n c^n| \leq M$ for all n. Then $0 \leq |a_n z^n| \leq M|z/c|^n$ for all n. Since $\sum_n M|z/c|^n$ is a convergent geometric series, the result follows by comparison.

Theorem. For any power series $\sum_{n=0}^{\infty} a_n z^n$, one of the following possibilities must hold:

1. $\sum a_n z^n$ converges only for $z = 0$;

2. $\sum a_n z^n$ converges absolutely for all $z \in \mathbb{C}$;

3. there is a number $R > 0$ such that $\sum a_n z^n$ converges absolutely if $|z| < R$ and diverges if $|z| > R$.

Proof. Define $S = \{x \in \mathbb{R} : \sum a_n w^n \text{ converges for some } w \text{ with } |w| = x\}$.
If S is not bounded above, then possibility 2 holds. For given any $z \in \mathbb{C}$, we can find $x \in S$ such that $x > |z|$. By definition of S, this means that there is a complex number w such that $\sum a_n w^n$ converges and $|w| > |z|$. But then the Lemma implies that $\sum a_n z^n$ converges absolutely.
If, on the other hand, S is bounded above, put $R = \sup(S)$. If $R = 0$, then $S = \{0\}$, and possibility 1 holds. So suppose that $R > 0$. Then if $|z| < R$, there exists $x \in S$ with $|z| < x$ (by definition of sup), so as before $\sum a_n z^n$ converges absolutely; while if $|z| > R$, then $|z| \notin S$, so $\sum a_n z^n$ diverges by definition of S.

Remark. A special case of a complex power series is a real power series $\sum_{n=0}^{\infty} a_n x^n$, where $x \in \mathbb{R}$. It has a radius of convergence exactly as in the complex case. The question is whether we can differentiate and integrate such a series term-by-term; we shall see below that we can. For general series of functions, however, this is not valid: for example, it is *not* true that the derivative of $\sum_n u_n(x)$ is $\sum_n du_n(x)/dx$. So power series have special properties that more general series of functions do not have.

Theorem (Proof omitted). Suppose that the real power series $\sum_{n=0}^{\infty} a_n x^n$ converges for $x = t$, and that $0 < c < t$. Then for $x \in [-c, c]$, we have

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

And for $-c \le \alpha \le \beta \le c$, we have

$$\int_{\alpha}^{\beta}\left(\sum_{n=0}^{\infty} a_n x^n\right) dx = \sum_{n=0}^{\infty} a_n(\beta^{n+1} - \alpha^{n+1})/(n+1).$$

Taylor Series. For certain functions $f(x)$, and appropriate ranges of x, we can write $f(x) = \sum_{n=0}^{\infty} a_n x^n$, where $a_n = f^{(n)}(0)/n!$.

Examples.

1. A polynomial $f(x) = a_k x^k + \ldots + a_1 x + a_0$ is its own Taylor series.
2. $\exp x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots$ is valid for all $x \in \mathbb{R}$.
3. $\cosh x = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \ldots$ and $\sinh x = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \ldots$ are valid for all $x \in \mathbb{R}$.
4. $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \ldots$ and $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots$ are valid for all $x \in \mathbb{R}$.

5. $\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots$ is valid for all $-1 < x \leq 1$.

Example *(the Binomial expansion).* For $|x| < 1$, and any $c \in \mathbb{R}$, we have

$$(1+x)^c = 1 + \sum_{n=1}^{\infty} \begin{pmatrix} c \\ n \end{pmatrix} x^n,$$

where

$$\begin{pmatrix} c \\ n \end{pmatrix} = \frac{c(c-1)(c-2)\ldots(c-n+1)}{n!}$$

are the *binomial coefficents.* Here are some special cases.

1. If c is a positive integer, then $\begin{pmatrix} c \\ n \end{pmatrix} = 0$ for $c < n$, and so the series terminates (only the first $n+1$ terms can be non-zero).
2. Since $\begin{pmatrix} -1 \\ n \end{pmatrix} = (-1)^n$, we have $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$, which we already knew (geometric series).
3. $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{3}{16}x^3 - \ldots$ for $|x| < 1$.
4. In relativity, the energy of an object of mass m travelling with speed v is $E = m\gamma c^2$, where c is the speed of light, and $\gamma = 1/\sqrt{1 - v^2/c^2}$. If we expand E in powers of v/c, we get $E = mc^2 + \frac{1}{2}mv^2 + \frac{3}{8}m\frac{v^4}{c^2} + \ldots$. This is valid for $|v/c| < 1$, ie for v less than the speed of light. The first term in this expansion of E is the rest-energy of the object, the second term is its Newtonian kinetic energy, and the subsequent terms are relativistic corrections (which tend to infinity as $|v| \to c$).

Remark. For a function like $f(x) = \exp(x^2)$, it's much easier to substitute $y = x^2$ into the Taylor series for $\exp y$ than to calculate the Taylor coefficients directly: clearly we have $\exp(x^2) = 1 + x^2 + x^4/2 + \ldots$. Here is another example.

Example. $[\log(1-2x)]^2 = [2x + 2x^2 + 8x^3/3 + 4x^4 + \ldots]^2 = 4x^2 + 8x^3 + (4 + 32/3)x^4 + \ldots$, which is valid for $-1 < -2x \leq 1$.

8 Limits of Functions, and Continuity.

Definition. Let $f : [c, \infty) \to \mathbb{R}$ be a function. We say that $f(x) \to L$ as $x \to \infty$, also written as $\lim_{x\to\infty} f(x) = L$, if: given $\varepsilon > 0$, there exists a number K such that $|f(x) - L| < \varepsilon$ for all $x > K$.

Remark. Clearly this is related to the idea of a limit of a sequence, and many of the same techniques can be used, such as the following.

Theorem (*Calculus of Limits Theorem*). If $f(x) \to L$ and $g(x) \to H$ as $x \to \infty$, and if A and B are constants, then

(i) $Af(x) + Bg(x) \to AL + BH$ as $x \to \infty$;

(ii) $f(x)g(x) \to LH$ as $x \to \infty$;

(iii) $f(x)/g(x) \to L/H$ as $x \to \infty$, provided $H \neq 0$ and $g(x) \neq 0$ for all x.

Sample Proof. We prove (i) with $A = B = 1$. Let $\varepsilon > 0$ be given. Find P such that $|f(x) - L| < \varepsilon/2$ for all $x > P$, and find Q such that $|g(x) - H| < \varepsilon/2$ for all $x > Q$. Take $K = \max(P, Q)$. Then for $x > K$, we have

$$\begin{aligned} |(f(x) + g(x)) - (L + H)| &= |(f(x) - L) + (g(x) - H)| \\ &\leq |f(x) - L| + |g(x) - H| \quad \text{by } \Delta \text{ inequality} \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon; \end{aligned}$$

and so we have shown that $f(x) + g(x) \to L + H$ as $x \to \infty$.

Facts. "Exponentials beat powers, and powers beat logs." In other words,

$$x^c/p^x \to 0 \quad \text{and} \quad x^{-b}\log x \to 0 \quad \text{as } x \to \infty,$$

for all $c \in \mathbb{R}$, $p > 1$ and $b > 0$.

Examples.

$$\frac{(x+1)^2 - (x-1)^2}{(x+2)^2 - (x-1)^2} = \frac{4x}{6x+3} = \frac{4}{6+3/x} \to \frac{2}{3} \quad \text{as } x \to \infty.$$

$$\begin{aligned} \sqrt{x}\left(\sqrt{x+1} - \sqrt{x-1}\right) &= \frac{\sqrt{x}\left(\sqrt{x+1} - \sqrt{x-1}\right)\left(\sqrt{x+1} + \sqrt{x-1}\right)}{\sqrt{x+1} + \sqrt{x-1}} \\ &= \frac{2\sqrt{x}}{\sqrt{x+1} + \sqrt{x-1}} \to 1 \quad \text{as } x \to \infty. \end{aligned}$$

$$\frac{\log(x^3 + \mathrm{e}^{2x})}{x+3} = \frac{2 + x^{-1}\log(1 + x^3\mathrm{e}^{-2x})}{1 + 3/x} \to 2 \quad \text{as } x \to \infty.$$

$$f(x) = \frac{x^2 \cos x}{2x^3 + 3} \quad \text{satisfies} \quad |f(x)| < \frac{1}{2x} \to 0, \quad \text{so} \quad f(x) \to 0 \quad \text{as } x \to \infty, \text{ by squeezing.}$$

$$f(x) = \frac{x^2}{2x^3 \sin^2 x + 1} \quad \text{satisfies} \quad f(n\pi) = n^2\pi^2 \quad \text{for } n \in \mathbb{N}; \text{ so no limit.}$$

$$f(x) = \frac{x^2}{2x^2 + x \sin x} = \frac{1}{2 + x^{-1} \sin x} \to \frac{1}{2} \quad \text{as } x \to \infty.$$

Definition. Let $f(x)$ be a real-valued function defined for $a < x < c$ and for $c < x < b$. We say that $f(x) \to L$ as $x \to c$, also written as $\lim_{x \to c} f(x) = L$, if: given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $|x - c| < \delta$ [or, more correctly, for all $x \in ((a, c) \cap (c - \delta, c)) \cup ((c, b) \cap (c, c + \delta))$.]

Example. The function $f(x) = 2x$ satisfies $\lim_{x \to 1} f(x) = 2$.
Proof. Note that $|f(x) - 2| = 2|x - 1|$. So we can take $\delta = \varepsilon/2$; for then $|x - 1| < \delta \Rightarrow |f(x) - 2| < \varepsilon$.

Definition. Let $f(x)$ be a real-valued function defined for $c < x < b$. We say that $f(x) \to L$ as $x \to c_+$ (pronounced "x tends to c from the right"), if: given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ for all $x \in (c, c + \delta)$. Similarly, one can define $f(x) \to K$ as $x \to c_-$.

Theorem. Let $f(x)$ be a real-valued function defined for $a < x < c$ and for $c < x < b$. Then $\lim_{x \to c} f(x) = L$ if and only if both $\lim_{x \to c_+} f(x) = L$ and $\lim_{x \to c_-} f(x) = L$.
Proof. First, suppose that both $\lim_{x \to c_+} f(x) = L$ and $\lim_{x \to c_-} f(x) = L$. Given $\varepsilon > 0$, find δ_1 such that $|f(x) - L| < \varepsilon$ for all $x \in (c - \delta_1, c)$, and find δ_2 such that $|f(x) - L| < \varepsilon$ for all $x \in (c, c + \delta_2)$. Put $\delta = \min\{\delta_1, \delta_2\}$. Then $|f(x) - L| < \varepsilon$ whenever $|x - c| < \delta$.
Conversely, if $\lim_{x \to c} f(x) = L$, then the corresponding one-sided limits work immediately, with the same δ.

Example. Define $f(x)$, for $x \neq 0$, by: $f(x) = 0$ for $x < 0$ and $f(x) = 1$ for $x > 0$. Then $\lim_{x \to 0_+} f(x) = 1$ and $\lim_{x \to 0_-} f(x) = 0$, but $\lim_{x \to 0} f(x)$ does not exist.

Theorem. Let $f(x)$ be an increasing function on (a, b) which is bounded above. Then $f(x)$ tends to a limit $L = \sup(f)$ as $x \to b_-$.
Proof. Note that $\sup(f)$ exists, by completeness of $\mathbb{R}$; and $f(x) \leq L = \sup(f)$ for all $x \in (a, b)$. Given $\varepsilon > 0$, we can find a number $c \in (a, b)$ such that $f(c) > L - \varepsilon$ (by definition of sup). Write $\delta = b - c$. Since f is increasing, we have $L - \varepsilon < f(c) \leq f(x) \leq L < L + \varepsilon$, and hence $|f(x) - L| < \varepsilon$, for all $x \in (b - \delta, b)$.

Definition. We say that $f(x)$ is *continuous* at $x = c$ if $f(x)$ is defined on an interval of the form $(c - \delta, c + \delta)$, and if $\lim_{x \to c} f(x) = f(c)$. Combining with the definition of limit gives the following equivalent version: a function $f(x)$ defined on an open interval containing

c is said to be *continuous* at $x = c$ if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$.

Theorem. Let $f(x)$ be continuous at $x = c$, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence with $\lim_{n\to\infty} x_n = c$. Then $\lim_{n\to\infty} f(x_n) = f(c)$.

Proof. Let $\varepsilon > 0$ be given. First, find $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$. Then find N such that $|x_n - c| < \delta$ for all $n \geq N$. Putting these together, we get $|f(x_n) - f(c)| < \varepsilon$ for all $n \geq N$.

Remark. The converse is also true: namely, if f has the property that $\lim_{n\to\infty} f(x_n) = f(c)$ whenever $\lim_{n\to\infty} x_n = c$, then it follows that $f(x)$ is continuous at $x = c$. The proof of this is left as an exercise.

Example. Give an ε-δ proof that $f(x) = (x + 2)/(x - 1)$ is continuous at $x = -1$.

Solution. First, compute $|f(x) - f(-1)| = \frac{3|x+1|}{2|x-1|}$. Choose $\delta \leq 1$; then
$|x + 1| < \delta \Rightarrow -1 < x + 1 < 1 \Rightarrow -3 < x - 1 < -1 \Rightarrow 1/|x - 1| < 1$.
So we can take $\delta = \min\{1, 2\varepsilon/3\}$; it follows that $|x + 1| < \delta \Rightarrow |f(x) - f(-1)| < \varepsilon$.

Theorem. If $f(x)$ and $g(x)$ are continuous at $x = c$, and if A and B are constants, then

(i) $Af(x) + Bg(x)$ is continuous at $x = c$;

(ii) $f(x)g(x)$ is continuous at $x = c$;

(iii) $f(x)/g(x)$ is continuous at $x = c$, provided $g(x)$ is non-zero near c;

(iv) $h \circ f(x)$ is continuous at $x = c$, provided $h(y)$ is continuous at $y = f(c)$.

Sample Proof. (i)-(iii) are just the Calculus of Limits Theorem again. Here we prove (iv). Let $\varepsilon > 0$ be given. Write $y_0 = f(c)$. Find $\alpha > 0$ such that $|h(y) - h(y_0)| < \varepsilon$ for $|y - y_0| < \alpha$. Then find $\delta > 0$ such that $|f(x) - f(c)| < \alpha$ for $|x - c| < \delta$. It then follows that $|h \circ f(x) - h \circ f(c)| < \varepsilon$ for $|x - c| < \delta$.

Remark. We now prove a two important theorems involving continuous functions, both using the technique of bisection. Note this *Definition:* we say that $f : X \to \mathbb{R}$ is continuous if $f(x)$ is continuous for all $x \in X$.

Theorem *(Intermediate Value Theorem).* If $f : [a, b] \to \mathbb{R}$ is continuous, and $f(a) < 0 < f(b)$, then there exists a number $c \in (a, b)$ such that $f(c) = 0$.

Proof. Write $a_1 = a$ and $b_1 = b$. Begin by bisecting the interval $[a_1, b_1]$ at its midpoint $c_1 = (a_1 + b_1)/2$. If $f(c_1) = 0$, then we are done. Otherwise, let a_2 and b_2 be the endpoints of the half-interval on which f changes sign. Now continue by bisecting $[a_2, b_2]$, etc. If we hit a midpoint at which $f(x) = 0$, then that is the answer; so suppose that we go on forever. In other words, we get an infinite sequence $\{[a_n, b_n]\}_{n=1}^{\infty}$ of nested intervals. By construction, the sequence $\{a_n\}_{n=1}^{\infty}$ is increasing and bounded above (for example by b_1).

So $c = \lim_{n\to\infty} a_n$ exists. By continuity, $f(c) = \lim_{n\to\infty} f(a_n)$. Also, $f(a_n) < 0$ for all $n \in \mathbb{N}$, and so it follows that $f(c) \leq 0$.
Now we do the same with the b_n, and we find $b_n \to c'$ as $n \to \infty$, with $f(c') \geq 0$. Finally, note that

$$0 \leq |c' - c| = |c' - b_n + b_n - a_n + a_n - c| \leq |c' - b_n| + |b_n - a_n| + |a_n - c|,$$

and each of these three terms tends to 0 as $n \to \infty$; so $c = c'$. Since we have both $f(c) \leq 0$ and $f(c) \geq 0$, so it follows that $f(c) = 0$.

Remark. By applying this to the function $g(x) = f(x) - k$, we get the more general form of the Intermediate Value Theorem: if $g : [a, b] \to \mathbb{R}$ is continuous, and $g(a) < k < g(b)$, then there exists a number $c \in (a, b)$ such that $g(c) = k$.

Theorem If $f : [a, b] \to \mathbb{R}$ is continuous, then f is bounded.

Proof. Suppose f is continuous, but unbounded. So f is either unbounded below or unbounded above; let us suppose the latter. Bisect as in the previous theorem. Then f must be unbounded above on at least one half-interval: let $[a_2, b_2]$ be such a half-interval. Continue in this way. We end up with two sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$, both having a limit c. Since f is not bounded above on $[a_n, b_n]$, we can find (for each n) a number $x_n \in [a_n, b_n]$ such that $f(x_n) > n$; clearly the sequence $\{f(x_n)\}$ has no limit as $n \to \infty$. But by squeezing we have $x_n \to c$, and hence by continuity we have $f(x_n) \to f(c)$ as $n \to \infty$. This contradiction shows that f must be bounded above. Similarly, f must be bounded below.

Remark. So if $f : [a, b] \to \mathbb{R}$ is continuous, then $\sup(f)$ exists. Claim: f attains its supremum; in other words, there is a number $c \in [a, b]$ such that $f(c) = \sup(f)$.

Proof. Write $S = \sup(f)$, and suppose that $f(x) < S$ for all $x \in [a, b]$. We show that this leads to a contradiction. Define a new function $g(x) = 1/[S - f(x)]$. Note this is well-defined and continuous on $[a, b]$ (since the denominator is positive for all x). So by the previous theorem, the function g is bounded above: there exists a number $K > 0$ such that $g(x) \leq K$ for all x. But this implies that $f(x) \leq S - 1/K$ for all $x \in [a, b]$, which contradicts the fact that S is the *least* upper bound of f.

9 Uniform Convergence

So far studied convergence and divergence of infinite series.

We also discussed, a little, infinite series of functions and considered their convergence (in particular, power series). An example of such series is the Taylor series expansion of a function.

Now we will consider more general series of functions $\{u_n(x)\}$ $n = 1, 2, ...$

Definition: Let $\{u_n\}$, $n = 1, 2, \cdots$ be a sequence of functions on $[a, b]$. The sequence is said to converge to $f(x)$ in $[a, b]$ ($i.e$ have $f(x)$ as its limit) if for any $\epsilon > 0$ and each $x \in [a, b]$ we can find $N > 0$ such that

$$|u_n(x) - f(x)| < \epsilon, \quad \text{for} \quad \text{all} \quad n > N.$$

Note then we can write $f(x) = \lim_{n\to\infty} u_n(x)$.

Remark: Clearly, N depends on ϵ and, in general, also depends on x.

If N does **not depend on** x we say that the sequence converges to $f(x)$ uniformly in $[a, b]$ (or to be uniformly convergent in $[a, b]$).

The same is true for a series of functions

$$\sum_{n=1}^{\infty} u_n(x) \; = \; u_1(x) + u_2(x) + u_3(x) + ...$$

In this case we define $S_k(x) = \sum_{n=1}^{k} u_n(x)$, $k = 1, 2, ...$ and consider the limit $k \to \infty$.

Definition: $S_k(x) = \sum_{n=1}^{k} u_n(x)$ converges to $S(x)$ in $[a, b]$ if for every $\epsilon > 0$ and for each $x \in [a, b]$ we can find $N > 0$ such that

$$|S_k(x) - S(x)| < \epsilon$$

for all $k > N$. If $N = N(\epsilon)$ only, the series is uniformly convergent.

Remark: Since can define $S(x) - S_n(x) = R_n(x)$ (the remainder after n terms) we have, equivalently, that $\sum u_n(x)$ is unformly convergent in $[a, b]$ if for all $\epsilon > 0$ we can find N (dependent on ϵ) but **not** on x such that $|R_n(x)| < \epsilon$ for all $n > N$ and all $x \in [a, b]$.

Examples:

- $$F_n(x) \; = \; \frac{\sin(x)}{n} \; + \; x^2$$

  Clearly $F_n \to x^2$. But is this a uniform convergence?

  Yes, as $|F_n(x) - x^2| = |\frac{\sin(x)}{n}| < \epsilon$ so given that $|\sin(x)| < 1$ we see that we can take $N > \frac{1}{\epsilon}$ and we have **uniform convergence**.

- $$f_n(x) = x^n \qquad \text{on} \quad x \in [0,1]$$

  Clearly (plots) $f_n(1) = 1$ and $f_n(x) \to 0$ for $x \neq 1$ and so the limiting function P is discontinuous.

  Can we find N etc..?

  $$|x^n - P| < \epsilon,$$

  where P is this discontinuous function. But this translates to $x^n < \epsilon$ so we need to take

  $$N > \left|\frac{\log \epsilon}{\log x}\right|$$

  and so $N = N(x)$. Moreover, $N \to \infty$ as $x \to 1$, so **NO** uniform convergence.

  Note, however, that this result is true as we are considering $x \in [0,1]$

  We could consider, say, $x \in [0, \frac{1}{2}]$. Then the sequence goes to a continuous function $P(x) = 0$ and the convergence is uniform. We follow the same steps as before but this time we can take

  $$N > \left|\frac{\log \epsilon}{\log \frac{1}{2}}\right|$$

  as for any $x < \frac{1}{2} \qquad |\log \frac{1}{2}| < |\log(x)|$.

Remark: Uniform convergence of a sequence of functions or of a series is very similar. There are some tests for it.

Weierstrass test (theorem)

If a sequence of positive constants $M_1, M_2, ...M_n, ...$ can be found so that for $b \geq x \geq a$ and

$$a)\ |u_n(x)| \leq M_n, \qquad b)\ \sum M_n \quad \text{converges}$$

then $\sum_n u_n$ is uniformly and absolutely convergent in this interval.

Example:

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}$$

is uniformly and absolutely convergent in $[0, 2\pi]$ since $|\frac{\cos(nx)}{n^2}| \leq \frac{1}{n^2}$ and $\sum_n \frac{1}{n^2}$ converges.

Various theorems on uniformly convergent series.

- 1. If $u_n(x)$, $n = 1, 2, ...$ are continuous in $[a, b]$ and, if $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly to $S(x)$ then $S(x)$ is continuous in $[a, b]$. Thus

  $$\lim_{x \to x_0} \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \lim_{x \to x_0} u_n(x) = \sum_{n=1}^{\infty} u_n(x_0) = S(x_0.)$$

- 2. If $\{u_n(x)\}$, $n = 1, 2, ...$ are continuous in $[a, b]$ and $\sum u_n$ converges uniformly to $S(x)$ in $[a, b]$ then
$$\int_a^b S(x)\,dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)\,dx,$$
($i.e.$ we can integrate term by term).

- 3 If $\{u_n(x)\}$, $n = 1, 2, ...$ are continuous and have continuous derivatives and $S(x) = \sum u_n(x)$ and $\sum u'(x)$ converge uniformly in $[a, b]$ then
$$S'(x) = \sum u_n'(x)$$
$i.e.$ we can differentiate the series, term by term.

Comment: We stated that this is also true for the power series, *ie*, for the series of monomials $\sum a_n(x - x_0)^n$ **within** their radius of convergence, *ie* for $|x - x_0| < R$. This is true because of the following theorem:

Theorem: *Uniform convergence of power series.*
A power series $\sum_{m=0}^{\infty} a_m(z - z_0)^m$, with a nonzero radius of convergence R, is uniformly convergent in every circular disc $|z - z_0| \leq r$ of radius $r < R$.

Remark: There are series that converge absolutely but not uniformly and there are series that converge uniformly but not absolutely. (hence no relation).

Examples: 1. Consider the series
$$x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \frac{x^2}{(1+x^2)^3} + ...,$$
where x is real. As $1 + q + q^2 + q^3 + ... + q^n = \frac{1}{1-q} - \frac{q^{n+1}}{1-q}$ we see that
$$S_n(x) = x^2 \left[\frac{1+x^2}{x^2} - \frac{(1+x^2)}{x^2(1+x^2)^{n+1}}\right] = 1 + x^2 - \frac{1}{(1+x^2)^n},$$
and so $S_n \to 1 + x^2$ for $x \neq 0$ and $S_n \to 1 - 1 = 0$ for $x = 0$.

So, we note that $S_n \to S$ which is discontinuous at $x = 0$ and so this series converges absolutely but **NOT** uniformly.

2.
$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{x^2 + m} = \frac{1}{x^2+1} - \frac{1}{x^2+2} + \frac{1}{x^3+3} - ....$$
This series converges uniformally but not absolutely for any x. *Proof* By Leibnitz rule (if

have an alternating series $x_1 - x_2 + x_2 + x_3 - x_4 + ...$, where $x_i \to 0$ then the remainder $|R_n| < x_{n+1}$) we have

$$|R_n(x)| \leq \frac{1}{x^2+n+1} < \frac{1}{n} < \epsilon \qquad \text{if} \quad n > N(\epsilon) = \frac{1}{\epsilon}.$$

So this series is uniformly convergent (as $N(\epsilon)$ is not a function of x). But the series is **not** absolutely convergent as

$$\sum \left| \frac{(-1)^{m-1}}{x^2+m} \right| = \sum \frac{1}{x^2+m} > \sum_{m>m_0} \frac{1}{m} \quad \text{diverges.}$$

10 The Riemann Integral.

Remark. There are various different ways of defining $\int f(x)\,dx$. They give the same answer if f is sufficiently "nice", for example if f is continuous; but can differ if f is "nasty" (in many applications, one needs to be able to integrate nasty functions). The best-known integrals are the Riemann Integral and the Lebesgue Integral; some functions are Lebesgue-integrable but not Riemann-integrable.

Definition. A *partition* $\mathcal{P}$ of a closed interval $[a,b]$ is a finite set of points $\{x_0, x_1, \ldots x_n\}$ satisfying $a = x_0 < x_1 < x_2 < \ldots < x_n = b$.

Definition. Let f be defined and bounded on $[a,b]$, and let $\mathcal{P}$ be a partition of $[a,b]$. Then the *upper Riemann sum* of f relative to $\mathcal{P}$ is

$$U(\mathcal{P}) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}), \quad \text{where} \quad M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\},$$

and the *lower Riemann sum* of f relative to $\mathcal{P}$ is

$$L(\mathcal{P}) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}), \quad \text{where} \quad m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}.$$

Examples.

1. The function $f(x) = x$, with the partition $\mathcal{P} = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$. This gives

$$U(\mathcal{P}) = \frac{1}{n}\sum_{j=1}^{n} \frac{j}{n} = \frac{n+1}{2n}, \quad L(\mathcal{P}) = \frac{1}{n}\sum_{j=0}^{n-1} \frac{j}{n} = \frac{n-1}{2n}.$$

2. The function $f(x) = x^2$, with the same partition $\mathcal{P}$. This gives

$$U(\mathcal{P}) = \frac{1}{n}\sum_{j=1}^{n} \frac{j^2}{n^2} = \frac{(n+1)(2n+1)}{6n^2}, \quad L(\mathcal{P}) = \frac{1}{n}\sum_{j=0}^{n-1} \frac{j^2}{n^2} = \frac{(n-1)(2n-1)}{6n^2}.$$

Definition. Let f be defined and bounded on $[a,b]$, and put

$$\mathcal{L} = \sup\{L(\mathcal{P}) : \mathcal{P} \text{ is a partition of } [a,b]\}, \qquad \mathcal{U} = \inf\{U(\mathcal{P}) : \mathcal{P} \text{ is a partition of } [a,b]\}.$$

Then f is *Riemann-integrable* on $[a,b]$ if and only if $\mathcal{L} = \mathcal{U}$. If this is the this case, we say that $\mathcal{L}$ is the integral of f on $[a,b]$ and we denote it by $\int_a^b f(x)\,dx$.

Example. Let f be the function on $[0,1]$ defined by: $f(x) = 1$ if x is rational, $f(x) = 0$ if x is irrational. Then clearly $U(\mathcal{P}) = 1$ and $L(\mathcal{P}) = 0$ for any partition $\mathcal{P}$; therefore $\mathcal{U} = 1$

and $\mathcal{L} = 0$. So f is not Riemann-integrable. [It is, however, Lebesgue-integrable, and its Lebesgue integral equals zero.]

Lemma. If $\mathcal{P}_1$ and $\mathcal{P}_2$ are two partitions of $[a, b]$ satisfying $\mathcal{P}_1 \subset \mathcal{P}_2$, then for any function f bounded on $[a, b]$, we have

$$L(\mathcal{P}_1) \leq L(\mathcal{P}_2) \leq U(\mathcal{P}_2) \leq U(\mathcal{P}_1).$$

Proof. Let's do it for $\mathcal{P}_2 = \mathcal{P}_1 \cup \{x'\}$, ie adding just one point (visualize $x_0 < x' < x_1$). Then it is clear that $U(\mathcal{P}_2) \leq U(\mathcal{P}_1)$ and $L(\mathcal{P}_2) \geq L(\mathcal{P}_1)$.

Theorem. Let f be defined and bounded on $[a, b]$. Then f is Riemann-integrable on $[a, b]$ if and only if, for every $\varepsilon > 0$, there exists a partition $\mathcal{P}$ of $[a, b]$ such that $U(\mathcal{P}) - L(\mathcal{P}) < \varepsilon$.
Proof. First, suppose that f is Riemann-integrable, so $\mathcal{L} = \mathcal{U}$. Then for any $\varepsilon > 0$ there are partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $L(\mathcal{P}_1) > \mathcal{L} - \varepsilon/2$, and $U(\mathcal{P}_2) < \mathcal{U} + \varepsilon/2$. With $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$, we can apply the Lemma to deduce that

$$\mathcal{L} - \varepsilon/2 < L(\mathcal{P}_1) \leq L(\mathcal{P}) \leq U(\mathcal{P}) \leq U(\mathcal{P}_2) < \mathcal{U} + \varepsilon/2,$$

so that $U(\mathcal{P}) - L(\mathcal{P}) < \varepsilon$.
Now for the converse. If given any $\varepsilon > 0$ we can find $\mathcal{P}$ such that $U(\mathcal{P}) - L(\mathcal{P}) < \varepsilon$, then since $\mathcal{U} - \mathcal{L} \leq U(\mathcal{P}) - L(\mathcal{P})$, we have $\mathcal{U} - \mathcal{L} < \varepsilon$ for any $\varepsilon > 0$, so that $\mathcal{U} \leq \mathcal{L}$. Now we show that $\mathcal{L} \leq \mathcal{U}$. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be arbitrary partitions, and let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Then by the Lemma, $L(\mathcal{P}_1) \leq L(\mathcal{P}) \leq U(\mathcal{P}) \leq U(\mathcal{P}_2)$, so that $L(\mathcal{P}_1) \leq \mathcal{U}$, and so $\mathcal{L} \leq \mathcal{U}$. So $\mathcal{U} = \mathcal{L}$.

Example. Prove that $f(x) = x^2$ is Riemann-integrable on $[0, 1]$.
Solution. Let $\mathcal{P}_n$ be the partition $\mathcal{P}_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$. Then, as we saw previously, $U(\mathcal{P}) = (n+1)(2n+1)/(6n^2)$ and $L(\mathcal{P}) = (n-1)(2n-1)/(6n^2)$. Thus $U(\mathcal{P}) - L(\mathcal{P}) = 1/n$, and this can be made smaller than any given ε by taking n large enough. Thus by the previous theorem, f is Riemann-integrable. We also get the value of the integral: since $L(\mathcal{P}_n) \leq \mathcal{L} \leq U(\mathcal{P}_n)$ for all n, and $L(\mathcal{P}_n)$ and $U(\mathcal{P}_n)$ both tend to $1/3$ as $n \to \infty$, we deduce by squeezing that $\int_0^1 x^2\, dx = \mathcal{L} = 1/3$.

Theorem. If f is an increasing function on $[a, b]$, then f is Riemann-integrable on $[a, b]$. [Of course, the same will hold if f is decreasing.]
Proof. Let $\mathcal{P}_n$ be the partition $\mathcal{P}_n = \{x_i = a + i(b-a)/n\}_{i=0}^n$, ie dividing the interval into n equal sub-intervals. On each interval $[x_{i-1}, x_i]$, we have $\sup f([x_{i-1}, x_i]) = f(x_i)$ and $\inf f([x_{i-1}, x_i]) = f(x_{i-1})$. Thus

$$U(\mathcal{P}_n) = \frac{b-a}{n} \sum_{i=1}^{n} f(x_i) \quad \text{and} \quad L(\mathcal{P}_n) = \frac{b-a}{n} \sum_{i=1}^{n} f(x_{i-1}).$$

Thus

$$U(\mathcal{P}_n) - L(\mathcal{P}_n) = \frac{b-a}{n} \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})] = \frac{b-a}{n} [f(b) - f(a)].$$

Given any $\varepsilon > 0$, it is then possible to choose n such that $U(\mathcal{P}_n) - L(\mathcal{P}_n) < \varepsilon$, and so f is Riemann-integrable.

Example. The function f is defined on $[0, 1]$ by:
$f(x) = 1 - 1/n$ for $x \in [1 - 1/(n-1), 1 - 1/n)$ where $n = 2, 3, \ldots$; and $f(1) = 1$.
Then f has infinitely many discontinuities; but it is bounded and increasing, and therefore Riemann-integrable. In fact,

$$\int_0^1 f(x)\, dx = \sum_{n=2}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} - 1.$$

Theorem *(Proof omitted).* If f is continuous on $[a, b]$, then f is Riemann-integrable on $[a, b]$.

Theorem. (Properties of the Riemann Integral) If f and g are Riemann-integrable on $[a, b]$, then the integrals below all exist, and

1. $\displaystyle\int_a^b [Af(x) + Bg(x)]\, dx = A\int_a^b f(x)\, dx + B\int_a^b g(x)\, dx \qquad (A, B \in \mathbb{R});$
2. $\displaystyle\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx \qquad (c \in [a, b]);$
3. if $f(x) \geq 0$ in $[a, b]$, then $\displaystyle\int_a^b f(x)\, dx \geq 0;$
4. if $f(x) \leq g(x)$ in $[a, b]$, then $\displaystyle\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx;$
5. $\displaystyle\left|\int_a^b f(x)\, dx\right| \leq \int_a^b |f(x)|\, dx.$
6. $\displaystyle\int_a^b f(x)\, dx \;=\; -\int_b^a f(x)\, dx.$

Proof (partial).

3. If $f(x) \geq 0$, then $m_i \geq 0 \Rightarrow L(\mathcal{P}) \geq 0 \Rightarrow \mathcal{L} \geq 0$.
4. Apply previous part to $g - f$.
5. From $f(x) \leq |f(x)|$ we get $\displaystyle\int_a^b f(x)\, dx \leq \int_a^b |f(x)|\, dx$; and from $-|f(x)| \leq f(x)$ we get $\displaystyle-\int_a^b |f(x)|\, dx \leq \int_a^b f(x)\, dx.$

Remark. Essentially all the results of this section hold for complex-valued functions $f : [a, b] \to \mathbb{C}$. We define the integral by integrating the real and imaginary parts of f separately. Items 3 and 4 in the previous theorem no longer make sense; but item 5 remains true.

11 Improper Integrals.

Definition. Let $f(x)$ be continuous for $x \geq c$, and write $F(s) = \int_c^s f(x)\,dx$. If $F(s) \to L$ as $s \to \infty$, we say that the integral $\int_c^\infty f(x)\,dx$ *converges* to L. If $F(s)$ has no limit as $s \to \infty$, we say that $\int_c^\infty f(x)\,dx$ *diverges*.

Examples.

1. $f(x) = x^{-p}$, with $c = 1$. If $p = 1$, then $F(s) = \log s$; while if $p \neq 1$, then $F(s) = (s^{1-p} - 1)/(1-p)$. So clearly $\int_1^\infty x^{-p}\,dx$ converges [to $1/(p-1)$] if and only if $p > 1$. For example, $\int_1^\infty dx/x^2 = 1$, but $\int_1^\infty dx/\sqrt{x}$ diverges.

2. $f(x) = \cos x$, with $c = 0$. Here $F(s) = \sin s$, and so the integral $\int_0^\infty \cos x\,dx$ diverges.

3. $f(x) = \exp(-tx)$, with $c = 0$ and where t is a constant. Here $F(s) = [1 - \exp(-ts)]/t$; so $\int_0^\infty \exp(-tx)\,dx$ converges to $1/t$ if $t > 0$, and diverges otherwise.

Theorem *(Linearity).* If $\int_c^\infty f(x)\,dx$ converges to L and $\int_c^\infty g(x)\,dx$ converges to K, and if A and B are constants, then $\int_c^\infty [Af(x) + Bg(x)]\,dx$ converges to $AL + BK$. Also, if $f(x)$ is continuous for $x \geq c$, and if $b \geq c$, then $\int_c^\infty f(x)\,dx$ converges if and only if $\int_b^\infty f(x)\,dx$ converges.

Proof. Apply the corresponding results for limits of functions to the "partial integrals" $\int^s$.

Theorem *(Comparison Test).* Suppose that $f(x)$ and $g(x)$ are continuous for $x \geq c$, and that $0 \leq f(x) \leq g(x)$ for all $x \geq c$. Then $\int_c^\infty g(x)\,dx$ convergent implies that $\int_c^\infty f(x)\,dx$ converges. Or, equivalently, $\int_c^\infty f(x)\,dx$ divergent implies that $\int_c^\infty g(x)\,dx$ diverges.

Proof. Since f and g are positive, the functions $F(s) = \int_c^s f(x)\,dx$ and $G(s) = \int_c^s g(x)\,dx$ are increasing functions of s. Also, we have $F(s) \leq G(s)$ and $G(s) \leq K = \int_c^\infty g(x)\,dx$; so $F(s) \leq K$ for all $s \geq c$. Thus $F(s)$ is an increasing function which is bounded above, and so it tends to a limit as $s \to \infty$.

Examples.

1. $\int_0^\infty f(x)\,dx$, where $f(x) = 1/(1 + \mathrm{e}^x)$. We can do this in two ways.
   First, we can do the integral explicitly: $\int_0^s f(x)\,dx = \log[2/(1 + \mathrm{e}^{-s})]$, and so clearly the integral converges (to $\log 2$).
   Or, we can use comparison: $0 < f(x) < \mathrm{e}^{-x}$, and $\int_0^\infty \mathrm{e}^{-x}\,dx = 1$, so the given integral converges by comparison (to a value ≤ 1).

2. $\int_1^\infty f(x)\,dx$, where $f(x) = \exp(-x^2)$. In this case, we cannot do the integral explicitly. Note that $h(x) = x^2 f(x) \to 0$ as $x \to \infty$, so $h(x)$ is bounded above: there is a number M such that $h(x) \leq M$ for all $x \geq 1$. This implies that $0 < f(x) \leq M/x^2$. But $\int_1^\infty M\,dx/x^2$ converges, so the given integral converges by comparison.
   [Remark: the related integral $\int_0^\infty \exp(-x^2)\,dx$ can actually be done explicitly, and its value is $\sqrt{\pi}/2$.]

3. $\int_1^\infty f(x)\,dx$, where $f(x) = x^{-2}\log x$. Note that $h(x) = x^{-1/2}\log x \to 0$ as $x \to \infty$, so $h(x)$ is bounded above: there is a number M such that $h(x) \le M$ for all $x \ge 1$. This implies that $0 < f(x) \le M/x^{3/2}$. But $\int_1^\infty M\,dx/x^{3/2}$ converges, so the given integral converges by comparison.
   [In fact, we can take $M = 1$, and so we deduce that the value of the given integral is ≤ 2. As an exercise, do the integral explicitly, and show that its value is 1.]

4. $\int_0^\infty f(t)\,dt$, where $f(t) = t/\sqrt{t^4+1}$. For $t \ge 1$, we have $t^4 + 1 \le t^4 + t^4 = 2t^4$, and so $f(t) \ge 1/(\sqrt{2}t)$. But $\int_1^\infty dt/(\sqrt{2}t)$ diverges, therefore so does $\int_1^\infty f(t)\,dt$ by comparison, therefore so does $\int_0^\infty f(t)\,dt$.
   [This case can also be done explicitly: $2\int_0^s f(t)\,dt = \sinh^{-1}(s^2)$, which has no limit as $s \to \infty$.]

Remark. If $\int_c^\infty f(x)\,dx$ converges, it does *not* necessarily follow that $f(x) \to 0$ as $x \to \infty$. In this respect, integrals are different from series.

Definition. If $\int_c^\infty |f(x)|\,dx$ converges, then we say that $\int_c^\infty f(x)\,dx$ *converges absolutely*. If $\int_c^\infty f(x)\,dx$ converges but $\int_c^\infty |f(x)|\,dx$ diverges, then we say that $\int_c^\infty f(x)\,dx$ *converges conditionally*.

Theorem *(Absolute Convergence Theorem).* If $\int_c^\infty f(x)\,dx$ converges absolutely, then it converges.

Proof. We are given that $\int_c^\infty |f(x)|\,dx$ converges, and hence so does $\int_c^\infty 2|f(x)|\,dx$. Put $g(x) = |f(x)| - f(x)$: then $0 \le g(x) \le 2|f(x)|$ and so $\int_c^\infty g(x)\,dx$ converges by comparison. Finally, $f(x) = |f(x)| - g(x)$, so $\int_c^\infty f(x)\,dx$ converges by linearity.

Example. $\displaystyle\int_\pi^\infty \frac{\cos x}{x^2}\,dx$ is absolutely convergent, by the Comparison Test, since $0 \le \left|\dfrac{\cos x}{x^2}\right| \le \dfrac{1}{x^2}$, and $\displaystyle\int_\pi^\infty \frac{dx}{x^2}$ converges.

Example. $\displaystyle\int_\pi^\infty \frac{\sin x}{x}\,dx$ is conditionally convergent. We can prove this as follows.
First, we show that the integral converges. Integration by parts gives

$$\int_\pi^s \frac{\sin x}{x}\,dx = -\left[\frac{\cos x}{x}\right]_\pi^s - \int_\pi^s \frac{\cos x}{x^2}\,dx.$$

The first term tends to a finite limit (in fact $-1/\pi$) as $s \to \infty$; and the second term also tends to a finite limit, by the previous example.
Now we have to show that $\displaystyle\int_\pi^\infty \left|\frac{\sin x}{x}\right|\,dx$ diverges. Write

$$I_n = \int_{n\pi}^{(n+1)\pi} \left|\frac{\sin x}{x}\right|\,dx = \int_0^\pi \frac{\sin y}{y+n\pi}\,dy,$$

(using the substitution $x = y + n\pi$); and note that

$$\int_{\pi}^{M\pi} \left| \frac{\sin x}{x} \right| dx = \sum_{n=1}^{M-1} I_n.$$

So by comparison, we need only show that $\sum_{n=1}^{\infty} I_n$ diverges. Now $\dfrac{1}{y + n\pi} \geq \dfrac{1}{(n+1)\pi}$ for $0 \leq y \leq \pi$, so

$$I_n \geq \frac{1}{(n+1)\pi} \int_0^{\pi} \sin y \, dy = \frac{2}{(n+1)\pi}.$$

And $\sum_{n=1}^{\infty} \dfrac{2}{(n+1)\pi}$ diverges (harmonic series), so $\sum_{n=1}^{\infty} I_n$ diverges by comparison.

Remark. Another kind of improper integral is where the range of integration is finite, but the function being integrated is unbounded. The definitions and features of these two types of improper integral are similar (but there are also important differences).

Definition. Let $f(x)$ be continuous for $a \leq x < b$, and write $F(s) = \int_a^s f(x)\, dx$ for $a \leq s < b$. If $F(s) \to L$ as $s \to b_-$, we say that the integral $\int_a^b f(x)\, dx$ *converges* to L. If $F(s)$ has no limit as $s \to b_-$, we say that $\int_a^b f(x)\, dx$ *diverges*. Similarly, if $f(x)$ is continuous for $a < x \leq b$, and $F(s) = \int_s^b f(x)\, dx$, then we say that $\int_a^b f(x)\, dx$ converges to L if $F(s) \to L$ as $s \to a_+$. Finally, if $a < c < b$, and if $f(x)$ is continuous for $x \in [a, b]$ except possibly at $x = c$, then we say that $\int_a^b f(x)\, dx$ converges if both $\int_a^c f(x)\, dx$ and $\int_c^b f(x)\, dx$ converge.

Examples.

1. $f(x) = x^{-p}$ on $(0, 1]$. If $p = 1$, then $F(s) = \log s$; while if $p \neq 1$, then $F(s) = (1 - s^{1-p})/(1 - p)$. So clearly $\int_0^1 x^{-p}\, dx$ converges [to $1/(1-p)$] if and only if $p < 1$. For example, $\int_0^1 dx/\sqrt{x} = 2$, but $\int_0^1 dx/x^2$ diverges.

2. If $f(x)$ is an odd function, then $\int_{-1}^1 f(x)\, dx = 0$. So one might have guessed that $\int_{-1}^1 dx/x$ exists, and equals zero. But in fact, this integral diverges according to our definition.

3. $f(x) = \log x$ on $(0, 1]$. Put $F(s) = \int_s^1 \log x\, dx = [x \log x - x]_s^1 = -1 + s - s \log s$. Note that $s \log s \to 0$ as $s \to 0_+$ (put $s = 1/y$ and use $y^{-1} \log y \to 0$ as $y \to \infty$). So $\int_0^1 \log x\, dx$ converges to -1.

4. $\displaystyle\int_1^3 \frac{dx}{(3-x)^4}$ diverges, whereas $\displaystyle\int_1^3 \frac{dx}{\sqrt{3-x}}$ converges to $2\sqrt{2}$.

Theorem *(Linearity).* If $f(x)$ and $g(x)$ are continuous for $a \le x < b$, and if $\int_a^b f(x)\,dx$ converges to L and $\int_a^b g(x)\,dx$ converges to K, and if A and B are constants, then $\int_a^b [Af(x) + Bg(x)]\,dx$ converges to $AL + BK$.

Theorem *(Comparison Test).* Suppose that $f(x)$ and $g(x)$ are continuous for $a \le x < b$, and that $0 \le f(x) \le g(x)$ for all $a \le x < b$. Then $\int_a^b g(x)\,dx$ convergent implies that $\int_a^b f(x)\,dx$ converges. Or, equivalently, $\int_a^b f(x)\,dx$ divergent implies that $\int_a^b g(x)\,dx$ diverges.

Proof. Since f and g are positive, the functions $F(s) = \int_a^s f(x)\,dx$ and $G(s) = \int_a^s g(x)\,dx$ are increasing functions of s. Also, we have $F(s) \le G(s)$ and $G(s) \le K = \int_a^b g(x)\,dx$; so $F(s) \le K$ for all $s \in [a, b)$. Thus $F(s)$ is an increasing function which is bounded above, and so it tends to a limit as $s \to b_-$.

Definition. If $f(x)$ is continuous for $a \le x < b$, and $\int_a^b |f(x)|\,dx$ converges, then we say that $\int_a^b f(x)\,dx$ *converges absolutely*. If $\int_a^b f(x)\,dx$ converges but $\int_a^b |f(x)|\,dx$ diverges, then we say that $\int_a^b f(x)\,dx$ *converges conditionally*.

Theorem *(Absolute Convergence Theorem).* If $\int_a^b f(x)\,dx$ converges absolutely, then it converges.

Proof. Same as for $\int_c^\infty$.

Examples.

1. $\displaystyle\int_0^{2\pi} \frac{\cos x}{\sqrt{x}}\,dx$ converges absolutely, by the Comparison Test,
   since $\displaystyle\left|\frac{\cos x}{\sqrt{x}}\right| \le \frac{1}{\sqrt{x}}$ for $0 < x \le 2\pi$, and $\displaystyle\int_0^{2\pi} \frac{dx}{\sqrt{x}}$ converges.

2. $\int_0^1 f(x)\,dx$, where $f(x) = \mathrm{e}^x x^{-p}$. Since $x^{-p} \le f(x) \le \mathrm{e}x^{-p}$ for all $x \in (0, 1]$, we can run the Comparison Test both ways to deduce that the integral converges if and only if $p < 1$.

3. $\int_0^1 f(x)\,dx$, where $f(x) = 1/\sqrt{x(2-x)}$. Exercise: do this by explicitly evaluating the integral. Alternatively, we can do a comparison:
   on $[0, 1]$, we have $1/\sqrt{2} \le 1/\sqrt{2-x} \le 1$, and so the integral converges by comparison with the convergent integral $\int_0^1 dx/\sqrt{x}$.

Observation. If $f(x)$ is continuous for $x > c$, then $\int_c^\infty f(x)\,dx$ converges if, for some number $b > c$, both $\int_c^b f(x)\,dx$ and $\int_b^\infty f(x)\,dx$ converge.

Remark. It is easy to see that the choice of b is irrelevant.

Examples.

1. $\int_0^\infty f(x)\,dx$, where $f(x) = (\log x)/(1+x^4)$. First, look at $\int_0^1$. On $(0,1]]$, we have $0 < |f(x)| < -\log x$, so $\int_0^1 f(x)\,dx$ converges absolutely, by comparison with the convergent integral $\int_0^1 \log x\,dx$.
   Next, look at $\int_1^\infty$. Since $x^{-1}\log x \to 0$ as $x \to \infty$, we know that there exists a number K such that $x^{-1}\log x \leq K$ for all $x \geq 1$. It follows that $0 \leq f(x) \leq Kx/(1+x^4) < K/x^3$. Since $\int_1^\infty dx/x^3$ converges, we deduce by comparison that $\int_1^\infty f(x)\,dx$ converges.
   Putting these two together, we conclude that $\int_0^\infty f(x)\,dx$ converges.

2. $\int_0^\infty f(x)\,dx$, where $f(x) = 1/(\sqrt{x}+x^2)$. First, look at $\int_0^1$. On $(0,1]]$, we have $0 < f(x) < 1/\sqrt{x}$, so $\int_0^1 f(x)\,dx$ converges by comparison with the convergent integral $\int_0^1 dx/\sqrt{x}$.
   Next, look at $\int_1^\infty$. For $x \geq 1$, we have $0 < f(x) < 1/x^2$, and $\int_1^\infty dx/x^2$ converges, so $\int_1^\infty f(x)\,dx$ converges by comparison.
   Putting these two together, we conclude that $\int_0^\infty f(x)\,dx$ converges.

12 Differentiability.

Definition. Let $f(x)$ be defined in an open interval containing c. We say that f is *differentiable* at $x = c$ if $\lim_{h\to 0} h^{-1}[f(c+h) - f(c)]$ exists. The value of the limit is the derivative $f'(c)$. Another way to write it is $f'(c) = \lim_{x\to c}[f(x) - f(c)]/(x-c)$. Since the limit itself has an ε-δ definition, a third formulation is: $f(x)$ is differentiable at $x = c$ and has derivative L there if, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left|\frac{f(x) - f(c)}{x - c} - L\right| < \varepsilon$$

whenever $|x - c| < \delta$ (and $x \neq c$).

Theorem. If $f(x)$ is differentiable at $x = c$, then $f(x)$ is continuous at $x = c$.
Proof: $|f(c+h) - f(c)| = h \cdot h^{-1}|f(c+h) - f(c)| \to 0 \cdot f'(c) = 0$ as $h \to 0$.

Remark. The converse is not necessarily true; for example, $f(x) = |x|$ is continuous but not differentiable at $x = 0$.

Theorem *(Fundamental Theorem of Calculus).* If $f(x)$ is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ is differentiable on (a, b), and $F'(x) = f(x)$.
Proof. Let $c \in (a, b)$. Given $\varepsilon > 0$, find $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$. Then for $0 < h < \delta$, we have

$$\begin{aligned} \left|\frac{F(c+h) - F(c)}{h} - f(c)\right| &= \left|\frac{1}{h}\int_c^{c+h} f(x)\,dx - f(c)\right| \\ &= \left|\frac{1}{h}\int_c^{c+h} [f(x) - f(c)]\,dx\right| \\ &\leq \frac{1}{h}\int_c^{c+h} |f(x) - f(c)|\,dx \\ &< \varepsilon. \end{aligned}$$

And a similar argument works for $-\delta < h < 0$.

Remark. So the operations of differentiation and integration are inverses of each other. Note, however, that differentiability is a rather restrictive condition, whereas integrability (for example Riemann integrability) is much weaker one — one can integrate functions which are highly-discontinuous.

Remark. There is another sense in which one can "differentiate an integral", namely:

$$\frac{d}{dx}\left[\int_a^b f(x, y)\,dy\right] = \int_a^b \frac{\partial f(x, y)}{\partial x}\,dy,$$

provided the function $f(x,y)$ satisfies some appropriate smoothness conditions. Here $\partial/\partial x$ is the *partial derivative* with respect to x.
Example:

$$\begin{aligned}\frac{d}{dx}\left[\int_0^x \sin(xy)\,dy\right] &= [\sin(xy)]_{y=x} + \int_0^x \frac{\partial}{\partial x}\sin(xy)\,dy \\ &= \sin(x^2) + \int_0^x y\cos(xy)\,dy.\end{aligned}$$

Exercise: check this by doing the integrals explicitly.

Theorem. If $f(x)$ and $g(x)$ are differentiable at x, then so is their product fg, and the derivative of fg at x is $f(x)\,g'(x) + g(x)\,f'(x)$.
Proof:

$$\begin{aligned}\frac{1}{h}\left[f(x+h)g(x+h) - f(x)g(x)\right] &= f(x+h)\frac{g(x+h)-g(x)}{h} + g(x)\frac{f(x+h)-f(x)}{h} \\ &\to f(x)\,g'(x) + g(x)\,f'(x) \text{ as } h \to 0.\end{aligned}$$

Definition. Let $f(x)$ be defined in some open interval (a,b), and let $c \in (a,b)$. We say that f has a *local maximum* at c if there exists a number $\delta > 0$ such that $f(x) \le f(c)$ for all $x \in (c-\delta, c+\delta)$. There is a similar definition for local minimum.

Theorem. If $f(x)$ is differentiable at $x = c$, and has a local maximum or minimum at c, then $f'(c) = 0$.
Proof. Suppose that f has a local maximum at c (the proof for local minimum is similar). So there exists $\delta > 0$ such that $f(c+h) \le f(c)$ provided $|h| < \delta$. Consider the function $R(h) = [f(c+h) - f(c)]/h$. When $h > 0$, we have $R(h) \le 0$, and when $h < 0$, we have $R(h) \ge 0$. But since f is differentiable at c, $\lim_{h\to 0} R(h)$ exists and is $f'(c)$. If we take a sequence of values $\{h_n\}$ with $h_n < 0$ and $h_n \to 0$ as $n \to \infty$, we get $\lim_{n\to\infty} R(h_n) = f'(c) \ge 0$; and if we take a sequence of values $\{h_n\}$ with $h_n > 0$, tending to 0, then we get $\lim_{n\to\infty} R(h_n) = f'(c) \le 0$. Thus $f'(c) = 0$.

Theorem *(Rolle's Theorem).* Let f be differentiable on (a,b), and continuous on $[a,b]$; and suppose that $f(a) = f(b)$. Then there exists $c \in (a,b)$ such that $f'(c) = 0$.
Proof. Since f is continuous on $[a,b]$, it is bounded and attains its bounds in $[a,b]$; in other words, there are points c_1 and c_2 such that $f(c_1) \le f(x) \le f(c_2)$ for all $x \in [a,b]$. If $f(c_1) = f(c_2)$, then f is constant and so $f'(x) = 0$ for all $x \in (a,b)$. Otherwise, at least one of c_1 and c_2 must lie in (a,b). Call this point c. Then c is a local maximum or a local minimum, and so $f'(c) = 0$.

Theorem *(The Mean value Theorem).* Let f be differentiable on (a,b) and continuous on $[a,b]$. Then there exists $c \in (a,b)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

Proof: Apply Rolle to $g(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}\right] x$.

Theorem *(Another useful theorem).*
Let f and g be differentiable on (a, b) and continuous on $[a, b]$ and $g(x) \geq 0, \quad m \leq f(x) \leq M$ (or $g \leq 0$).

Then there exists μ, $m \leq \mu \leq M$ such that
$\int_a^b f(x)g(x)\,dx = \mu \int_a^b g(x)dx.$

Proof. Since $g(x) \geq 0$ then $mg(x) \leq f(x)g(x) \leq Mg(x)$.

Integrate over x and divide by $\int_a^b g(x)dx \neq 0$ and find

$$m \leq \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)dx} \leq M.$$

Hence define

$$\mu = \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)dx}$$

and have the result.

Finally, we have two more useful theorems (proofs are straightforward)

Theorem *For Continuous functions.*
If f is continuous on $[a, b]$ then there exists $c \in (a, b)$ such that
$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx.$

Theorem *Change of variables*
Consider $\int_a^b f(x)dx$, where $f(x)$ is continuous in (a, b). Put $x = \varphi(t)$ where

1. $\varphi(t)$ is defined for $t \in (\alpha, \beta)$ and its values lie in (a, b)
2. $\varphi(\alpha) = a, \qquad \varphi(\beta) = b$
3. $\varphi(t)$ has a continuous derivative $\dot{\varphi}(t)$ for $\alpha \leq t \leq \beta$

Then $\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\,\dot{\varphi}(t)\,dt.$