The slingshot effect

R. C. Johnson*
Department of Mathematical Sciences,
University of Durham, Durham DH1 3LE, England.

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Abstract

How spacecraft accelerate in planetary flybys is explained clearly. For a given speed change often two different encounters are possible.

1 Introduction

On 15 October 1997, NASA launched the probe *Cassini* on a 6.7-year voyage to Saturn [1, 2]. A Titan/Centaur booster sent the 5700-kg spacecraft from Earth with speed 4 km/s.

But Saturn is high up the Sun's gravitational potential well, and to reach it from Earth's orbit a spacecraft needs at least 10 km/s.

So *Cassini*'s flight plan involves acquiring extra speed from four intermediate planetary encounters — with Venus, Venus again, Earth, and Jupiter (VVEJ trajectory) — see Fig. 1.

In each of these flybys, *Cassini* is boosted by a gravitational '*slingshot*' interaction with a planet moving in orbit round the Sun at speeds from 13 km/s (Jupiter) to 35 km/s (Venus).

The first flyby at Venus on 26 April 1998 gave *Cassini* an extra 7 km/s or so [4], the third — at Earth on 17 August 1999 — added 5.5 km/s [5].

The fourth and last flyby, at Jupiter on 30 December 2000 [6], added 2 km/s and set Cassini on course to arrive at Saturn on 1 July 2004.

It is estimated that all four encounters together save about 75 tons of fuel [1]. NASA now routinely uses such planetary "gravity assists" for economy on missions to the outer solar system [7], and future Martian expeditions may benefit from Lunar flybys [8].

At first sight the underlying slingshot mechanism seems puzzling, for there is an uneasy feeling that something is conjured from nothing. It was presumed

* *bob.johnson@dur.ac.uk*

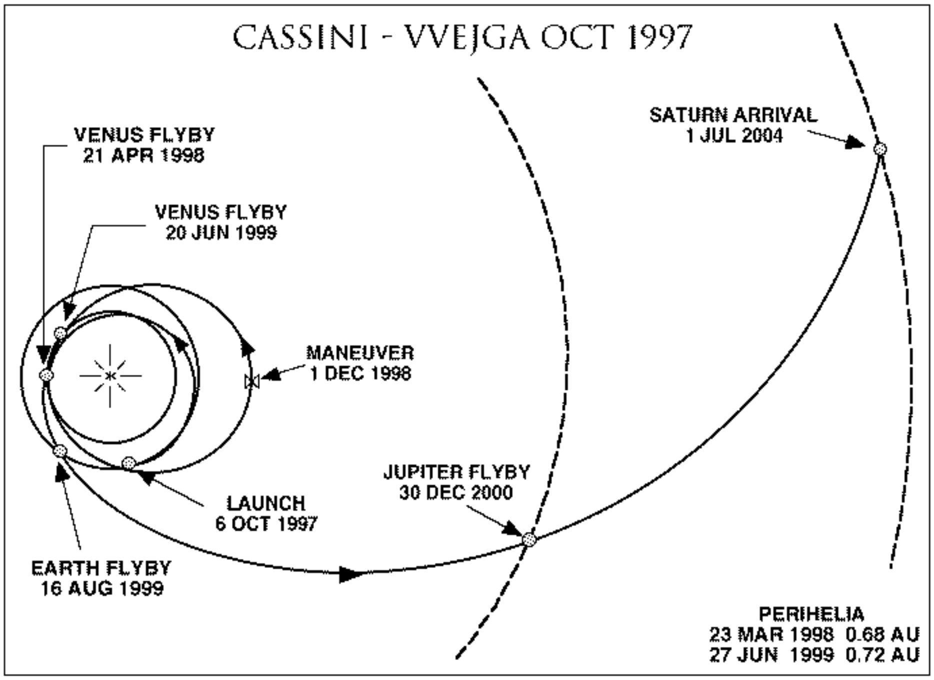

Figure 1: *Cassini*'s VVEJ Gravitational Assist trajectory, as given in ref. [3]

to be rather sophisticated when, despite its obvious importance, NASA's public information was either highly technical [9] or somewhat confusing [10].

Although more recently NASA has improved its educational materials [11], a clear elementary mathematical treatment is less accessible than it might be — given that by its usage the slingshot effect is a modern triumph of Newtonian Mechanics.

Few undergraduate texts mention the subject. Of those that do, for instance Alonso and Finn [13, Ex. 6.4] consider only a 1-dimensional interaction, and Lockett's discussion [14, p. *35*] is cryptic, to say the least.

Scientific American [12] gives a non-technical introduction, avoiding mathematics — while the article by Diehl [15] at least includes a version of the appropriate velocity-addition diagram[1].

Marion and Thornton [16, pps. *314–5*] make an exemplary statement of the underlying principles (energy and momentum conservation) and give a clearer velocity-addition diagram — but omit further development.

A detailed calculation appears in the book by Barger and Olsson[2] [17]. This deals with a gravity assist at Jupiter en route to Uranus[3]. However the formidable length of the 13-page manipulation [17, pps. *131–44*] only reinforces perception of the slingshot's subtlety. A shorter calculation in similar vein by Roy [18,

[1]And several references to historical developments.
[2]Published just before the first *Voyager* launch [19].
[3]The emphasis is on reducing transit time rather than saving fuel.

pps. *366–7*] is a masterpiece of obscurity.

Bartlett and Hord [20] give more insight, putting the slingshot into context of analogous physical effects[4] and mentioning more early sources — *eg.* [21]. But again the explicit example is uncomfortably lengthy [20, Secs. V–VII].

A middle course is taken here. First we set out the basic principles as clearly as possible:

- momentum conservation — the enormously more massive planet imparts significant speed to the spacecraft without measurable change in its own velocity;

- energy conservation — the pull of gravity simply rotates the spacecraft's distant-velocity vector in the planet frame, leaving its magnitude unaffected. This acceleration is a speed change relative to the Sun.

Details are in Sec. 2 and Sec. 3, and are summarised in a single diagram — see Fig. 2 below.

Then we go beyond Marion and Thornton [16, pps. *314–5*] by including relevant kinematical formulas in Sec. 4. These in fact can apply to any elastic interaction between objects of very different mass; the dynamics specific to gravity are given in Sec. 5 for a point planet, and in Sec. 6 for one of finite size.

Some discussions (*eg.* [10], [12]) allude to change of angular momentum about the Sun as a fundamental part of the slingshot effect. This is plainly misleading, and Sec. 7 briefly considers the essential consequences of planet and spacecraft within the Solar System.

2 Momentum conservation

An interaction between a spacecraft of mass m and a planet of mass M, whether hard or soft (crash or landing, launch or flyby) obeys Newton's Third Law and momentum is conserved:

$$m\mathbf{v}_\mathrm{i} + M\mathbf{V}_\mathrm{i} = m\mathbf{v}_\mathrm{f} + M\mathbf{V}_\mathrm{f}.$$

Here $(\mathbf{v}_\mathrm{i},\ \mathbf{V}_\mathrm{i})$ and $(\mathbf{v}_\mathrm{f},\ \mathbf{V}_\mathrm{f})$ are respective velocities before and after the encounter.

Then

$$\mathbf{V}_\mathrm{f} - \mathbf{V}_\mathrm{i} = \frac{m}{M}(\mathbf{v}_\mathrm{i} - \mathbf{v}_\mathrm{f}).$$

Since m is of order 10^3 kg while M is typically 10^{24}–10^{27} kg (Venus–Jupiter range) the mass ratio m/M is very small indeed: 10^{-21}–10^{-24}.

So, for relevant velocities, we have

$$\mathbf{V}_\mathrm{f} = \mathbf{V}_\mathrm{i} \stackrel{\mathrm{def}}{=} \mathbf{V} \qquad \text{to 1 part in } 10^{21} \text{ or better.} \tag{1}$$

[4] Such as adiabatic heating and cooling of a gas.

This is extremely accurate — typically, interaction with a spacecraft affects a planet at least 25 million times less than impact of a 1-microgram gnat perturbs the progress of a 40-tonne truck.

NASA remarks that a *Voyager* flyby [19] slowed Jupiter by about 1 foot every trillion years, while *Galileo* [22] slowed Earth by 5 billionths of an inch per year.

Bartlett and Hord's discussion of 'The Natural Order of Things' [20, Sec. VIII] puts this alongside other momentum-changing perturbations to planets — such as meteor, comet and asteroid impact. And of course, spacecraft launch itself perturbs Earth.

In all of what follows, eq. (1) is taken to hold without further comment.

3 Energy conservation

Next, consider energy conservation, as it applies in the planet frame.

The spacecraft's distant-approach velocity $\mathbf{u}_{\rm i} = \mathbf{v}_{\rm i} - \mathbf{V}$ relative to the planet is deflected by gravitational pull to a distant-departure velocity $\mathbf{u}_{\rm f}$ which has the same magnitude:

$$|\mathbf{u}_{\rm f}| = |\mathbf{u}_{\rm i}| \,. \tag{2}$$

Then $\mathbf{v}_{\rm f} = \mathbf{u}_{\rm f} + \mathbf{V}$ in the space frame.

Fig. 2 gives an example, where the triangle construction is equivalent to Diehl's Fig.1 [15, p. *676*] and to Barger and Olsson's Fig. 4.12 [17, p. *139*]. It summarises the two useful diagrams appearing in NASA's discussion [10, 11], and is essentially the same as Marion and Thornton's Fig. 8-13(b).

Fig. 2 shows that as the spacecraft passes behind the planet, a modest gravitational deflection aligns its velocity closer to that of the planet. Then $|\mathbf{v}_{\rm f}| > |\mathbf{v}_{\rm i}|$ — *ie,* the spacecraft is swept along, gaining speed.

This is how each encounter in Fig. 1 is arranged[5].

Inspection of similar velocity diagrams shows the accessible range of $\mathbf{v}_{\rm f}$ for given $(\mathbf{v}_{\rm i}, \mathbf{V})$, including:

- maximum boost is when $\mathbf{v}_{\rm f}$ aligns exactly with $\mathbf{V}$ (this is expressed by eq. (4) below);
- any given submaximal boost can be achieved from either of two flybys, when corresponding vectors $\mathbf{v}_{\rm f}$ are related by reflection in $\mathbf{V}$;
- too large a deflection brakes the spacecraft: $|\mathbf{v}_{\rm f}| < |\mathbf{v}_{\rm i}|$ (see eq. (5) below);
- the maximum possible gain is $2\,|\mathbf{V}|$, for a head-on collision with rotation through $180°$.

[5] And this is one way that the Moon has over aeons accelerated and ejected dust and debris from Earth's neighbourhood.

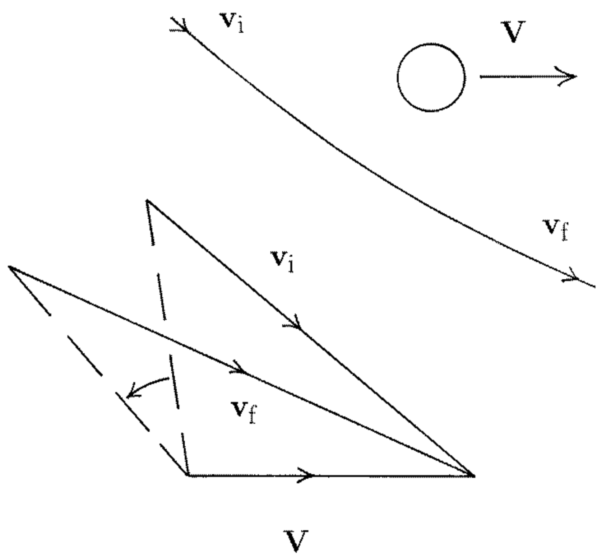

Figure 2: Velocity triangles for the flyby depicted above them. The triangles' common base is the planet's constant velocity $\mathbf{V}$. In the planet frame, the spacecraft's distant-approach velocity $\mathbf{v}_\mathrm{i} - \mathbf{V}$ (dashed) is rotated as arrowed by gravity to a distant-departure velocity — with magnitude unaltered since energy is conserved. Addition of $\mathbf{V}$ gives $\mathbf{v}_\mathrm{f}$. Plainly, $|\mathbf{v}_\mathrm{f}| > |\mathbf{v}_\mathrm{i}|$.

If the spacecraft is deflected oppositely — passing in front of the planet[6] — then in small-angle encounters its rotated planet-frame velocity tends to oppose $\mathbf{V}$ and so it is hauled back, losing speed[7]. Fig. 3 illustrates this situation.

Slingshot braking of *Cassini* presumably saves fuel again at Saturn Orbit Insertion [23]. The *Galileo* probe at Jupiter [22] braked with the help of the satellite moon Io [10] (see also Bartlett and Hord [20, Sec. X]).

4 Kinematics

Explicit formulas corresponding to the observations in Sec. 3 are as follows.

Referring to Fig. 2, let $(\alpha,\, \alpha')$ be the angles between the positive directions of $\mathbf{V}$ and $(\mathbf{v}_\mathrm{i},\, \mathbf{v}_\mathrm{f})$ respectively, and let β be the deflection of the spacecraft in the planet frame — the positive rotation angle arrowed between the dashed lines. Let

$$v_\mathrm{i} = |\mathbf{v}_\mathrm{i}|\,, \quad v_\mathrm{f} = |\mathbf{v}_\mathrm{f}| \qquad \text{and} \qquad V = |\mathbf{V}|\,.$$

[6] Or behind in a hypothetical repulsive interaction.

[7] It is reduced to rest if $\mathbf{v}_\mathrm{i} = 2\mathbf{V}$ in a 'tail-on' collision.

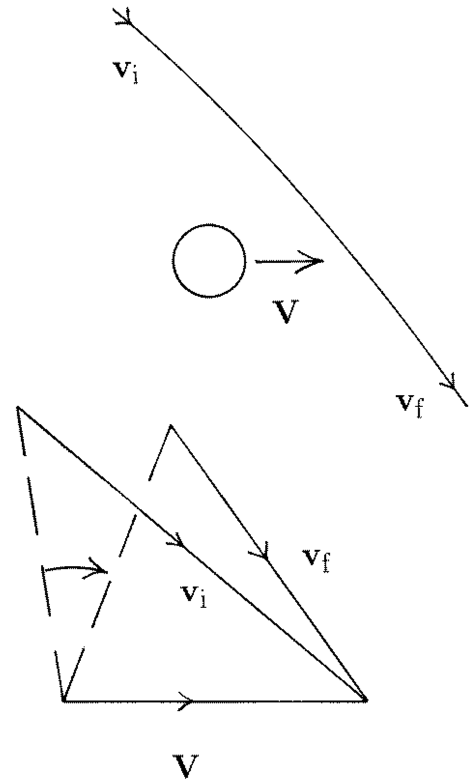

Figure 3: As Fig. 2, but for a braking manouevre, where the spacecraft passes in front of the planet and so gravity rotates oppositely the spacecraft's planet-frame velocity. The formulas of Sec. 4 apply with $\beta < 0$.

Then straightforward trigonometry gives

$$v_\mathrm{f}^2 = v_\mathrm{i}^2 + 2V\left\{V(1-\cos\beta) + v_\mathrm{i}\left[\cos(\alpha-\beta) - \cos\alpha\right]\right\}, \tag{3}$$

along with

$$v_\mathrm{f}\cos\alpha' = V(1-\cos\beta) + v_\mathrm{i}\cos(\alpha-\beta),$$
$$v_\mathrm{f}\sin\alpha' = V\sin\beta + v_\mathrm{i}\sin(\alpha-\beta).$$

Of course, $v_\mathrm{f} = v_\mathrm{i}$ and $\alpha' = \alpha$ at $\beta = 0$.

As Fig. 4 shows, with (v_i, V, α) fixed, outgoing speed v_f first increases with β. It reaches its maximum at $\beta = \beta_\mathrm{max}$ where

$$\tan\beta_\mathrm{max} = \frac{v_\mathrm{i}\sin\alpha}{v_\mathrm{i}\cos\alpha - V}, \tag{4}$$

when $\alpha' = 0$ and $\mathbf{v}_\mathrm{f}$ aligns with $\mathbf{V}$.

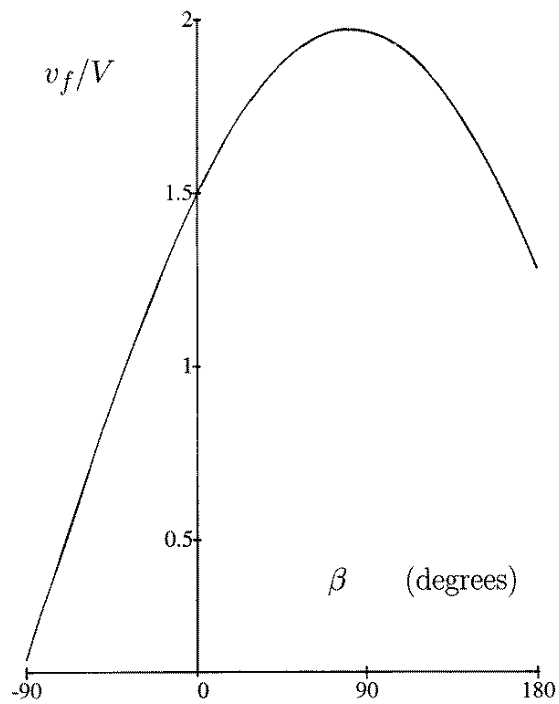

Figure 4: Dependence of v_f/V on β given by eq. (3), with $(v_\mathrm{i}/V,\, \alpha) = (1.5,\, 40°)$ corresponding to the examples in Fig. 2 and Fig. 3.

Fig. 2, Fig. 3 and Fig. 4 are all for a case where $v_\mathrm{i} = 1.5V$ and $\alpha = 40°$, when $\beta_\mathrm{max} = 81°$.

Further increase of β decreases v_f, and $v_\mathrm{f} = v_\mathrm{i}$ again at $\beta = \beta_0$ where

$$\beta_0 = 2\beta_\mathrm{max}. \tag{5}$$

Then $\alpha' = -\alpha$ and $\mathbf{v}_\mathrm{f}$ and $\mathbf{v}_\mathrm{i}$ are related by reflection in $\mathbf{V}$.

Note therefore (Fig. 4) that there are two potential deflection angles $\beta_{1,2}$ for each possible submaximal speed boost. They are related by

$$\beta_1 + \beta_2 = 2\beta_\mathrm{max}. \tag{6}$$

The smaller is often preferred in practice since it turns out to involve a more distant flyby, as Sec. 5 shows.

In the example of Fig. 2, where $\beta = \beta_1 = 33°$ and $v_\mathrm{f}/v_\mathrm{i} = 1.2$, the same 20% boost is achieved with $\beta = \beta_2 = 129°$.

For angles $\beta > \beta_0$ $(= 162°$ in Fig. 4) we have $v_\mathrm{f} < v_\mathrm{i}$ and the spacecraft loses speed in the encounter. Again, practical preference is for a more distant smaller-angle flyby, braking with $\beta < 0$ as in Fig. 3. Note that eq. (6) applies.

In fact Fig. 3 shows $\beta = \beta_1 = -30°$ and $v_f/v_i = 0.75$, when identical 25% braking is obtained with $\beta = \beta_2 = 192°$. However, angles $\beta > 180°$ are not attainable in an inverse-square interaction — see Sec. 5.

5 Dynamics

The kinematics of Sec. 4 applies to any energy-conserving (elastic) interaction, and is easily modified for a repulsion.

A spacecraft, however, swings round a planet under gravitational attraction and in the planet frame follows the Keplerian orbit [24, §15]

$$r(\theta) = \frac{h^2/GM}{1+\epsilon\cos\theta}, \qquad \text{where} \qquad \epsilon = \sqrt{1+\frac{2\mathcal{E}h^2}{G^2M^2}}. \tag{7}$$

Here $(r,\,\theta)$ are polar coordinates centred on the planet in the plane of $(\mathbf{v}_i,\,\mathbf{V})$, and $(h,\,\mathcal{E})$ are the spacecraft's angular momentum per unit mass and energy per unit mass respectively, both constant. As before M is the planet's mass, and G is the universal gravitational constant.

For an open (hyperbolic) orbit, energy $\mathcal{E} > 0$ and eccentricity $\epsilon > 1$.

The spacecraft's closest approach to the planet is at

$$r = r_{\text{min}} = \frac{h^2/GM}{1+\epsilon}, \tag{8}$$

corresponding to $\theta = 0$ by choice of reference line.

At large distance, $r \to \infty$ as polar angle $\theta \to \pm\arccos(-1/\epsilon)$. Then the encounter deflects the spacecraft through an angle

$$\beta = 2\arccos(-1/\epsilon) - \pi. \tag{9}$$

This is the rotation angle between $\mathbf{u}_f$ and $\mathbf{u}_i$ — *ie,* that arrowed in Fig. 2.

Of the two angles β corresponding to a given speed boost (Sec. 4) the smaller evidently involves larger r_{min}.

The reason is that, for given $\mathcal{E}$, the values of $(\epsilon,\,r_{\text{min}},\,\beta)$ together are controlled by the size of angular momentum parameter h. Inspection of eq. (7), eq. (8) and eq. (9) shows that smaller h means both smaller r_{min} and larger β.

Very large h gives negligible deflection, while the head-on limit is $h = 0$, giving maximum deflection $\beta = \pi$, when $\epsilon = 1$ — *ie,* a parabolic orbit.

6 Finite-size planet

For a given accessible boost, the more distant of the two possible planetary flybys is usually preferred. For it allows more margin for error — and indeed sometimes may be the only option.

The reason is that the planet is not a fixed point — *ie,* although its mass is effectively infinite, its radius R is not zero — the spacecraft crashes unless $r_{\text{min}} > R$.

With $GM = gR^2$, where g is gravitational acceleration at the planet's surface, from eq. (8) this condition is

$$r_{\text{min}} = \frac{h^2}{gR^2 + \sqrt{g^2R^4 + 2\mathcal{E}h^2}} > R, \qquad \text{or} \qquad h^2 > 2R^2(\mathcal{E} + gR).$$

If the spacecraft approaches the planet with distant speed $u \stackrel{\text{def}}{=} |\mathbf{u}_{\text{i}}| = |\mathbf{v}_{\text{i}} - \mathbf{V}|$ on a line to miss by impact parameter b in the absence of gravity, then

$$\mathcal{E} = \tfrac{1}{2}u^2 \qquad \text{and} \qquad h = bu,$$

and so the condition $r_{\text{min}} > R$ is

$$b > R\sqrt{1 + \frac{2gR}{u^2}}. \tag{10}$$

The flyby in Fig. 2 for a spacecraft at Jupiter, where $R = 71400$ km, would involve $b = 2.7$ million km and $r_{\text{min}} = 2.0$ million km. The same boost with $\beta = \beta_2 = 129°$ has $r_{\text{min}} = 87500$ km, perhaps a little close for comfort — unless the opportunity for observation is judged too good to miss.

Note that for a finite planet a parabolic flyby with $h = bu = 0$ is always ruled out. Consequently $\beta < \pi$ and the maximum boost available from any gravity assist[8] is less than $2\,|\mathbf{V}|$.

In summary, the spacecraft's outgoing velocity $\mathbf{v}_{\text{f}}$ is found from $(\mathbf{v}_{\text{i}}, \mathbf{V})$ with the equations in Sec. 4, provided angle β is given. This comes from the equations of Sec. 5 after also impact parameter b is specified — and supposing that eq. (10) holds.

7 Real life

The above description of the slingshot effect is idealised as follows:

- the spacecraft/planet system is isolated — in particular, the planet's velocity $\mathbf{V}$ is constant;
- the spacecraft's velocities $(\mathbf{v}_{\text{i}}, \mathbf{v}_{\text{f}})$ are asymptotic quantities.

[8] Maximum boost is available in a repulsive interaction such as *eg.* a soccer ball bouncing directly from the front of a fast-moving truck [12, 13, 20] or a gas molecule bouncing from an advancing piston [20].

In reality, *Cassini* and other probes encounter planets with near-circular orbits round the Sun, whose velocity vectors $\mathbf{V}$ therefore change direction steadily. The spacecraft is also orbiting the Sun, and its gravitational interactions are infinite-range and occupy infinite time.

To maintain simplicity of description:

- vectors ($\mathbf{v}_\mathrm{i}$, $\mathbf{v}_\mathrm{f}$) must be taken as incoming and outgoing velocities at transition between the Sun and the planet as dominant influence;

- the flyby then occupies a moderate intervening interval (a few days) during which $\mathbf{V}$ rotates little (a degree or two) and the spacecraft exchanges negligible energy with the Sun — *ie,* eq. (2) effectively remains.

These approximations are used in numerical examples [17, 18, 20].

Then, ignoring much more distant planets *etc*[9], somewhat blurred velocity triangles of Fig. 2 and Fig. 3 apply in practice.

On the scale of the Solar System the flyby is localised and, for the purpose of estimating the spacecraft's subsequent trajectory, quite well approximated by a point event that instantaneously changes kinetic energy and angular momentum about the Sun.

NASA's control of *Cassini* (*eg.* [4]) uses very much more detailed computations, and engine-burns for fine correction.

8 Conclusion

Classical Mechanics originated from attempts to understand planetary and lunar motion. Its description of spacecraft orbits — including gravity assists — is a fitting contemporary development.

The diagram in Fig. 2 makes clear the simplicity of the slingshot effect, using basic conservation laws — of linear momentum, and energy. The main corrections in practical situations are relatively minor, and the description is suitable for elementary courses and textbooks.

Acknowledgements

Thanks to John Lucey for supplying several references.

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