# Unicycles and Bifurcations*

R. C. Johnson†
*Department of Mathematical Sciences,*
*University of Durham, Durham DH1 3LE, England.*

October 22, 2002

**Abstract**

A model of balancing upright on a unicycle has pitchfork and Hopf bifurcations, and describes riders from novice to expert with both good and not-so-good reactions.

## 1 Introduction

There is no real mystery about how unicyclists [1] stay upright — they pedal so as to keep their point of contact with the ground under their centre of gravity.

But pedalling is confined to the plane of the wheel, and so a sideways fall has to be countered by first turning the wheel-plane. This may be done by upper-body rotation, using angular momentum conservation and wheel/ground friction.

As a result, a competent rider can control the machine near upright by continual small adjustments of the wheel-plane plus minor pedalling to and fro. This may be either static balancing ('idling') or a subsidiary component of steady progress.

Here we consider details of the planar part of the stabilising motion, and develop a feedback model for the rider's reactions to falling backwards or forwards in the wheel-plane.

The corresponding equation of motion, eq. (1) below, is written in terms of the unicycle's angle $\phi$ of lean from vertical. It involves a function $h$ specifying the rider's acceleration of the wheel as a reaction to non-zero $\phi$.

**Am. J. Phys.* **66**(7), pps. *589–92*, 1998.
†bob.johnson@dur.ac.uk

A model for $h(\phi, \ldots)$ necessarily includes parameters measuring strength and delay of response, which influence stability of upright equilibrium, $\phi = 0$.

Consequently the equation shows bifurcations from $\phi = 0$. It allows description of a hierarchy of riders — beginners (who fall off), improvers (who can balance upright) and experts (who may drive swaying motion for showmanship).

## 2 Equation of motion

Consider a unicyclist riding near upright on a level surface [1]. Motion in the wheel plane is modelled by a planar inverted pendulum with horizontally moving support.

Variables are $\phi(t)$ = angle from vertical, and $z(t)$ = horizontal displacement of the wheel in the same sense. See Fig. 1.

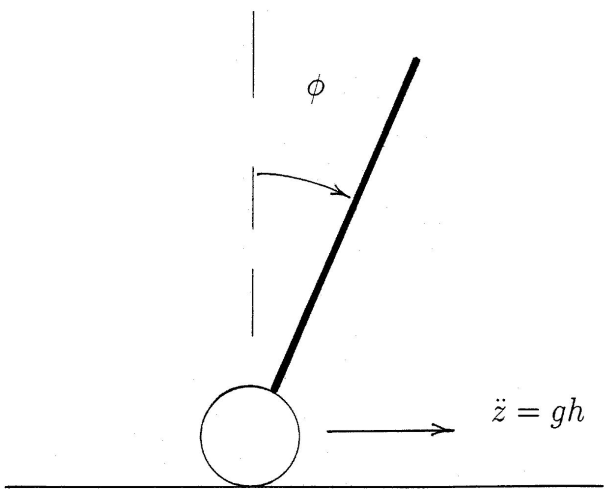


Figure 1: Unicycle as an inverted pendulum.

Straightforward mechanics [2] gives the equation of motion (with mass cancelled)

$$l\ddot{\phi} = g \sin\phi - \ddot{z}\cos\phi,$$

where $l$ = length of the equivalent simple pendulum, $g$ = acceleration of gravity. Friction is ignored in the wheel plane.

The rider controls the wheel's linear acceleration $\ddot{z}$ by pedalling. Let $\ddot{z} = gh$, where the dimensionless function $h$ describes this essential part of the dynamics.

Then with $g/l = 1$, so that the pendulum period is $2\pi$, we have the unicycle equation

$$\ddot{\phi} = \sin\phi - h\cos\phi. \tag{1}$$

We consider the motion it describes for $0 \leq |\phi| < \frac{1}{2}\pi$.

For example, a mindless rider might just oscillate the wheel back and forth:

$$h(t) = h(0)\cos(\omega t). \tag{2}$$

This model (like the rider) falls at the first hurdle, for then upright equilibrium $\phi = 0$ is not a solution of eq. (1).

Moreover, solutions $\phi(t)$ near $\phi = 0$ grow exponentially, as linearisation ($\sin\phi \approx \phi$, $\cos\phi \approx 1$) quickly shows [3].

A feedback (autonomous) model is better, where the rider accelerates the wheel intelligently to counter falling. Then $h$ depends on $\phi$ and how it changes.

Feedback stabilisation of an inverted planar pendulum in its simplest aspects is a textbook problem, of course, used for illustration in both control theory (*eg.* Chap. 2 of [4]) and dynamical systems (*eg.* p. *277* of [5]).

But application to unicycling is apparently novel [6], as is consideration of delayed response, and of non-linear effects.

# 3 Feedback

As the first and simplest feedback model, let $h = a\phi$, where $a$ is constant. That is, suppose the rider reacts only to angle of lean and not its rate of change, and in the most straightforward linear way.

Also, assume an ideal rider with perfect reactions, so there is no time lag $t_0$ and $h$ depends on $\phi(t)$ and not $\phi(t - t_0)$.

In fact, with or without time-lag:

- $h = 0$ at $\phi = 0$ — *ie,* $\phi = 0$ solves eq. (1), and at this upright equilibrium the rider lets well alone;

- the dynamics are forward-backward symmetric — *ie,* the rider reacts equally to $\phi$ negative and positive.

These desirable properties constrain the model for $h$.

The unicycle equation, eq. (1), becomes

$$\ddot{\phi} = \sin\phi - a\phi\cos\phi \tag{3}$$

and, linearising for $\phi \approx 0$, we have approximately

$$\ddot{\phi} = (1-a)\phi.$$

This has growing real exponential solutions for $a < 1$ [3], when equilibrium $\phi = 0$ is unstable.

However, a strong enough positive rider reaction to $\phi \neq 0$ — namely $a > 1$ — creates a restoring force; then motion near upright is regular swaying — simple-harmonic motion about $\phi = 0$ [3] with period

$$T(a) = \frac{2\pi}{\sqrt{a-1}}. \tag{4}$$

A value $a \sim 2$ makes $T$ comparable with the unicycle's pendulum period of $2\pi$ [7]. This sets the scale for $a$ — when horizontal acceleration from pedalling is the same order as vertical gravitational acceleration.

For given $a > 1$, there is simple-harmonic swaying about upright for all small amplitudes, and the motion does not damp to zero. Technically, upright equilibrium is stable but not asymptotically stable (*eg.* Sec. 1.3 of [5]).

This version of the feedback model appears to accommodate beginning riders, who either fall off (weak reaction, $a < 1$) or can react enough to keep swaying about upright under control, without eliminating it entirely.

# 4 Pitchfork bifurcation

Before refining the model, consider more closely the changes occuring at $a = 1$.

With $h = a\phi(t)$, eq. (3) describes conservative motion and there is the energy integral

$$\tfrac{1}{2}\dot{\phi}^2 + V(\phi; a) = const \qquad \text{where} \qquad V(\phi; a) = (1+a)\cos\phi + a\phi\sin\phi.$$

Near upright, for small $\phi$,

$$V(\phi; a) = 1 + a + \tfrac{1}{2}(a-1)\phi^2 + \dots,$$

confirming that the turning-point $\phi = 0$ is a maximum for $a < 1$ and a minimum for $a > 1$.

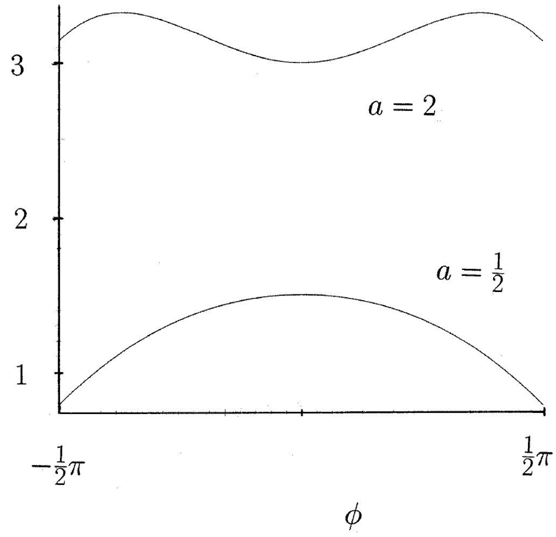


Figure 2: Potential function $V(\phi; a)$.

Fig. 2 shows $V(\phi; a)$ for $a = \frac{1}{2}$ and $a = 2$ over the whole range from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$. Besides illustrating the change in upright stability, the figure includes the extra information that for $a > 1$ there are more turning-points in $V(\phi)$, and that these are two symmetrically-placed maxima.

Indeed Fig. 2 shows that eq. (1) with $h = a\phi$ has a pitchfork bifurcation at $a = 1$, where unstable equilibrium solutions $\phi = \pm\phi_*$ emerge from $\phi = 0$ [8].

This bifurcation is 'subcritical' (*eg.* pps. *34–5* of [5]). By contrast, the more familiar supercritical pitchfork bifurcation has a minimum of potential energy that changes to a maximum as a symmetrical pair of minima emerges from it. Then the offset minima of $V$ give unsymmetrical stable static configurations — this is spontaneous symmetry-breaking [9].

# 5 Delay

Now make the model more realistic, by allowing for a rider's imperfect reactions through a time-delay:

$$h = a\phi(t - t_0), \qquad \text{where} \qquad t_0 > 0.$$

We may surmise that $t_0$ is relatively small.

One accessible estimate of human delay $t_0$ is given by the reaction-time of athletes at the start of a sprint race. Here electronics record the time lag between firing the start-gun and a certain change of pressure on the athletes' blocks; a value less than 0.1 sec is defined to be a false start.

For example, the finalists in the mens' 100m dash at the 1994 European Championship in Helsinki showed reaction-times between 0.15 and 0.18 sec [10]. In the 60m dash at the 1995 World Indoor Championship in Barcelona the figures were 0.11–0.13 sec [11].

The unicycle time-scale is set by the pendulum period $2\pi\sqrt{l/g}$. For equivalent length $l$ of 1 meter and $g \approx 10$ meters/sec/sec, this is roughly 2 sec.

On this scale, time $t_0$ of order 0.1–0.2 sec is indeed quite small.

To see the effect of small $t_0$, use

$$\phi(t - t_0) \approx \phi(t) - t_0\dot{\phi}(t).$$

The corresponding version of eq. (1) is then

$$\ddot{\phi} = \sin\phi - a(\phi - t_0\dot{\phi})\cos\phi,$$

with small-angle approximation

$$\ddot{\phi} = (1 - a)\phi + at_0\dot{\phi}. \tag{5}$$

Solutions to this differential equation [3] with $a > 1$ are oscillations multiplied by a factor

$$\exp \tfrac{1}{2}at_0t$$

and so, with $a$ and $t_0$ both positive, they inevitably grow with $t$, making upright equilibrium $\phi = 0$ unstable.

Even the smallest imperfection in reactions implies that the rider falls off!

Larger values of $t_0$ are unlikely to improve matters, and so we conclude that if $h$ is simply proportional to $\phi(t-t_0)$ then the rider cannot stay upright.

# 6 Refinement

A more sophisticated rider reacts to rate as well as angle of fall, and so may be able to overcome the effects of a finite reaction time.

Consider for instance

$$h\left(\phi,\, \dot{\phi}\right) = a\phi + b\dot{\phi}, \tag{6}$$

where $a$ and $b$ are constants. This is the simplest such refinement that maintains desirable forward-backward symmetry, and eq. (1) becomes

$$\ddot{\phi} = \sin\phi - (a\phi + b\dot{\phi})\cos\phi, \tag{7}$$

which still has the upright equilibrium solution $\phi = \dot{\phi} = 0$.

Note that with no time-delay, the small-angle approximation is simply

$$\ddot{\phi} = (1-a)\phi - b\dot{\phi}, \tag{8}$$

similar in form to eq. (5).

Then if $a > 1$ the solutions [3] are oscillations multiplied by

$$\exp(-\tfrac{1}{2}bt);$$

therefore they grow if $b < 0$ and decay if $b > 0$, and the equilibrium solution $\phi = 0$ is respectively unstable and stable.

Now, with the delay $t_0 > 0$ for both $\phi$ and $\dot{\phi}$ we have

$$h = a\phi(t-t_0) + b\dot{\phi}(t-t_0),$$

or

$$h \approx a\phi(t) + (b - at_0)\dot{\phi}(t) - bt_0\ddot{\phi}(t)$$

if $t_0$ is small.

Then the small-$\phi$ version of eq. (7) is

$$(1-bt_0)\ddot{\phi} = (1-a)\phi + (at_0 - b)\dot{\phi}, \tag{9}$$

where as in eq. (5) and eq. (8), the stability of $\phi = 0$ is controlled by the sign of the coefficient of $\dot{\phi}$.

With $a$ of order 1 or 2 and $t_0$ small, if $b$ and $at_0$ are comparable then certainly $1 - bt_0 > 0$, and the coefficient of $\ddot{\phi}$ on the left-hand side of eq. (9) merely renormalises magnitudes and leaves signs unchanged.

That is, small-angle solutions to eq. (7) with $a > 1$ are oscillation with a growing or decaying exponential factor as the combination $at_0 - b$ is respectively positive or negative.

So the conclusion is that a sufficient positive response to angular velocity as well as to angle itself — *ie,* $b > at_0$ along with $a > 1$ — is enough to overcome a small delay in reaction, and then the rider's upright equilibrium is stable [12].

Indeed, the equilibrium $\phi = \dot{\phi} = 0$ is asymptotically stable (*eg.* Sec. 1.3 of [5]) since nearby oscillations die away to zero.

This version of the model therefore accommodates a quite proficient unicyclist, able to control and damp out unwanted swaying.

# 7 Hopf bifurcation

When $b \neq 0$ the two-parameter unicycle equation eq. (7) is not conservative and there is no potential energy $V$ as in Sec. 4. Even so, for any value of $b$ it's easily shown to have a subcritical pitchfork bifurcation at $a = 1$ from $\phi = 0$ [13].

The other stability change of upright equilibrium — at $b = 0$ for $a > 1$ — involves no extra equilibria and has small oscillations changing between exponentially damped and exponentially growing. This is characteristic of a Hopf bifurcation(*eg.* Sec. 11.2 of [5]).

Often when a system's equilibrium loses stability at a Hopf bifurcation, small oscillations about it do not grow indefinitely but settle to a steady moderate-amplitude periodicity (*eg.* p. *344* of [5]). However, this is not inevitable, and happens only when larger-amplitude excursions from equilibrium remain damped [14].

The unicycle equation of eq. (7) has no such larger-amplitude damping, since for $|\phi| < \frac{1}{2}\pi$ we have $0 < \cos\phi \leq 1$, and so the damping term involving $\dot{\phi}$ has its sign fixed only by $b$. Therefore there is no steady oscillation.

That is, with the model of eq. (6), a rider incompetent enough to allow the growth of small swaying inevitably loses control completely and falls off.

Only a unicyclist who strongly damps large angles can allow $\phi = 0$ to be unstable. This suggests how the model can accommodate expert riders, who may wish deliberately to encourage swaying about upright to exhibit their skill to spectators.

# 8 Expert

As an example, consider the further refinement of the model:

$$h\left(\phi,\, \dot{\phi}\right) = a\phi + b\dot{\phi} + c\dot{\phi}\phi^2, \tag{10}$$

where $(a,\, b,\, c)$ are constants. The unicycle equation of motion is now

$$\ddot{\phi} = \sin\phi - (a\phi + b\dot{\phi} + c\dot{\phi}\phi^2)\cos\phi. \tag{11}$$

Both forward-backward symmetry and the equilibrium solution $\phi = \dot{\phi} = 0$ are maintained, while expert unicyclists are accommodated as follows.

To start with, ignore any delay $t_0$ of the rider's reactions.

Then with $a > 1$, by linearisation as before, small oscillations about $\phi = 0$ grow if $b < 0$. However larger ones, where the non-linear term $\dot{\phi}\phi^2$ is significant, are damped by $c > 0$.

Steady 'self-excited' oscillations (*eg.* Sec. 5.6 of [15]) are consequently trapped between. Since damping is zero at

$$b + c\phi^2 = 0,$$

they have amplitude of order

$$\sqrt{\frac{-b}{c}} \tag{12}$$

and period estimated by eq. (4) [16].

Evidently they appear as $b$ is decreased through zero and $\phi = 0$ loses stability. The amplitude's square-root dependence on the parameter $b$ is typical of the Hopf bifurcation (*eg.* Sec. 11.3 of [5]).

Now consider a small time-delay $t_0$.

With $b$ negligible, the expert rider can make $\phi = 0$ unstable since, in eq. (11) with $a > 1$ and $b = 0$, a small delay operates as in Sec. 5 to generate an additional term $-at_0\dot{\phi}$ that gives exponentially growing small-angle oscillation.

At the same time, the presence of $c\dot{\phi}\phi^2$ to damp large angles generates only a renormalisation as in eq. (9) — by order-$t_0$ corrections to $\ddot{\phi}$.

And this does not affect the amplitude of the rider's steady oscillation about upright, eq. (12), which involves a ratio from which such renormalisation cancels.

For example, with $a = 2$, $b = 0$ and $c = 3$, time delays of $t_0 = 0.05$ and 0.1 (typically 0.1 and 0.2 sec, according to Sec. 5) generate stable swaying about upright equilibrium with amplitudes of 21.2° and 30.0° respectively [17].

Treating the delay as effective $b = -at_0$, and ignoring any renormalisation, the standard theory of self-excited oscillation [16] predicts an amplitude of $2\sqrt{at_0/c}$, or 20.9° and 29.6° respectively.

In summary, skillful and extrovert riders may drive swaying about upright by delaying their reactions a little, while making very sure to damp large-angle motion.

# 9 Conclusion

Finally, we have the model of eq. (11), which for various values of parameters ($a$, $b$, $c$) includes all cases considered, and describes the essential planar part of the unicycle balancing mechanism.

It accommodates the spectrum of riding ability, including unicyclists with both perfect and, more realistically, somewhat imperfect reactions.

Quite reasonably, it shows that any small time-lag in the rider's reactions tends to increase the instability of upright equilibrium. However, expert riders may exploit this to show off their skill.

The model's stability-changes illustrate nicely two standard phenomena — a supercritical Hopf bifurcation and a subcritical pitchfork bifurcation.

While the former is commonplace, the latter is seen less often. Physical systems more readily exhibit the supercritical sort of pitchfork bifurcation, responsible for spontaneous symmetry-breaking [9].

# References


[1] See *eg. The Unicycle Page*, `http://www.unicycling.org/`.

[2] L. D. Landau and E. M. Lifshitz, *Mechanics* (Butterworth/Heinemann, Oxford, UK, 3rd Edition, 1976) p. *11*, problems 2 and 3.

[3] Standard properties of solutions to second-order linear differential equations with constant coefficients may be found in *eg.* Chap. 2 of E. Kreyszig, *Advanced Engineering Mathematics* (Wiley, New York, 7th Edition, 1993).

[4] O. I. Elgard, *Control Systems Theory* (McGraw-Hill, New York, 1967).

[5] J. Hale and H. Koçak, *Dynamics and Bifurcations* (Springer-Verlag, New York, 1991).

[6] Although the case described by eq. (2) is mentioned by W. T. Grandy, Jr., and M. Schöck, "Simulations of non-linear pivot-driven pendula," Am. J. Phys. **65**, 376 (1997).

[7] In practice, strong pushing on the pedals tends to turn the wheel-plane out of line — an added complication for a nervous novice.

[8] The angle $\phi_* > 0$ is fixed by $V'(\phi) = 0$ as the positive solution less than $\frac{1}{2}\pi$ to the equation $a\phi = \tan\phi$. A simple sketch graph of line $a\phi$ and curve $\tan\phi$ shows that $\phi_*$ exists for all $a > 1$. It rapidly approaches $\frac{1}{2}\pi$ as $a \to \infty$, and goes to 0 as $a \to 1$ from above.

[9] I. N. Stewart and M. Golubitsky, *Fearful Symmetry* (Penguin Books, London, UK, 1993).

[10] *Athletics Weekly* (Emap Publishing, Peterborough, UK) **47**, no. 33, p. *6* (1994).

[11] *ibid.* **48**, no. 12, p. *5* (1995).

[12] At this point, comparing $t_0$ with time-scale $2\pi\sqrt{l/g}$ (as in Sec. 5) shows why taller unicycles tend to be easier to ride.

[13] The change in number and stability of equilibrium solutions follows for $b < 0$ by straightforward phase space methods (*eg.* Sec. 10.1 of [5]), while a more detailed centre manifold analysis (*eg.* Sec. 10.3 of [5]) is needed for $b > 0$.

[14] In fact this describes a *supercritical* Hopf bifurcation. The subcritical case involves an equilibrium losing stability by absorbing unstable oscillations — a feature of *eg*. the Lorenz equations (*eg*. Chap. 7 of [15]).

[15] E. Atlee Jackson, *Perspectives of Nonlinear Dynamics* (Cambridge U.P., Cambridge, UK, 1990).

[16] An averaging calculation (as in *eg*. Sec. 5.5 of [15]) gives amplitude $2\sqrt{-b/c}$, and period as in eq. (4) with small corrections proportional to $b/c$.

[17] The calculation used the package by B. Ermentrout, *PhasePlane: the Dynamical Systems Tool, version 3.0* (Brooks/Cole, Pacific Grove, CA, 1990).