Riemannian Geometry IV

Solutions, set 17.

Exercise 42. We know from Exercise 41 that the tensor \( R' \) is parallel, i.e., \( \nabla R' = 0 \). We conclude from Exercise 39 that \( R = f R' \), and therefore

\[
\]

The Second Bianchi Identity tells us that

\[
\nabla R(X, Y, Z, W, T) + \nabla R(X, Y, W, T, Z) + \nabla R(X, Y, T, Z, W) = 0,
\]

which yields, using the definition of \( R' \):

\[
0 = (T f)((X, W)\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle)
+ (Z f)((X, T)\langle Y, W\rangle - \langle X, W\rangle\langle Y, T\rangle)
+ (W f)((X, Z)\langle Y, T\rangle - \langle X, T\rangle\langle Y, Z\rangle).
\]

Using the relations \( \langle Z(p), W(p)\rangle = \langle Z(p), Y(p)\rangle = \langle Y(p), W(p)\rangle = 0, \|Y(p)\| = 1 \) and \( T = Y \), we conclude that, at \( p \)

\[
0 = (T f)(p)((X(p), W(p)) \cdot 0 - \langle X(p), Z(p)\rangle \cdot 0)
+ (Z f)(p)((X(p), T(p)) \cdot 0 - \langle X(p), W(p)\rangle \cdot 1)
+ (W f)(p)((X(p), Z(p)) \cdot 1 - \langle X(p), T(p)\rangle \cdot 0)
= ((W f)(p)Z(p) - (Z f)(p)W(p), X(p)).
\]

Since \( Z(p) \) and \( W(p) \) are linearly independent and \( X(p) \in T_pM \) was arbitrary, we conclude that both \( (W f)(p) = 0 \) and \( (Z f)(p) = 0 \). Since \( Z(p) \) was arbitrary, \( f \) must be locally constant. Since \( M \) is connected, \( f \) is globally constant.
COLLECTIVE HOMEWORK OVER PREVIOUS WEEKS

Exercise 36.

(a) Let \( \text{grad } f(p) = \sum_{i=1}^{n} \alpha_i e_i \). In order to calculate the coefficients \( \alpha_i \), we take inner product with \( e_k \):

\[
\alpha_k = \langle \text{grad } f(p), e_k \rangle = e_k(f).
\]

This proves (a).

(b) We have

\[
\text{div } (f X)(p) = \sum_{i=1}^{n} \langle \nabla_{e_i} f X, e_i \rangle = \sum_{i=1}^{n} \langle e_i(f) X(p), e_i \rangle + f(p) \sum_{i=1}^{n} \langle \nabla_{e_i} X, e_i \rangle = \langle X(p), \sum_{i=1}^{n} e_i(f) e_i \rangle + f(p) \text{div } X(p) = \langle X(p), \text{grad } f(p) \rangle + f(p) \text{div } X(p).
\]

(c) We have

\[
\Delta f(p) = -\text{div} \left( \sum_{i=1}^{n} E_i(f) E_i \right) = -\sum_{i=1}^{n} \langle \text{grad } E_i(f)(p), e_i \rangle - \sum_{i=1}^{n} e_i(f) \text{div } E_i(p)
\]

\[
= -\sum_{i=1}^{n} \langle e_j(E_i(f)) e_j, e_i \rangle - \sum_{i=1}^{n} e_i(f) \sum_{j=1}^{n} \langle \nabla_{e_j} E_i, e_j \rangle
\]

\[
= -\sum_{i=1}^{n} \langle e_i(E_i(f)) + \sum_{i,j=1}^{n} e_i(f) \langle e_i, \nabla_{e_j} E_j \rangle
\]

\[
= -\sum_{i=1}^{n} \langle \langle e_i(E_i(f)) - \langle \text{grad } f, \nabla_{e_i} E_i \rangle \rangle - \sum_{i=1}^{n} \langle \langle e_i(E_i(f)) - \nabla_{e_i} E_i(f) \rangle \rangle.
\]

(d) We have

\[
\Delta (f g) = -\text{div}(\text{grad } (f g)) = -\text{div}(f \text{ grad } g) - \text{div}(g \text{ grad } f)
\]

\[
= -\langle \text{grad } f, \text{ grad } g \rangle - f \text{ div } (\text{grad } g) - \langle \text{grad } g, \text{ grad } f \rangle - g \text{ div } (\text{grad } f)
\]

\[
= f \Delta g + g \Delta f - 2\langle \text{grad } f, \text{ grad } g \rangle.
\]

Finally, we have

\[
(\Delta f, g) = \int_M (\Delta f) g \, dv = -\int_M \text{div} (\text{grad } f) g \, dv
\]

\[
= -\int_M \text{div}(g \text{grad } f) - \langle \text{grad } f, \text{ grad } g \rangle \, dv = \int_M \langle \text{grad } f, \text{ grad } g \rangle \, dv = (\text{grad } f, \text{ grad } g).
\]
The full result follows now by symmetry between $f$ and $g$.

**Exercise 40.**

Let $M$ be $n$-dimensional, and assume $K(\Sigma) = K_0$ for all 2-dimensional subspaces of $TM$. Then, by Exercise 39, we have

$$\langle R(v_1, v_2)v_3, v_4 \rangle = K_0 \left( \langle v_1, v_4 \rangle \langle v_2, v_3 \rangle - \langle v_1, v_3 \rangle \langle v_2, v_4 \rangle \right).$$

Let $p \in M$ and $e_1, \ldots, e_n$ be an orthonormal basis of $T_p M$. Then

$$\text{Ric}(v, w) = \sum_{i=1}^n \langle R(e_i, v)w, e_i \rangle = K_0 \sum_{i=1}^n \left( \langle e_i, e_i \rangle \langle v, w \rangle - \langle e_i, w \rangle \langle v, e_i \rangle \right) = K_0 \left( n \langle v, w \rangle - \langle w, v \rangle \right) = (n - 1)K_0 \langle v, w \rangle,$$

i.e., $M$ is an Einstein manifold with constant $(n - 1)K_0$. Above, we used

$$\sum_{j=1}^n \langle w, e_j \rangle \sum_{k=1}^n \langle v, e_k \rangle \delta_{jk} = \sum_i \langle w, e_i \rangle \langle v, e_i \rangle.$$

**Exercise 43.**

Since $l(s) = \int_a^b \frac{\partial F}{\partial t}(s, t), \frac{\partial F}{\partial t}(s, t) \rangle^{1/2} dt$, we obtain using the Riemannian property,

$$l'(s) = \int_a^b \frac{1}{\|F\|_{s,t}} \left( \frac{D \partial F}{ds \partial t} (s, t), \frac{\partial F}{\partial t} (s, t) \right).$$

Differentiating the integrand with respect to $s$, using the Symmetry Lemma, and setting then $s = 0$ yields

$$- \frac{1}{\|c'(t)\|^3} \left( \frac{D \partial F}{dt} (0, t), c'(t) \right)^2 + \frac{1}{\|c'(t)\| s=0} \left( \frac{D \partial F}{dt} (s, t), \frac{\partial F}{\partial t} (s, t) \right).$$

Using $\|c'\| = 1$ and $\frac{\partial F}{ds}(0, t) = X(t)$ yields

$$l''(0) = \int_a^b \left( - \left( \frac{D \partial F}{dt} X(t), c'(t) \right)^2 + \left. \frac{\partial}{\partial s} \right|_{s=0} \left( \frac{D \partial F}{dt} (s, t), \frac{\partial F}{\partial t} (s, t) \right) \right) dt.$$
Using, again, the Riemannian property and the Symmetry Lemma, we conclude that
\[
\frac{\partial}{\partial s} \left. \left| \frac{D}{dt} \frac{\partial F}{\partial s}(s,t), \frac{\partial F}{\partial t}(s,t) \right| \right|_{s=0} = \left\langle \frac{D}{ds} \frac{D}{dt} \frac{\partial F}{\partial s}(s,t), c'(t) \right\rangle + \left\langle \frac{D}{dt} X(t), \frac{D}{dt} X(t) \right\rangle.
\]
Now we make use of Lemma 7.4 to interchange the order of covariant derivatives, and obtain
\[
\frac{\partial}{\partial s} \left. \left| \frac{D}{dt} \frac{\partial F}{\partial s}(s,t), \frac{\partial F}{\partial t}(s,t) \right| \right|_{s=0} = \left\langle R \left( \frac{\partial F}{\partial s}(0,t), \frac{\partial F}{\partial t}(0,t) \right) \frac{\partial F}{\partial s}(0,t), c'(t) \right\rangle + \left\langle \frac{D}{dt} X(t), \frac{D}{dt} X(t) \right\rangle^2 + \left\langle \frac{D}{dt} \frac{D}{ds} \frac{\partial F}{\partial s}(s,t), c'(t) \right\rangle + \left\langle \frac{D}{dt} X(t), \frac{D}{dt} X(t) \right\rangle^2.
\]
Now, since \( X^\perp = X - \left\langle X, c' \right\rangle c' \) and \( c \) is a geodesic, we obtain
\[
\frac{DX^\perp}{dt} = \frac{DX}{dt} - \left\langle \frac{DX}{dt}, c' \right\rangle c',
\]
and, consequently,
\[
\left\| \frac{DX^\perp}{dt} \right\|^2 = \left\| \frac{DX}{dt} \right\|^2 - 2\left\langle \frac{DX}{dt}, c' \right\rangle^2 + \left\langle \frac{DX}{dt}, c' \right\rangle^2 \left\| c' \right\|^2 = \left\| \frac{DX}{dt} \right\|^2 - \left\langle \frac{DX}{dt}, c' \right\rangle^2.
\]
Putting everything together, we obtain
\[
l''(0) = \int_a^b \left( - \left\langle \frac{D}{dt} X(t), c'(t) \right\rangle^2 + \frac{\partial}{\partial s} \left. \left| \frac{D}{dt} \frac{\partial F}{\partial s}(s,t), \frac{\partial F}{\partial t}(s,t) \right| \right|_{s=0} \right) dt
\]
\[
= \int_a^b \left( \left\| \frac{DX^\perp}{dt} \right\|^2 + \left\langle R(X(t), c'(t)) X(t), c'(t) \right\rangle + \left\langle \frac{D}{dt} \frac{D}{ds} \frac{\partial F}{\partial s}(s,t), c'(t) \right\rangle \right) dt.
\]
Since $\langle R(v_1, v_2) v_3, v_4 \rangle$ vanishes if $v_1 = v_2$ or $v_3 = v_4$, and $\langle R(v_1, v_2) v_3, v_4 \rangle = -\langle R(v_1, v_2) v_4, v_3 \rangle$, we conclude that

$$\langle R(X, c') X, c' \rangle = -\langle R(X^\perp, c') c', X^\perp \rangle$$

$$= -K(\text{span}\{X^\perp, c'\}) \left( \|X^\perp\|^2 \|c'\| - \langle X^\perp, c' \rangle^2 \right) = -K(\text{span}\{X^\perp, c'\}) \|X^\perp\|^2.$$

Moreover, since $c$ is a geodesic, we have

$$\int_a^b \langle \frac{D}{dt} \frac{D}{ds} |_{s=0} \frac{\partial F}{\partial s}(s, t), c'(t) \rangle \, dt = \int_a^b \frac{\partial}{\partial t} \left( \langle \frac{D}{ds} |_{s=0} \frac{D}{ds} (s, t), c'(t) \rangle \right) \, dt$$

$$= \left\langle \frac{D}{ds} |_{s=0} \frac{\partial F}{\partial s} (s, b), c'(b) \right\rangle - \left\langle \frac{D}{ds} |_{s=0} \frac{\partial F}{\partial s} (s, a), c'(a) \right\rangle.$$

Since $F$ is a proper variation, we have $\frac{\partial F}{\partial s}(s, a) = 0$ and $\frac{\partial F}{\partial s}(s, b) = 0$, and we conclude that

$$\int_a^b \langle \frac{D}{dt} \frac{D}{ds} |_{s=0} \frac{\partial F}{\partial s}(s, t), c'(t) \rangle \, dt = 0.$$

Combining these results, we end up with

$$l''(0) = \int_a^b \left( \left\| \frac{D X^\perp}{dt} \right\|^2 - K(\text{span}\{X^\perp, c'\}) \|X^\perp\|^2 \right) \, dt.$$