Riemannian Geometry IV

Solutions, set 13.

Exercise 29. We have
\[ c'(s) = \frac{\partial F}{\partial s}(s, r(s)) + \frac{\partial F}{\partial t}(s, r(s))r'(s), \]
by the chain rule. Note that \( \|\frac{\partial F}{\partial t}(s, t)\| = \|v(s)\| = 1 \) (since \( t \mapsto F(s, t) \) is a geodesic with initial vector \( v(s) \)) and \( \frac{\partial F}{\partial s}(s, t) \perp \frac{\partial F}{\partial t}(s, t) \), by the Gauß-Lemma.

Therefore,
\[ \|c'(s)\| = \sqrt{|r'(s)|^2 + \|\frac{\partial F}{\partial s}(s, r(s))\|^2} \geq |r'(s)|, \]
and we conclude that
\[ l(c) = \int_a^b \|c'(s)\|ds \geq \int_a^b |r'(s)|ds \geq \int_a^b r'(s)ds = |r(b) - r(a)|, \]
with equality in the first inequality if and only if \( \|\frac{\partial F}{\partial s}(s, r(s))\| \equiv 0 \) and equality in the second inequality if and only if \( r' \geq 0 \) or \( r' \leq 0 \) on \([a, b]\). Hence: We have equality if and only if \( r \) is monotone and \( v(s) \) is a constant function \( \equiv v \), i.e., \( c(s) = \exp_p r(s)v \).

Exercise 30. (a) Note that \( \varphi(p) = 0 \), so
\[ \frac{\partial}{\partial x_i} \bigg|_p = \frac{d}{dt} \bigg|_{t=0} \varphi^{-1}(0 + te_i) = \frac{d}{dt} \bigg|_{t=0} \exp_p(tv_i) = v_i. \]

This implies that
\[ g_{ij}(p) = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle_p = \langle v_i, v_j \rangle_p = \delta_{ij}. \]

(b) We have
\[ c(t) = \varphi^{-1}(tw_1, \ldots, tw_n) = \exp_p(t \sum_j w_jv_j). \tag{1} \]
Let \( v = \sum_{j} w_j v_j \in T_p M \). Then (1) shows that \( c \) is a geodesic with initial vector \( v \). Let \( (c_1, \ldots, c_n) = \varphi \circ c \), i.e., \( c_j(t) = tw_j, \ c'_j(t) = w_j \) and \( c''_j(t) = 0 \). Let \( \frac{D}{dt} \) denote covariant derivative along \( c \). Since \( c \) is a geodesic, we have

\[
0 = \frac{D}{dt} c' = \frac{D}{dt} \sum_{j} c'_j \left( \frac{\partial}{\partial x_j} \circ c \right) = \sum_{j} w_j \nabla_{c'} \frac{\partial}{\partial x_j}
\]

\[
= \sum_{i,j} w_i w_j \left( \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} \right) \circ c = \sum_{k} \left( \sum_{i,j} w_i w_j (\Gamma^k_{ij} \circ c) \right) \frac{\partial}{\partial x_k} \circ c.
\]

Using the fact that \( \frac{\partial}{\partial x_k} \) form a basis, we conclude that

\[
\sum_{i,j} w_i w_j \Gamma^k_{ij}(c(t)) = 0,
\]

for all \( k \in \{1, \ldots, n\} \).

(c) Evaluating (2) at \( t = 0 \), we obtain

\[
\sum_{i,j} w_i w_j \Gamma^k_{ij}(p) = 0 \quad \text{for all } w \in \mathbb{R}^n.
\]

The choice \( w = e_i + e_j \) yields

\[
2 \Gamma^k_{ij}(p) = 0,
\]

so we conclude that all Christoffel symbols vanish at \( p \). Consequently, we have

\[
\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}(p) = 0.
\]

**Exercise 31.** We first show that

\[
\Psi(T_v SM) \subset \{(w_1, w_2) \in T_p M \times T_p M \mid w_2 \perp v \text{ w.r.t } g_p\}.
\]

The result follows then immediately from dimension considerations, since both vector spaces have dimension \( 2n - 1 \).

Let \( X : (-\epsilon, \epsilon) \to SM \) be a curve with \( X(0) = v \in S_p M \), representing a tangent vector \( X'(0) \in T_v SM \). Let \( c = \pi \circ X : (-\epsilon, \epsilon) \to M \) be the corresponding projected curve. Let \( \frac{D}{dt} \) denote the covariant derivative along
c. Then $X \in \mathcal{X}_c(M)$ and we have, using the Riemannian property of the Levi-Civita connection,

$$0 = \frac{d}{dt}_{t=0} \|X(t)\|^2 = 2 g_c(t) \left( \frac{D}{dt} X(t), X(t) \right).$$

Evaluating at $t = 0$ yields

$$0 = 2 g_p \left( \frac{D}{dt} X(0), v \right),$$

which implies that

$$\Psi(X'(0)) = \left( w_1 = c'(0), w_2 = \frac{D}{dt} X(0) \right)$$

with $g_p(w_2, v) = 0$, i.e., $w_2 \perp v$. 