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Riemannian Geometry IV

Solutions, set 11.

Exercise 26. Let $N = \mathrm{dim} G$ and $n = \mathrm{dim} H$.

(a) We first show that $T_eH \subset \ker D\pi(e)$. Let $v \in T_eH$. Then there exists a curve $c : (-\epsilon, \epsilon) \to H$ such that $c(0) = e$ and $c'(0) = v$. The image curve $\pi \circ c : (-\epsilon, \epsilon) \to G/H$ is constant because of $c(t)H = eH$ for all $t \in (-\epsilon, \epsilon)$. This implies that

$$D\pi(e)(v) = \frac{d}{dt}|_{t=0} \pi \circ c(t) = 0 \in T_{eH}G/H.$$

$D\pi(e) : T_eG \to T_{eH}G/H$ is surjective, and we have by the dimension formula:

$$\mathrm{dimker} D\pi(e) + \mathrm{dim} T_{eH}G/H = \mathrm{dim} T_eG,$$

i.e., $\mathrm{dimker} D\pi(e) = N - (N - n) = n$. Since $\mathrm{dim} T_eH = n$, we conclude that $T_eH = \ker D\pi(e)$.

(b) Note first that $\mathrm{dim} V = \mathrm{dim} T_eG - \mathrm{dimker} D\Pi(e) = N - n$ and $\mathrm{dim} T_{eH}G/H = N - n$, so we are done if we prove that Φ is surjective (then it is also injective, for dimensional reasons). We know that $D\pi(e) : T_eG \to T_{eH}G/H$ is surjective. For a given $v \in T_{eH}G/H$ let $v_1 \in T_eG$ such that $D\pi(e)(v_1) = v$. Let $v_1 = u_1 + w_1 \in T_eH \perp V$. Since $T_eH = \ker D\pi(e)$, we have $D\pi(v_1) = D\pi(w_1) = \Phi(w_1)$. This shows surjectivity of Φ.

(c) We first show that T_eH is $Ad(H)$ invariant. Let $v \in T_eH = \ker D\pi(e)$. Then there is a curve $c : (-\epsilon, \epsilon) \to H$ such that $c(0) = e$ and $c'(0) = v$, and we have

$$D\pi(e)(Ad(h)v) = \frac{d}{dt}|_{t=0} \pi(\underbrace{hc(t)h^{-1}}_{\in H}) = 0 \in T_{eH}G/H,$$

i.e., $Ad(h)v \in \ker D\pi(e) = T_eH$. Recall that $\langle \cdot, \cdot rangle_e$ is $Ad(H)$-invariant. Let $v \in V$. We need to show that $Ad(h)v \perp T_eH$. Let $h \in H$ and $w \in T_eH$. Then

$$\langle Ad(h)v, w\rangle_e = \langle Ad(h^{-1})Ad(h)v, Ad(h^{-1})w\rangle_e = \langle \underbrace{v}_{\in V}, \underbrace{Ad(h^{-1})w}_{\in T_eH}\rangle_e = 0.$$

Here we used $Ad(h_1)Ad(h_2) = Ad(h_1h_2)$, which we finally show:

$$\begin{aligned} Ad(h_1)Ad(h_2)v &= Ad(h_1)\frac{d}{dt}|_{t=0}h_2Exp(tv)h_2^{-1} \\ &= \frac{d}{dt}|_{t=0}h_1\left(h_2Exp(tv)h_2^{-1}\right)h_1^{-1} \\ &= \frac{d}{dt}|_{t=0}(h_1h_2)Exp(tv)(h_1h_2)^{-1} = Ad(h_1h_2)v. \end{aligned}$$

Exercise 27. (a) The curve

$$c(t) = Exp(t\begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}) = \begin{pmatrix} \cos\alpha t & \sin\alpha t \\ -\sin\alpha t & \cos\alpha t \end{pmatrix} \in H$$

satisfies $c(0) = e$ and $c'(0) = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}$. Thus we have

$$\{\begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix} \mid \alpha \in \mathbb{R}\} \subset T_eH.$$

Equality follows from the fact that both vector spaces are one-dimensional.

(b) For the symmetry, observe that $\mathrm{tr}(U) = \mathrm{tr}(U^\top)$. Thus

$$\langle A, B\rangle_e = 2\mathrm{tr}(AB^\top) = 2\mathrm{tr}(BA^\top) = \langle B, A\rangle_e.$$

Let $X \in H$. Then (see Example 30):

$$Ad(X)A = XAX^{-1}, \quad Ad(X)B = XBX^{-1}.$$

Using $X^{-1} = X^\top$ (since $X \in SO(2)$), this implies

$$\begin{aligned} \langle Ad(X)A, Ad(X)B\rangle_e &= 2\mathrm{tr}\left((XAX^{-1})(XBX^{-1})^\top\right) \\ &= 2\mathrm{tr}\left(XAB^\top X^{-1}\right) = 2\mathrm{tr}\left(X^{-1}XAB^\top\right) \\ &= 2\mathrm{tr}(AB^\top) = \langle A, B\rangle_e, \end{aligned}$$

where we used $\mathrm{tr}(UV) = \mathrm{tr}(VU)$ in the second to last line above.

(c) We have

$$\begin{aligned} \langle\begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}, \begin{pmatrix} 0 & \gamma \\ -\gamma & 0 \end{pmatrix}\rangle &= 2\mathrm{tr}\left(\begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}\begin{pmatrix} 0 & -\gamma \\ \gamma & 0 \end{pmatrix}\right) \\ &= 2\mathrm{tr}\begin{pmatrix} \beta\gamma & -\alpha\alpha \\ -\alpha\gamma & -\beta\gamma \end{pmatrix} = 0. \end{aligned}$$

(d) We have

$$\begin{aligned}
\langle A, A\rangle_2 &= 2\mathrm{tr}\left(\begin{pmatrix}\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{pmatrix}\right) = 2\mathrm{tr}\begin{pmatrix}\frac{1}{4} & 0\\ 0 & \frac{1}{4}\end{pmatrix} = 1,\\
\langle A, B\rangle_2 &= 2\mathrm{tr}\left(\begin{pmatrix}\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}\right) = 2\mathrm{tr}\begin{pmatrix}0 & \frac{1}{4}\\ -\frac{1}{4} & 0\end{pmatrix} = 0,\\
\langle B, B\rangle_2 &= 2\mathrm{tr}\left(\begin{pmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}\begin{pmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}\right) = 2\mathrm{tr}\begin{pmatrix}\frac{1}{4} & 0\\ 0 & \frac{1}{4}\end{pmatrix} = 1.
\end{aligned}$$

(e) Let

$$\begin{aligned}
c_1(t) &= Exp(tA) = Exp\begin{pmatrix}\frac{t}{2} & 0\\ 0 & -\frac{t}{2}\end{pmatrix} = \begin{pmatrix}e^{t/2} & 0\\ 0e^{-t/2} & \end{pmatrix} \in SL(2,\mathbb{R}),\\
c_2(t) &= Exp(tB) = Exp\begin{pmatrix}0 & \frac{t}{2}\\ \frac{t}{2} & 0\end{pmatrix} = \begin{pmatrix}\cosh(t/2) & \sinh(t/2)\\ \sinh(t/2) & \cosh(t/2)\end{pmatrix} \in SL(2,\mathbb{R}).
\end{aligned}$$

Then $c_1(0) = c_2(0) = e$ and $c_1'(0) = A$ and $c_2'(0) = B$. We calculate the tangent vectors of the image curves

$$\begin{aligned}
\gamma_1(t) &= f_{c_1(t)}(i) = \frac{e^{t/2}i + 0}{e^{-t/2}} = e^t i \in \mathbb{H}^2,\\
\gamma_2(t) &= f_{c_2(t)}(i) = \frac{\cosh(t/2)i + \sinh(t/2)}{\sinh(t/2)i + cosh(t/2)} \in \mathbb{H}^2,
\end{aligned}$$

at $t = 0$. Then $\gamma_1(0) = \gamma_2(0) = i$ and

$$\gamma_1'(t) = e^t i \in T_{\gamma_1(t)}\mathbb{H}^2,$$

i.e., $\gamma_1'(0) = i \in T_i\mathbb{H}^2$, and

$$\begin{aligned}
\gamma_2'(t) &= \frac{1}{2(\sinh(t/2)i + \cosh(t/2))^2}\cdot\\
&\qquad \left((\sinh(t/2)i + \cosh(t/2))^2 - (\cosh(t/2)i + \sinh(t/2))^2\right)\\
&= \frac{1}{(\sinh(t/2)i + \cosh(t/2))^2} \in T_{\gamma_2(t)}\mathbb{H}^2,
\end{aligned}$$

i.e., $\gamma_2'(0) = 1 \in T_i\mathbb{H}^2$. Note that $1, i \in T_i\mathbb{H}^2$ form an orthonormal base with respect to the hyperbolic Riemannian metric on $\mathbb{H}^2$.