Riemannian Geometry IV

Problems, set 8 (to be handed in on 6 December 2010 in the afternoon lecture).

Exercise 19. Let $M$ be a differentiable manifold, $\mathcal{X}(M)$ be the vector space of smooth vector fields on $M$, and $\nabla$ be a general affine connection (we do not require a Riemannian metric on $M$ and the "Riemannian property", and also not the "torsionless property" of the Levi-Civita connection). We say, a map

$$A : \mathcal{X}(M) \times \cdots \times \mathcal{X}(M) \to C^\infty(M) \text{ or } \mathcal{X}(M)$$

is a tensor, if it is linear in each argument, i.e.,

$$A(X_1, \ldots, fX_i + gY_i, \ldots, X_r) =$$

$$fA(X_1, \ldots, X_i, \ldots, X_r) + gA(X_1, \ldots, Y_i, \ldots, X_r),$$

for all $X, Y \in \mathcal{X}(M)$ and $f, g \in C^\infty(M)$.

(a) Show that

$$T : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M),$$

$$T(X, Y) = [X, Y] - (\nabla_X Y - \nabla_Y X)$$

is a tensor (called the "torsion" of the manifold $M$).

(b) Let

$$A : \underbrace{\mathcal{X}(M) \times \cdots \times \mathcal{X}(M)}_{r \text{ factors}} \to C^\infty(M)$$

be a tensor. The covariant derivative of $A$ is a map

$$\nabla A : \underbrace{\mathcal{X}(M) \times \cdots \times \mathcal{X}(M)}_{r + 1 \text{ factors}} \to C^\infty(M),$$

defined by

$$\nabla A(X_1, \ldots, X_r, Y) =$$

$$Y(A(X_1, \ldots, X_r)) - \sum_{j=1}^r A(X_1, \ldots, \nabla_Y X_j, \ldots, X_r).$$

Show that $\nabla A$ is a tensor.
(c) Let \((M, g)\) be a Riemannian manifold and \(G : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{C}^\infty(M)\) be the Riemannian tensor, i.e., \(G(X, Y) = \langle X, Y \rangle\). Calculate \(\nabla G\). What does it mean that \(\nabla G \equiv 0\)?

**Exercise 20.** Given a curve \(c : [a, b] \to \mathbb{R}^3, c(t) = (f(t), 0, g(t))\) without self intersections and with \(f(t) > 0\) for all \(t \in [a, b]\), let \(M \subset \mathbb{R}^3\) denote the surface of revolution obtained by rotating this curve around the vertical \(Z\)-axis. Let \(\nabla\) denote the Levi-Civita connection of \(M\). An almost global coordinate chart is given by \(\varphi : U \to V := (a, b) \times (0, 2\pi), \varphi^{-1}(x_1, x_2) = (f(x_1) \cos x_2, f(x_1) \sin x_2, g(x_1))\).

(a) Calculate the Christoffel symbols of this coordinate chart and express

\[ \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} \]

in terms of the basis \(\frac{\partial}{\partial x_k}\).

(b) Let \(\gamma_1(t) = \varphi^{-1}(x_1 + t, x_2)\). \(\frac{D}{dt}\) denotes covariant derivative along \(\gamma_1\). Calculate

\[ \frac{D}{dt} \gamma_1' \]

Show that this vector field along \(\gamma_1\) vanishes if and only if the generating curve \(c\) of \(M\) is parametrized proportional to arc-length. Note that \(\gamma_1\) is obtained by rotation of \(c\) by a fixed angle. Derive from these facts that meridians of a surface of revolution are geodesics if they are parametrized proportional to arc length.

(c) Let \(\gamma_2(t) = \varphi^{-1}(x_1, x_2 + t)\). \(\frac{D}{dt}\) denotes covariant derivative along \(\gamma_2\). Calculate

\[ \frac{D}{dt} \gamma_2' \]

Show that this vector field along \(\gamma_2\) vanishes if and only if \(f'(x_1) = 0\). Explain that this implies that parallels of a surface of revolution are geodesics if they have locally maximal or minimal radius.

(d) Assume that the generating curve \(c\) of the surface of revolution is arc-length parametrized. Show in this particular case that

\[ \text{vol}(M) = 2\pi \int_a^b f(x_1) \, dx_1. \]