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Riemannian Geometry IV

Problems, set 8 (to be handed in on 6 December 2010 in the afternoon lecture).

Exercise 19. Let M be a differentiable manifold, $\mathcal{X}(M)$ be the vector space of smooth vector fields on M, and ∇ be a general affine connection (we do not require a Riemannian metric on M and the "Riemannian property", and also not the "torsionless property" of the Levi-Civita connection). We say, a map

$$A : \mathcal{X}(M) \times \cdots \times \mathcal{X}(M) \to C^\infty(M) \text{ or } \mathcal{X}(M)$$

is a tensor, if it is linear in each argument, i.e.,

$$A(X_1, \cdots, fX_i + gY_i, \cdots, X_r) = \\ fA(X_1, \cdots, X_i, \cdots, X_r) + gA(X_1, \cdots, Y_i, \cdots, X_r),$$

for all $X, Y \in \mathcal{X}(M)$ and $f, g \in C^\infty(M)$.

(a) Show that

$$\begin{aligned} T : \mathcal{X}(M) \times \mathcal{X}(M) &\to \mathcal{X}(M), \\ T(X,Y) &= [X,Y] - (\nabla_X Y - \nabla_Y X) \end{aligned}$$

is a tensor (called the "torsion" of the manifold M).

(b) Let

$$A : \underbrace{\mathcal{X}(M) \times \cdots \times \mathcal{X}(M)}_{r \text{ factors}} \to C^\infty(M)$$

be a tensor. The covariant derivative of A is a map

$$\nabla A : \underbrace{\mathcal{X}(M) \times \cdots \times \mathcal{X}(M)}_{r+1 \text{ factors}} \to C^\infty(M),$$

defined by

$$\nabla A(X_1, \ldots, X_r, Y) = \\ Y(A(X_1, \ldots, X_r)) - \sum_{j=1}^{r} A(X_1, \ldots, \nabla_Y X_j, \ldots, X_r).$$

Show that ∇A is a tensor.

(c) Let (M, g) be a Riemannian manifold and $G : \mathcal{X}(M) \times \mathcal{X}(M) \to C^\infty(M)$ be the Riemannian tensor, i.e., $G(X, Y) = \langle X, Y \rangle$. Calculate ∇G. What does it mean that $\nabla G \equiv 0$?

Exercise 20. Given a curve $c : [a, b] \to \mathbb{R}^3$, $c(t) = (f(t), 0, g(t))$ without self intersections and with $f(t) > 0$ for all $t \in [a, b]$, let $M \subset \mathbb{R}^3$ denote the surface of revolution obtained by rotating this curve around the vertical Z-axis. Let ∇ denote the Levi-Civita connection of M. An almost global coordinate chart is given by $\varphi : U \to V := (a, b) \times (0, 2\pi)$,

$$\varphi^{-1}(x_1, x_2) = (f(x_1) \cos x_2, f(x_1) \sin x_2, g(x_1)).$$

(a) Calculate the Christoffel symbols of this coordinate chart and express

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}$$

in terms of the basis $\frac{\partial}{\partial x_k}$.

(b) Let $\gamma_1(t) = \varphi^{-1}(x_1 + t, x_2)$. $\frac{D}{dt}$ denotes covariant derivative along γ_1. Calculate

$$\frac{D}{dt}\gamma_1'.$$

Show that this vector field along γ_1 vanishes if and only if the generating curve c of M is parametrized proportional to arc-length. Note that γ_1 is obtained by rotation of c by a fixed angle. Derive from these facts that meridians of a surface of revolution are geodesics if they are parametrized propotional to arc length.

(c) Let $\gamma_2(t) = \varphi^{-1}(x_1, x_2 + t)$. $\frac{D}{dt}$ denotes covariant derivative along γ_2. Calculate

$$\frac{D}{dt}\gamma_2'.$$

Show that this vector field along γ_2 vanishes if and only if $f'(x_1) = 0$. Explain that this implies that parallels of a surface of revolution are geodesics if they have locally maximal or minimal radius.

(d) Assume that the generating curve c of the surface of revolution is arc-length parametrized. Show in this particular case that

$$\mathrm{vol}(M) = 2\pi \int_a^b f(x_1)\, dx_1.$$