Dr. Norbert Peyerimhoff, Durham University 15/11/2010

Riemannian Geometry IV

Problems, set 6.

Exercise 14. Use the same coordinate chart of the torus T^2 as in Exercise 12 and derive (with methods from Differential Geometry) that the Gaussian curvature $K : T^2 \to \mathbb{R}$ is given by

$$K(\varphi^{-1}(x_1, x_2)) = \frac{\cos x_1}{r(R + r \cos x_1)}.$$

Explain geometrically why

$$K(\varphi^{-1}(\pi/2, x_2)) = K(\varphi^{-1}(3\pi/2, x_2)) = 0 \quad \text{and} \, K(\varphi^{-1}(\pi, x_2)) < 0.$$

Show by calculation that

$$\int_{T^2} K \, d\mathrm{vol} = 0.$$

Does this come as a surprise to you or do you have an explanation?

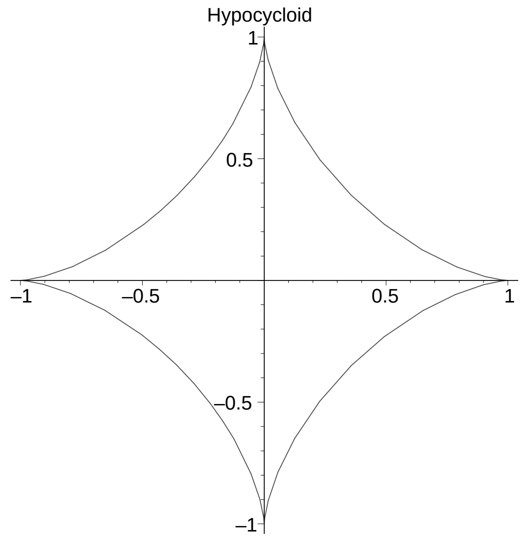
Exercise 15. Length calculations in the upper half plane model $\mathbb{H}^2$ of the hyperbolic plane.

(a) Let $0 < a < b$ and $c : [a, b] \to \mathbb{H}^2$, $c(t) = ti$. Calculate the arc-length reparametrization $\gamma : [0, \ln(b/a)] \to \mathbb{H}^2$ with the help of the sketch of proof of Proposition 2.16 in the lecture.

(b) Let $c : [0, \pi] \to \mathbb{H}^2$, given by

$$c(t) = \frac{ai \cos t + \sin t}{-ai \sin t + \cos t},$$

for some $a > 1$. Calculate $L(c)$.



Exercise 16. We interpret $\mathbb{R}^2$ as a two-dimensional Riemannian manifold with the standard Euclidean Riemannian metric. Find the length of the curve $c : [0, 2\pi] \to \mathbb{R}^2$, $c(t) = (\cos^3(t), \sin^3(t))$. (The above picture illustrates the shape of the curve.)

Exercise 17. In this exercise, we derive the following explicit formula for the hyperbolic distance function in $\mathbb{H}^2$:

$$\sinh(\frac{1}{2}d(z_1, z_2)) = \frac{|z_1 - z_2|}{2\sqrt{\mathrm{Im}(z_1)\mathrm{Im}(z_2)}}. \tag{1}$$

(a) Let $z_1, z_2 \in \mathbb{H}^2$ with $\mathrm{Re}(z_1) = \mathrm{Re}(z_2) = 0$. Check (**??**) for this case. You may use the well known distance (Example 19) in this case.

(b) Let $f_A(z) = \frac{az+b}{cz+d}$ for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$. Using the identity (from Exercise 11)

$$\mathrm{Im}(f_A(z)) = \frac{\mathrm{Im}(z)}{|cz + d|^2},$$

show that both left- and right-hand sides of (**??**) are invariant under f_A. You may use (without proof) the fact that isometries preserve the distance function.

(c) Let $z_1, z_2 \in \mathbb{H}^2$ with $\mathrm{Re}(z_1) = \mathrm{Re}(z_2) = x \neq 0$. Using the map $f(z) = z - x$, show the validity of (**??**) for the points z_1, z_2.

(d) Finally, let $z_1, z_2 \in \mathbb{H}^2$ with $\mathrm{Re}(z_1) \neq \mathrm{Re}(z_2)$. Then, obviously, the (unique) geodesic through z_1 and z_2 is a Euclidean semicircle (and not a vertical line). Let $x \in \mathbb{R}$ be the centre of this Euclidean semicircle and $R > 0$ its radius. Show that the map

$$f(z) = \frac{z - (x + R)}{z - (x - R)}$$

maps this semicircle to the positive imaginary axis. Again, using this map, conclude the validy of (**??**) for the points z_1, z_2.