Riemannian Geometry IV

Problems, set 5.

Exercise 10. Let \( \mathcal{W}^2 = \{ x \in \mathbb{R}^3 \mid q(x, x) = -1, x_3 > 0 \} \) with \( q(x, y) = x_1 y_1 + x_2 y_2 - x_3 y_3 \) be the hyperboloid model of the hyperbolic plane and \( \mathcal{B}^2 = \{ x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 < 1, x_3 = 0 \} \) be the embedding of the Poincaré ball model into \( \mathbb{R}^3 \). For every point \( p \in \mathcal{W}^2 \) let \( L_p \) denote the Euclidean straight line through \( p \) and \((0, 0, -1)\). Let \( f : \mathcal{W}^2 \rightarrow \mathcal{B}^2 \) be the stereographic projection, defined as follows: \( f(p) = L_p \cap \mathcal{B}^2 \).

(a) Calculate explicitly the maps \( f(X, Y, Z) \) for \((X, Y, Z) \in \mathcal{W}^2\) and \( f^{-1}(x, y, 0) \) for \((x, y, 0) \in \mathcal{B}^2\).

(b) A coordinate chart \( \varphi : U \rightarrow V \) of \( \mathcal{W}^2 \) is given by 
\[
\varphi^{-1}(x_1, x_2) = (\cos x_1 \sinh x_2, \sin x_1 \sinh x_2, \cosh x_2),
\]
for \((x_1, x_2) \in V = (0, 2\pi) \times (0, \infty)\). Let \( \psi = \varphi \circ f^{-1} \) be a coordinate chart of \( \mathcal{B}^2 \) with coordinate functions \( y_1, y_2 \). Calculate \( \psi^{-1} \) explicitly.

(c) Show that 
\[
\left. \frac{\partial}{\partial x_i} \right|_p \left. \frac{\partial}{\partial x_j} \right|_p = \left. \frac{\partial}{\partial y_i} \right|_{f(p)} \left. \frac{\partial}{\partial y_j} \right|_{f(p)} \right|_{f(p)} \text{ for all } p \in U.
\]
Using Lemma 2.4, this shows that \( f \) is an isometry.

Additional remark: To be precise, one would have to choose two coordinate charts of the above type with \( V_1 = (0, 2\pi) \times (0, \infty) \) and \( V_2 = (-\pi, \pi) \times (0, \infty) \) and an extra consideration for the linear map \( D f(0, 0, 1) : T_{(0,0,1)} \mathcal{W}^2 \rightarrow T_0 \mathcal{B}^2 \) to cover all of the hyperbolic plane and to fully prove that \( f \) is an isometry.

Exercise 11. Let 
\[
\mathbb{H}^2 = \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \}
\]
be the upper half plane model of the hyperbolic plane. The group $SL(2, \mathbb{R}) = \{ A \in M(2, \mathbb{R}) \mid \det A = 1 \}$ acts on $\mathbb{H}^2$ in the following way: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$. Then

$$f_A : \mathbb{H}^2 \to \mathbb{H}^2, \quad f_A(z) = \frac{az + b}{cz + d}.$$  

(a) Show that the maps $f_A : \mathbb{H}^2 \to \mathbb{H}^2$ are isometries of the upper half plane model. **Hint:** First prove

$$\text{Im}(f_A(z)) = \frac{\text{Im}(z)}{|cz + d|^2}.$$  

(b) Show that $\{f_A(i) \mid A \in SL(2, \mathbb{R})\} = \mathbb{H}^2$. **Hint:** Calculate $f_A(z)$ for $A = \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}$ and for $A = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$.

(c) Show that $\{A \in SL(2, \mathbb{R}) \mid f_A(i) = i\} = SO(2)$.

**Exercise 12.** An almost global coordinate chart of the torus of revolution $T^2 \subset \mathbb{R}^3$ is given by $\varphi : U \to V = (0, 2\pi) \times (0, 2\pi) \subset \mathbb{R}^2$,

$$\varphi^{-1}(x_1, x_2) = ((R + r \cos x_1) \cos x_2, (R + r \cos x_1) \sin x_2, r \sin x_1),$$

with $r > 0$ and $R > r$. Explain the precise geometric shape which is rotated around the vertical axis in $\mathbb{R}^3$ in order to obtain $T^2$. Show that

$$\text{vol}(T^2) = (2\pi r)(2\pi R) = 4\pi^2 r R.$$

**Exercise 13.** We derive the formula

$$\text{vol}(\Delta P_1 P_2 P_3) = \pi - \alpha - \beta$$

for a triangle $\Delta P_1 P_2 P_3$ in the upper half space model $\mathbb{H}^2$ with an ideal vertex $P_3$ and interior angles $\alpha, \beta$. Look at the following picture and denote the “ideal” point at infinity by $P_3$. 

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Show that

\[
\text{vol}(\Delta P_1 P_2 P_3) = \arcsin\left(\frac{x_0 - x_1}{r}\right) + \arcsin\left(\frac{x_2 - x_0}{r}\right).
\]

(You may use \( (\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \).) Conclude from this with \( \arcsin(x) + \arccos(x) = \pi/2 \) that

\[
\text{vol}(\Delta P_1 P_2 P_3) = \pi - \alpha - \beta.
\]