Riemannian Geometry IV

Problems, set 18.

Exercise 44. (See also Example 38) For \( r > 0 \), let \( S^2_r := \{ x \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 = r^2 \} \) and \( X : [-\pi/2, \pi/2] \to TS^2_r \) be a vector field along \( c : [-\pi/2, \pi/2] \to S^2_r \) with \( c(t) = (r \cos t, 0, r \sin t) \), defined by

\[
X(t) := (0, \cos t, 0).
\]

Let \( \frac{D}{dt} \) denote covariant derivative on \( S^2_r \) along \( c \).

(a) Calculate \( \frac{D}{dt} X(t) \) and \( \frac{D^2}{dt^2} X(t) \), using the formula for the induced covariant derivative for a surface in \( \mathbb{R}^3 \) (see Example 22).

(b) Show that \( X \) satisfies the Jacobi equation (using the results of Exercise 37).

Exercise 45. (Jacobi fields on manifold of constant curvature) Let \( M \) be a Riemannian manifold of constant sectional curvature \( K \), and \( c : [0, l] \to M \) be a geodesic satisfying \( \| c' \| = 1 \). Let \( J : [0, l] \to TM \) be a orthogonal Jacobi field along \( c \).

(a) Using Proposition 6.4, show that \( R(J, c')c' = KJ \).

(b) Let \( Z_1, Z_2 : [0, l] \to M \) be parallel vector fields along \( c \) with \( Z_1(0) = J(0), \ Z_2(0) = \frac{DJ}{dt}(0) \). Show that

\[
J(t) = \begin{cases} 
\cos(t\sqrt{K})Z_1(t) + \frac{\sin(t\sqrt{K})}{\sqrt{K}}Z_2(t) & \text{if } K > 0, \\
Z_1(t) + tZ_2(t) & \text{if } K = 0, \\
\cosh(t\sqrt{-K})Z_1(t) + \frac{\sinh(t\sqrt{-K})}{\sqrt{-K}}Z_2(t) & \text{if } K < 0.
\end{cases}
\]

Exercise 46. Let \( M \) be a Riemannian manifold with non-positive sectional curvatures.
(a) Let \( c : [a, b] \to M \) be a differentiable curve and \( J \) be a Jacobi field along \( c \). Let \( f(t) = \|J(t)\|^2 \). Show that \( f''(t) \geq 0 \), i.e., \( f \) is a convex function.

(b) Derive from (a) that \( M \) does not admit conjugate points.

**Exercise 47.** (Jacobi fields and conjugate points on locally symmetric spaces) A Riemannian manifold \((M, g)\) is called a *locally symmetric space* if \( \nabla R = 0 \). Let \((M, g)\) be an \( n \)-dimensional locally symmetric space and \( c : [0, \infty) \to M \) be a geodesic with \( p = c(0) \) and \( v = c'(0) \in T_p M \). Prove the following facts:

(a) Let \( X, Y, Z \) be parallel vector fields along \( c \). Show that \( R(X, Y)Z \) is also parallel.

(b) Let \( K_v : T_p M \to T_p M \) be the curvature operator, defined by
\[
K_v(w) = R(w, v)v.
\]
Show that \( K_v \) is symmetric, i.e.,
\[
\langle K_v(w_1), w_2 \rangle = \langle w_1, K_v(w_2) \rangle,
\]
for every pair of vectors \( w_1, w_2 \in T_p M \).

(c) Choose an orthonormal basis \( w_1, \ldots, w_n \in T_p M \) that diagonalises \( K_v \), i.e.,
\[
K_v(w_i) = \lambda_i w_i.
\]
Let \( W_1, \ldots, W_n \) be the parallel vector fields along \( c \) with \( W_i(0) = w_i \). Show that, for all \( t \in [0, \infty) \),
\[
K_{c'(t)}(W_i(t)) = \lambda_i W_i(t).
\]

(d) Let \( J(t) = \sum_i J_i(t)W_i(t) \) be a Jacobi field along \( c \). Show that Jacobi’s equation translates into
\[
J''_i(t) + \lambda_i J_i(t) = 0, \quad \text{for } i = 1, \ldots, n.
\]

(e) Show that the conjugate points of \( p \) along \( c \) are given by \( c(\pi k/\sqrt{\lambda}_i) \), where \( k \) is any positive integer and \( \lambda_i \) is a positive eigenvalue of \( K_v \).