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Riemannian Geometry IV

Problems, set 18.

Exercise 44. (See also Example 38) For $r > 0$, let $S^2_r := \{x \in \mathbb{R}^3 \mid {x_1}^2 + {x_2}^2 + {x_3}^2 = r^2\}$ and $X : [-\pi/2, \pi/2] \to TS^2_R$ be a vector field along $c : [-\pi/2, \pi/2] \to S^2_r$ with $c(t) = (r \cos t, 0, r \sin t)$, defined by

$$X(t) := (0, \cos t, 0).$$

Let $\frac{D}{dt}$ denote covariant derivative on S^2_r along c.

(a) Calculate $\frac{D}{dt}X(t)$ and $\frac{D^2}{dt^2}X(t)$, using the formula for the induced covariant derivative for a surface in $\mathbb{R}^3$ (see Example 22).

(b) Show that X satisfies the Jacobi equation (using the results of Exercise 37).

Exercise 45. (Jacobi fields on manifold of constant curvature) Let M be a Riemannian manifold of constant sectional curvature K, and $c : [0, l] \to M$ be a geodesic satisfying $\|c'\| = 1$. Let $J : [0, l] \to TM$ be a orthogonal Jacobi field along c.

(a) Using Proposition 6.4, show that $R(J, c')c' = KJ$.

(b) Let $Z_1, Z_2 : [0, l] \to M$ be parallel vector fields along c with $Z_1(0) = J(0)$, $Z_2(0) = \frac{DJ}{dt}(0)$. Show that

$$J(t) = \begin{cases} \cos(t\sqrt{K})Z_1(t) + \frac{\sin(t\sqrt{K})}{\sqrt{K}}Z_2(t) & \text{if } K > 0, \\ Z_1(t) + tZ_2(t) & \text{if } K = 0, \\ \cosh(t\sqrt{-K})Z_1(t) + \frac{\sinh(t\sqrt{-K})}{\sqrt{-K}}Z_2(t) & \text{if } K < 0. \end{cases}$$

Exercise 46. Let M be a Riemannian manifold with non-positive sectional curvatures.

(a) Let $c : [a, b] \to M$ be a differentiable curve and J be a Jacobi field along c. Let $f(t) = \|J(t)\|^2$. Show that $f''(t) \geq 0$, i.e., f is a convex function.

(b) Derive from (a) that M does not admit conjugate points.

Exercise 47. (Jacobi fields and conjugate points on locally symmetric spaces) A Riemannian manifold (M, g) is called a *locally symmetric space* if $\nabla R = 0$. Let (M, g) be an n-dimensional locally symmetric space and $c : [0, \infty) \to M$ be a geodesic with $p = c(0)$ and $v = c'(0) \in T_pM$. Prove the following facts:

(a) Let X, Y, Z be parallel vector fields along c. Show that $R(X, Y)Z$ is also parallel.

(b) Let $K_v : T_pM \to T_pM$ be the curvature operator, defined by

$$K_v(w) = R(w, v)v.$$

Show that K_v is symmetric, i.e.,

$$\langle K_v(w_1), w_2 \rangle = \langle w_1, K_v(w_2) \rangle,$$

for every pair of vectors $w_1, w_2 \in T_pM$.

(c) Choose an orthonormal basis $w_1, \dots, w_n \in T_pM$ that diagonalises K_v, i.e.,

$$K_v(w_i) = \lambda_i w_i.$$

Let $W_1, \dots, W_n$ be the parallel vector fields along c with $W_i(0) = w_i$. Show that, for all $t \in [0, \infty)$,

$$K_{c'(t)}(W_i(t)) = \lambda_i W_i(t).$$

(d) Let $J(t) = \sum_i J_i(t) W_i(t)$ be a Jacobi field along c. Show that Jacobi's equation translates into

$$J_i''(t) + \lambda_i J_i(t) = 0, \quad \text{for } i = 1, \dots, n.$$

(e) Show that the conjugate points of p along c are given by $c(\pi k / \sqrt{\lambda_i})$, where k is any positive integer and λ_i is a positive eigenvalue of K_v.