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Riemannian Geometry IV

Problems, set 15.

Do **Exercise 36** as homework for this week. The cumulative homework over the coming weeks will be collected and marked in a few weeks time.

Exercise 36. On a Riemannian manifold (M, g) let

$$\begin{aligned} \langle \operatorname{grad} f(p), X(p) \rangle &= X(f)(p), \\ (\operatorname{div} X)(p) &= \operatorname{tr}(\nabla_{\bullet} X)(p) = \sum_{i=1}^{n} \langle \nabla_{e_i} X, e_i \rangle, \\ \Delta f(p) &= -\operatorname{div} \operatorname{grad} f(p), \end{aligned}$$

where $f \in C^\infty(M)$, X is a vector field on M, and $e_1, \dots, e_n \in T_pM$ is an arbitrary orthonormal basis. Δ is called the **Laplace-Beltrami operator** of the Riemannian manifold. Let $U \subset M$ be an open set containing p, and $E_1, \dots, E_n : U \to TM$ be an arbitrary orthonormal frame with $E_i(p) = e_i$. Show the following identities.

(a) We have $\operatorname{grad} f(p) = \sum_{i=1}^{n} e_i(f) e_i$.

(b) We have $\operatorname{div}(fX)(p) = \langle \operatorname{grad} f(p), X(p) \rangle + f(p) \operatorname{div} X(p)$.

(c) We have $\Delta f(p) = -\sum_{i=1}^{n} \big(\big(e_i(E_i(f)) - (\nabla_{e_i} E_i)(f) \big)$.

(d) We have $\Delta(fg) = f(\Delta g) + g(\Delta f) - 2\langle \operatorname{grad} f, \operatorname{grad} g \rangle$.

Let (M, g) be from now on a **compact** Riemannian manifold. We introduce the following inner products on $C^\infty(M)$ and $\mathcal{X}(M)$:

$$(f, g) = \int_M f(p) g(p) \, d\mathrm{vol}(p), \qquad (X, Y) = \int_M \langle X(p), Y(p_) \rangle \, d\mathrm{vol}(p).$$

Use (without proof) **Gauß' Divergence Formula**

$$\int_M \operatorname{div} X(p) \, d\mathrm{vol}(p) = 0,$$

to prove the following result:

$$(\Delta f, g) = (\mathrm{grad}\, f, \mathrm{grad}\, g) = (f, \Delta g).$$

Exercise 37. A coordinate chart of the sphere $S^2 \subset \mathbb{R}^3$ of radius $r > 0$ is given by

$$\varphi^{-1}(x_1, x_2) = (r \cos x_1 \cos x_2, r \cos x_1 \sin x_2, r \sin x_1).$$

(a) Calculate
$$\nabla_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_1}, \nabla_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_2}, \nabla_{\frac{\partial}{\partial x_2}} \frac{\partial}{\partial x_1}, \nabla_{\frac{\partial}{\partial x_2}} \frac{\partial}{\partial x_2}.$$

(b) Let R denote the Riemannian curvature tensor. Calculate $R(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})\frac{\partial}{\partial x_2}$.

(c) How are the Riemannian curvature tensor and the Gaussian curvature in the case of surfaces related? Conclude from (b) that the Gaussian curvature of S_r is equal to $\frac{1}{r^2}$.

Exercise 38. Let (M, g) be a Riemannian manifold and R be the curvature tensor, defined by

$$R(X, Y, Z, W) := \langle R(X, Y)Z, W \rangle.$$

Prove the **Second Bianchi Identity**:

$$\nabla R(X, Y, Z, W, T) + \nabla R(X, Y, W, T, Z) + R(X, Y, T, Z, W) = 0,$$

for X, Y, Z, W, T vector fields on M.

Hint: Use the orthonormal frame $E_1, \ldots, E_n : B_\epsilon(p) \to TM$ introduced in Exercise 35 (then you know that $\nabla_{E_i} E_j(p) = 0$, which simplifies calculations considerably). For simplicity, let $e_i := E_i(p)$ and $E_{ij} := [E_i, E_j]$. Recall the definition of covariant derivative for tensors (see Exercise 19). Show first that

$$\nabla R(e_i, e_j, e_k, e_l, e_m) = \langle \nabla_{e_m} \nabla_{E_k} \nabla_{E_l} E_i - \nabla_{e_m} \nabla_{E_l} \nabla_{E_k} E_i - \nabla_{e_m} \nabla_{E_{kl}} E_i, e_j \rangle.$$

Using this and the Riemannian curvature tensor, derive

$$\nabla R(e_i, e_j, e_k, e_l, e_m) + \nabla R(e_i, e_j, e_l, e_m, e_k) + \nabla R(e_i, e_j, e_m, e_k, e_l) = \langle \nabla_{[E_{mk}, E_l] + [E_{kl}, E_m] + [E_{lm}, E_k]} E_i, e_j \rangle,$$

which implies the desired result, by Jacobi's identity and linearity.