Dr. Norbert Peyerimhoff, Durham University 11/10/2010

Riemannian Geometry IV

Problems, set 1.

Exercise 1. Let M be a differentiable manifold of dimension m and N be a differentiable manifold of dimension n. Show that the cartesian product

$$M \times N := \{(x,y) \mid x \in M, y \in N\}$$

is a differentiable manifold of dimension $m+n$.

Exercise 2. Let $R > r > 0$ and

$$M := \left\{ (x_1,x_2,x_3) \in \mathbb{R}^3 \mid \left(\sqrt{x_1^2+x_2^2} - R \right)^2 + x_3^2 = r^2 \right\}.$$

Show that the set M is the preimage of a regular value and, therefore, a 2-dimensional differentiable manifold. Since this set can be generated by a rotation of the circle $\{x \in \mathbb{R}^3 \mid (x_1-R)^2+x_3^2 = r^2, x_2 = 0\}$ in the x_1, x_3-plane around the x_3-axis, it is diffeomorphic to the cartesion product $S^1 \times S^1$, where $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$ is the unit circle. Give this diffeomorphism $\Phi : S^1 \times S^1 \to M$ explicitly.

Exercise 3. This exercise shows that the matrix group $SL(n,\mathbb{R}) = \{A \in M(n,\mathbb{R}) \mid \det A = 1\}$ is a differentiable manifold, using Theorem 1.5.

(a) Let $f : \mathbb{R}^k \to \mathbb{R}$ be a homogeneous polynomial of degree $m \geq 1$. Prove *Euler's relation*

$$\langle \operatorname{grad} f(x), x \rangle = m f(x),$$

where

$$\operatorname{grad} f(x) = \left(\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), \ldots, \frac{\partial f}{\partial x_k}(x) \right).$$

Hint: Differentiate $\lambda \mapsto f(\lambda x_1, \lambda x_2, \ldots, \lambda x_k)$ with respect to λ and use homogeneity.

(b) Let $f : \mathbb{R}^k \to \mathbb{R}$ be a homogeneous polynomial of degree $m \geq 1$. Show that every value $y \neq 0$ is a regular value of f.

(c) Use the fact that $\det A$ is a homogeneous polynomial in the entries of A in order to show that $SL(n,\mathbb{R})$ is a differentiable manifold.