ERGODIC PROPERTIES OF ISOPERIMETRIC DOMAINS IN SPHERES

GERHARD KNIEPER AND NORBERT PEYERIMHOFF

ABSTRACT. Let φ be a function on the unit tangent bundle of a compact manifold of negative curvature. We show that averages of φ over subdomains of increasing spheres converge to the horospherical mean if these domains satisfy an isoperimetric condition. We apply this result to spherical means with continuous density and, by using relations between the horospherical mean and the Patterson-Sullivan measure, we derive some kind of mixing properties.

1. INTRODUCTION

Convergence properties of spherical means have been studied for many types of non-positively curved manifolds (see, e.g., [Gu, Ra, Fa, Kn1, Pe]). In this article we study convergence of more general means, namely, averages over subsets of spheres satisfying a particular isoperimetric property. Let us first introduce some basic notations.

Let M be a compact Riemannian manifold of strictly negative curvature and $\tilde{M}$ be its universal covering. Let SM and $S\tilde{M}$ denote the corresponding unit tangent bundles and $\pi : S\tilde{M} \to \tilde{M}$ be the canonical footpoint projection. The sphere of radius $r > 0$ about $p \in \tilde{M}$ is denoted by $S_r(p)$ and $S_r^{\pm}(p) \subset S\tilde{M}$ is the set of outward/inward unit normal vectors of $S_r(p)$. All submanifolds of $S_r(p)$ inherit naturally Riemannian metrics and volumes via the embedding $S_r(p) \subset \tilde{M}$ and so do all submanifolds of $S_r^{\pm}(p)$ via the diffeomorphisms $\pi : S_r^{\pm}(p) \to S_r(p)$. We denote the induced Riemannian volume on the manifolds $S_r(p)$ and $S_r^{\pm}(p)$ by λ and the Riemannian volume on M by vol.

Our main result reads as follows:

THEOREM 1. *Let (M, g) be a n-dimensional compact Riemannian manifold of negative curvature and $\tilde{M}$ be its universal covering. Let (p_j, r_j) be a sequence of points $p_j \in \tilde{M}$ and radii with $r_j \to \infty$. Let $U_j \subset S_{r_j}^{+}(p_j)$ be a sequence of measurable subsets satisfying the isoperimetric property*

$$\frac{\lambda(\partial_{\epsilon_0} U_j)}{\lambda(U_j)} \to 0, \tag{1}$$

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for some $\epsilon_0 > 0$ *where*

$$\partial_{\epsilon_0} U_j = \{x \in S^+_{r_j}(p_j) \mid d(x, \partial U_j) < \epsilon_0\}$$

and d *denotes the intrinsic distance in* $S^+_{r_j}(p_j)$*. Then we have for all* $\varphi \in C(SM)$*:*

$$\lim_{j\to\infty} \frac{1}{\lambda(U_j)} \int\limits_{U_j} \tilde{\varphi}\, d\lambda = \int\limits_{SM} \varphi\, d\mu_{H^+}, \tag{2}$$

where $\tilde{\varphi} \in C(S\tilde{M})$ *is the periodic lift of* φ *and* μ_{H^+} *is the horospherical measure introduced in* [Kn1]*.*

(Note that the left hand side of (2) are spherical means on the unit tangent bundle if we choose $U_j = S^+_{r_j}(p_j)$.)

The same result holds for subsets of $S^-_{r_j}(p_j)$ if we replace the probability measure μ_{H^+} by μ_{H^-} in (2). The precise definition of the measures $\mu_{H^\pm}$ together with the proof of Theorem 1 is given in Section 2. An analogous result holds for isoperimetric subdomains of horospheres (see Theorem 11 in Section 3).

The remaining sections discuss applications of this formula. Let $\phi^t : S\tilde{M} \to S\tilde{M}$ denote the geodesic flow. In Section 4 we consider the case when the subdomains U_r are generated from a set $A \subset S_p\tilde{M}$ by the geodesic flow, i.e., $U_r := \phi^r(A) \subset S^+_r(p)$.

In Section 5 we introduce spherical means with a continuous density $f \in C(S_p\tilde{M})$. Let us explain this in more detail: Let $\varphi \in C(SM)$ and $\tilde{\varphi} \in C(S\tilde{M})$ be its periodic lift. Then the spherical mean $M_{r,f}(\varphi)$ with density f is defined by

$$M_{r,f}(\varphi) := \frac{1}{\lambda_f(S^+_r(p))} \int_{S^+_r(p)} \tilde{\varphi}(v) f(\phi^{-r}(v))\, d\lambda(v),$$

where

$$\lambda_f(S^+_r(p)) := \int_{S^+_r(p)} f(\phi^{-r}(v))\, d\lambda(v).$$

Corollary 18 in Section 5 states that spherical means with continuous density converge to the horospherical mean. This generalizes Theorem 3.2 of [Kn1]. (Note that Theorem 3.2 deals with the case $f \equiv 1$.)

Finally, the following kind of mixing property is proved in Section 6. Recall that Margulis' asymptotic function is defined as

$$a(p) := \lim_{r\to\infty} \frac{\lambda(S_r(p))}{e^{hr}},$$

where h denotes the topological entropy of the geodesic flow on SM.

THEOREM 2. *Let* (M, g) *be a compact Riemannian manifold of negative curvature and* $a(p)$ *denote Margulis' asymptotic function. Then we*

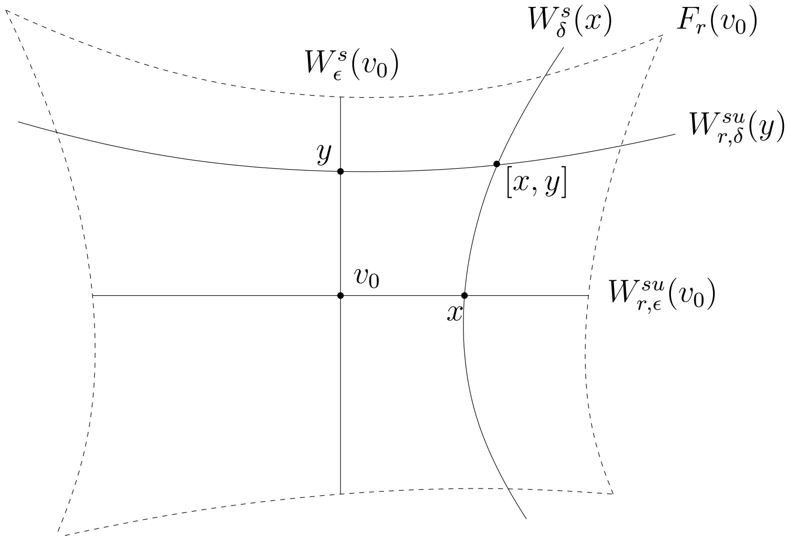

FIGURE 1. The foliation box $F_r(v_0)$

have, for every pair of continuous functions $\varphi, \psi \in C(SM)$:

$$\lim_{r\to\infty} \frac{1}{\int_M a\, dvol} \int_M a(p) \int_{S_pM} \varphi(\phi^r(v))\psi(v)\, d\theta^+_{r,p}(v)\, dvol(p) = \int_{SM} \varphi\, d\mu_{H^+} \int_{SM} \psi\, d\mu_{H^-},$$

where $\theta^+_{r,p}$ denotes the pullback of the normalized Riemannian measure on $S^+_r(p)$ via $\phi^r : S_pM \to S^+_r(p)$.

In fact, Theorem 2 states the usual mixing property if M is a locally compact symmmetric space of rank one. In this case $d\theta^+_{r,p}$ is equal to the normalized measure $d\theta_p$ on S_pM induced by the Riemannian metric and the measures $\frac{1}{\mathrm{vol}(M)}\, d\theta_p\, dvol, d\mu_{H^+}, d\mu_{H^-}$ coincide with the Liouville measure. Furthermore, by homogeneity, the Margulis function a is constant.

In due course we also explain analogues and differences to the case of a flat manifold M.

2. Isoperimetric domains in spheres

In this section we prove Theorem 1. We start with basic notations about foliations. W^s denotes the weak stable foliation of the geodesic flow on SM and $W^s(v)$ denotes the leaf through v. For fixed $r > 0$, the projection of the spheres $S^+_r(p)$ determines a spherical foliation of SM denoted by W^{su}_r. Since the spherical foliation converges uniformly to the strong unstable foliation, we extend this definition to $r = \infty$ and define W^{su}_∞ to be the strong unstable foliation. Let $W^s_\epsilon(v)$, respectively, $W^{su}_{r,\epsilon}(v)$ denote the ϵ-neighborhood of v in the leaf $W^s(v)$. (The leaves can be identified with horospheres and the distance is induced by the Riemannian metric.) For $\epsilon > 0$ small enough and $r \in (0, \infty]$ large enough, we can find a $\delta > 0$ such that we have the following intersection properties for all $v_0 \in SM$:

(i) For all $x \in W^{su}_{r,\epsilon}(v_0)$ and $y \in W^s_\epsilon(v_0)$, $W^s_\delta(x) \cap W^{su}_{r,\delta}(y)$ is a single point denoted by $[x, y]$ (see Figure 1).
(ii) For all $y_1, y_2 \in W^s_\epsilon(v_0)$, $y_1 \neq y_2$, we have $W^{su}_{r,\delta}(y_1) \cap W^{su}_{r,\delta}(y_2) = \emptyset$.

The corresponding foliation box $F_r(v_0) \subset SM$ is then defined by

$$F_r(v_0) := \{[x, y] \mid (x, y) \in W^{su}_{r,\epsilon}(v_0) \times W^s_\epsilon(v_0)\}.$$

Note that the ϵ-neighborhoods $W^s_\epsilon(v)$ are natural transversals of the strong unstable foliation.

Margulis [Ma] introduced a family of measures μ^s_v on the leaves $W^s(v)$ of the weak stable foliations which are invariant under the holonomy maps of the transversals $W^s_\epsilon(v)$ (see also [Kn1]). The horospherical measure μ_{H^+} is a probability measure which is locally defined by

$$\mu_{H^+}(U) := \int_{W^s_\epsilon(v_0)} \lambda^{su}_y(U \cap [W^{su}_\epsilon(v_0), y]) d\mu^s_{v_0}(y),$$

for all $U \subset F_\infty(v_0)$, where λ^{su}_y denotes the measure on $W^{su}(y)$ induced by lifting the Riemannian metric on the projected horosphere onto $W^{su}(y)$. Analogously, the probability measure μ_{H^-} is locally defined by

$$\mu_{H^-}(U) := \int_{W^u_\epsilon(v_0)} \lambda^{ss}_y(U \cap [W^{ss}_\epsilon(v_0), y]) d\mu^u_{v_0}(y),$$

where the roles of stable and unstable are interchanged. Here W^u denotes the weak unstable foliation and W^{ss} the strong stable foliation. Furthermore, μ^u is the Margulis measure on W^u and λ^{ss} the Riemannian volume on W^{ss}.

Examples It is known for flat spaces that spherical means on the unit tangent bundle converge to the (normalized) Liouville measure μ_L (see [Pe, Section 2]). Therefore, the natural measure replacing μ_{H^+} in (2) in the flat case would be the Liouville measure. However, the following counterexamples show that the stronger statement for isoperimetric subdomains (Theorem 1) does not hold in the flat case: Let $M = \mathbb{R}^n/\mathbb{Z}^n$, $n \geq 2$, and $\mathrm{pr} : \mathbb{R}^n \to M$ denote the canonical projection of the unit tangent bundles.

a) There are canonical identifications $S\mathbb{R}^n \cong \mathbb{R}^n \times S^{n-1}$ and $SM \cong M \times S^{n-1}$ such that the following diagram is commutative:

$$\begin{array}{ccc} S\mathbb{R}^n & \longrightarrow & \mathbb{R}^n \times S^{n-1} \\ {\scriptstyle D\mathrm{pr}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}\times\mathrm{id}} \\ SM & \longrightarrow & M \times S^{n-1} \end{array}$$

Let $\pi_2 : SM \to S^{n-1}$ be the associated projection. Choose an open subset $U \subset S^{n-1}$ with smooth boundary and $0 < \lambda(U) < \lambda(S^{n-1})$, where λ is the canonical Lebesgue measure on S^{n-1}. Let

$$U_r := \{v \in S_r^+(p) \mid \pi_2(v) \in U\}.$$

Note that U_r is the image of the set $\{p\} \times U \subset S_p\mathbb{R}^n$ under the geodesic flow Φ^r, and that U_r obviously satisfies (1) for all $\epsilon_0 > 0$. Let $A := S^{n-1}\backslash U$ and $\varphi \in C(SM)$ be a non-negative, non-vanishing function with $\operatorname{supp}\varphi \subset M \times A \subset SM$. Then the left hand side of (2) vanishes whereas the right hand side is obviously strictly positive.

b) An analogue of Theorem 1 cannot hold in the flat case even if we restrict ourselves to functions $\varphi \in C(SM)$ which are constant in the fibers: Choose $0 < \epsilon < 1$ and let

$$U_r := \{v \in S_r^+(re_n) \mid \pi(v) \in \mathbb{R}^{n-1} \times [0, \epsilon]\},$$

where $e_n \in \mathbb{R}^n$ is the n-th standard basis vector and $\pi : S\mathbb{R}^n \to \mathbb{R}^n$ is the footpoint projection. Obviously, U_r satisfies the isoperimetric property (1), as $r \to \infty$, but the sets $\operatorname{pr}(\pi U_r)$ don't even become dense in M. Consequently, (2) cannot be satisfied even for functions $\varphi \in C(SM)$ which are the (constant) lift of a function $\varphi_0 \in C(M)$ into the unit tangent bundle.

Let $V_j \subset SM$ denote the projection of the sequence $U_j \subset S\tilde{M}$ of Theorem 1. Let T be a transversal in $F_r(v)$, i.e., an open subset of $[x, W_\epsilon^s(v)]$ with $x \in W_{r,\epsilon}^{su}(v)$. We define on T the counting measures

$$\mu_j^T(A) = \frac{\operatorname{card}(A \cap V_j)}{\lambda(V_j)}$$

where $A \subset T$ is a Borel set. By using the isoperimetric property (1), we show that any weak limit of these counting measures is an invariant transversal measure of the strong unstable foliation. This construction goes back in its roots to Plante [Pl]. Note that the $\epsilon_0 > 0$ occuring in the following lemmata was introduced in Theorem 1.

Lemma 3. *Assume that $\delta < \epsilon_0$. Then there exists a constant $d > 0$ and $j_0 \in \mathbb{N}$ such that we have, for every $v \in SM$ and all transversals $T \subset F_r(v)$:*

$$\mu_j^T(T) \leq d \quad \textit{for all } j \geq j_0.$$

Proof. We write

$$\mu_j^T(T) = \frac{\operatorname{card}(T \cap V_j)}{\lambda(V_j \cup \partial_{\epsilon_0} V_j)} \frac{\lambda(V_j \cup \partial_{\epsilon_0} V_j)}{\lambda(V_j)}.$$

We have

$$\frac{\operatorname{card}(T \cap V_j)}{\lambda(V_j \cup \partial_{\epsilon_0} V_j)} \leq \frac{1}{\inf\{\lambda_{r_j,y}^{su}([W_{r_j,\epsilon}^{su}(v), y]) \mid y \in W_\epsilon^s(v)\}} =: C(r_j, v),$$

where $\lambda^{su}_{r_j,y}$ denotes the measure on $W^{su}_{r_j}(y)$ induced by lifting the Riemannian metric on the corresponding projected sphere onto $W^{su}_{r_j}(y)$.

Since spheres converge to horospheres, we can find a $d > 0$ such that we have, for all $v \in SM$ and j large enough: $C(r_j, v) \le d/2$. We also have

$$\frac{\lambda(V_j \cup \partial_{\epsilon_0} V_j)}{\lambda(V_j)} \le 1 + \frac{\lambda(\partial_{\epsilon_0} V_j)}{\lambda(V_j)} \le 2$$

for all j large enough. Therefore, we can find a $j_0 \in \mathbb{N}$ such that for all $j \ge j_0$:

$$\mu_j^T(T) \le d.$$

$\square$

Next, we fix a center $v \in SM$ of the foliation boxes $F_{r_j}(v)$ and consider the decomposition

$$\mu_j^T = \frac{1}{\lambda(V_j)} \sum_{v \in A_j} \delta_v + \frac{1}{\lambda(V_j)} \sum_{v \in B_j} \delta_v =: \nu_j^T + \beta_j^T,$$

where $A_j = \{[x,y] \in T : [W^{su}_{r_j,\epsilon}(v), y] \subset V_j\}$ and $B_j = (T \cap V_j) \setminus A_j$. ($A_j$ are the points in $T \cap V_j$ such that the corresponding plaques in $F_{r_j}(v)$ are completely contained in V_j.) Note that the measures ν_j^T and β_j^T depend on the center v of the foliation boxes, whereas μ_j^T only depends on the choice of T.

Lemma 4. *Assume that $3\delta < \epsilon_0$. Then every weak limit of β_j^T is 0.*

Proof. We recall that T is an open subset of a weak stable leaf, i.e., for each j there is an $x_j \in W^{su}_{r_j,\epsilon}(v)$ such that $T \subset [x_j, W^s_\epsilon(v)]$ (where the coordinate-brackets $[\cdot,\cdot]$ refer to the foliation box $F_{r_j}(v)$).

W.l.o.g, we may assume that the full sequence β_j^T converges weakly to β_∞^T. Since T is open,

$$\beta_\infty^T(T) \le \limsup_{j\to\infty} \beta_j^T(T).$$

If $y \in T \cap V_j$ and $[W^{su}_{r_j,\epsilon}(v), y] \not\subset V_j$ then there exists $z \in W^{su}_{r_j,\delta}(y) \cap \partial V_j$ and therefore

$$[x_j, y] \in \partial_{2\delta} V_j.$$

By choosing a new foliation box centered at $[x_j, y_1]$, we conclude from $[x_j, y_1], [x_j, y_2] \in T$, $y_1 \neq y_2$ that we have $W^{su}_{r_j,\delta}([x_j, y_1]) \cap W^{su}_{r_j,\delta}([x_j, y_2]) = \emptyset$. Therefore, we obtain

$$\mathrm{card}(B_j)\ \inf\{\lambda^{su}_{r_j,w}(W^{su}_{r_j,\delta}(w)) \mid w \in SM\} \le \lambda(\partial_{3\delta} V_j) \le \lambda(\partial_{\epsilon_0} V_j).$$

Obviously, there is a constant $C_0 > 0$ such that we have, for all $r > 0$ large enough

$$C_0 \le \inf\{\lambda^{su}_{r,w}(W^{su}_{r,\delta}(w)) \mid w \in SM\}.$$

Hence,

$$\mathrm{card}(B_j) \le \frac{\lambda(\partial_{\epsilon_0} V_j)}{C_0},$$

and

$$\beta_\infty^T(T) \le \limsup_{j\to\infty} \frac{1}{\lambda(V_j)}\,\mathrm{card}(B_j) \le \frac{1}{C_0}\lim_{j\to\infty}\frac{\lambda(\partial_{\epsilon_0} V_j)}{\lambda(V_j)} = 0.$$

□

As an immediate consequence, we obtain

Corollary 5. *The weak limits of μ_j^T and ν_j^T coincide.*

The next lemma states that the measures ν_j^T are holonomy invariant for holonomy maps within the same fixed foliation box $F_{r_j}(v)$.

Lemma 6. *Let T_1, T_2 be transversals in the foliation box $F_{r_j}(v)$ and $H_{12}^j : D(H_{12}^j) \to T_2$ be the associated holonomy map (determined by the plaques of $F_{r_j}(v)$). Then we have*

$$\nu_j^{T_2}(H_{12}^j(A)) = \nu_j^{T_1}(A)$$

for all Borel sets $A \subset D(H_{12}^j)$.

Proof. Let $T_1 \subset [x_1, W_\epsilon^s(v)]$ and $T_2 \subset [x_2, W_\epsilon^s(v)]$. Then for all $[x_1, y] \in D(H_{12}^j)$ we have $H_{12}^j([x_1,y]) = [x_2, y]$. Moreover,

$$\begin{aligned}
\nu_j^{T_1}(A) &= \frac{1}{\lambda(V_j)}\mathrm{card}(A \cap \{[x_1,y] \in T_1 \mid [W^{su}_{r_j,\epsilon}(v), y] \subset V_j\}) \\
&= \frac{1}{\lambda(V_j)}\mathrm{card}(H_{12}^j(A) \cap \{[x_2,y] \in T_2 \mid [W^{su}_{r_j,\epsilon}(v), y] \subset V_j\}) \\
&= \nu_j^{T_2}(H_{12}^j(A)).
\end{aligned}$$

□

Lemma 7. *Let T_1, T_2 be two transversals in $F_\infty(v)$ and $H_{12} : D(H_{12}) \to T_2$ be the holonomy map (determined by the plaques of the strong unstable foliation in $F_\infty(v)$). Let $\mu_\infty^{T_1}, \mu_\infty^{T_2}$ be weak limits of $\mu_j^{T_1}, \mu_j^{T_2}$ with respect to the same subsequence. Then*

$$\int_{T_1} f \circ H_{12}\, d\mu_\infty^{T_1} = \int_{T_2} f\, d\mu_\infty^{T_2}$$

for all $f \in C_c(T_2)$ with $\mathrm{supp}\, f \subset \overline{\mathrm{im}\, H_{12}}$.

Proof. W.l.o.g., we assume again that the full sequences $\mu_\infty^{T_j}$ converge weakly. Let H_{12}^j denote the corresponding holonomy maps of the foliations $W_{r_j}^{su}$. Uniform continuity of $f \in C_c(T_2)$ implies that

$$\| f \circ H_{12} - f \circ H_{12}^j \|_\infty \to 0.$$

By Corollary 5, the boundedness of the measures ν_j^T (see Lemma 3) and their invariance property (see Lemma 6) we conclude that

$$\begin{aligned}\int_{T_1} f \circ H_{12}\, d\mu_\infty^{T_1} &= \lim_{j\to\infty} \int_{T_1} f\circ H_{12}\, d\nu_j^{T_1} \\ &= \lim_{j\to\infty} \int_{T_1} f \circ H_{12}^j\, d\nu_j^{T_1} + \underbrace{\int_{T_1} f\circ H_{12} - f\circ H_{12}^j\, d\nu_j^{T_1}}_{\to 0} \\ &= \lim_{j\to\infty} \int_{T_2} f\, d\nu_j^{T_2} = \int_{T_2} f\, d\mu_\infty^{T_2}.\end{aligned}$$

□

Lemma 7 shows that each weak limit of the measures μ_j^T is an invariant transversal measure of the strong unstable foliation. Using [BoM] we conclude that this invariant transversal measure μ_∞^T is uniquely determined up to a constant factor and, therefore, agrees with the Margulis measures μ_v^s, restricted to the transversals $W_\epsilon^s(v)$ up to a constant factor. The possibility that a weak limit of the measures μ_j^T is zero is excluded by Lemma 9 below (see the comment after the proof of this lemma).

Next we define global measures μ_{U_j} on SM which approximate the averages over the sets U_j on the left hand side of (2). By choosing $\epsilon > 0$ small enough, we can assume that the foliated atlas given by the boxes $F_r(v)$ is regular, i.e., each plaque of $F_r(v_1)$ intersects at most one plaque of $F_r(v_2)$, for all large enough $r \in (0, \infty]$.

Definition 8. *Let j be large enough and $F_k = F_{r_j}(v_k)$, $k \in \{1, \dots, N\}$ be a regular cover of SM by finitely many foliation boxes and $T_k = W_\epsilon^s(v_k)$. Let $\phi_1, \dots, \phi_N$ be a partition of unity subordinated to the cover $SM = \bigcup_{k=1}^N F_k$. For every Borel set $A \subset SM$, we define*

$$\mu_{U_j}(A) := \sum_{k=1}^N \int_{T_k} \int_{W^{su}_{r_j,\delta}(y)} \phi_k(v)\chi_A(v)\, d\lambda^{su}_{r_j,y}(v)\, d\mu_j^{T_k}(y), \tag{3}$$

where χ_A is the characteristic function of A.

Lemma 9. *Assume that $2\delta < \epsilon_0$. Then there is a constant $C > 0$ and $j_1 \in \mathbb{N}$, such that we have, for all $j \geq j_1$:*

$$\left| \int_{SM} \varphi\, d\mu_{U_j} - \frac{1}{\lambda(U_j)} \int_{U_j} \tilde\varphi\, d\lambda \right| \leq C\, \frac{\lambda(\partial_{\epsilon_0} U_j)}{\lambda(U_j)}\, \|\varphi\|_\infty,$$

where φ is any continuous function on SM and $\tilde\varphi$ denotes its periodic lift onto $S\tilde M$.

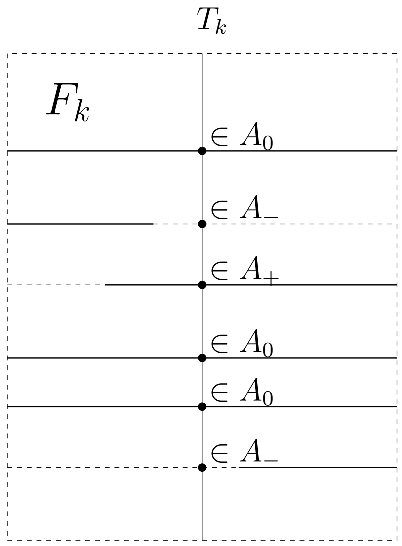

FIGURE 2. Illustation of the sets A_+, A_- and A_0

Proof. We have

$$\int_{SM} \varphi \, d\mu_{U_j} - \frac{1}{\lambda(U_j)} \int_{U_j} \tilde{\varphi} \, d\lambda =$$

$$\sum_{k=1}^{N} \left(\int_{T_k} \int_{W^{su}_{r_j,\delta}(y)} (\phi_k \varphi)(v) \, d\lambda^{su}_{r_j,y}(v) \, d\mu_j^{T_k}(y) - \frac{1}{\lambda(U_j)} \int_{U_j} \widetilde{\phi_k \varphi} \, d\lambda \right).$$

Therefore, it is enough to show that for all $g \in C(SM)$ with $\operatorname{supp} g \subset F_k$ we have for all j large enough

$$\left| \int_{T_k} \int_{W^{su}_{r_j,\delta}(y)} g(v) \, d\lambda^{su}_{r_j,y}(v) \, d\mu_j^{T_k}(y) - \frac{1}{\lambda(U_j)} \int_{U_j} \tilde{g} \, d\lambda \right| \leq C' \frac{\lambda(\partial_\epsilon V_j)}{\lambda(V_j)} \|g\|_\infty.$$

For later use, let us denote the left hand side of this inequality by $D_k(g)$.

Let $A = \{y \in T_k \mid [W^{su}_{r_j,\epsilon}(v_k), y] \cap V_j \neq \emptyset\}$. Note that A are those points of T_k whose plaques have non-empty intersection with the set V_j. Then we have

$$\int_{T_k} \int_{W^{su}_{r_j,\delta}(y)} g(v) \, d\lambda^{su}_{r_j,y}(v) \, d\mu_j^{T_k}(y) = \int_{A} \int_{W^{su}_{r_j,\delta}(y)} g(v) \, d\lambda^{su}_{r_j,y}(v) \, d\mu_j^{T_k}(y) \tag{4}$$

and

$$\frac{1}{\lambda(U_j)} \int\limits_{U_j} \tilde{g}\, d\lambda = \int\limits_{A} \int\limits_{W^{su}_{r_j,\delta}(y)} \chi_{V_j}(v) g(v)\, d\lambda^{su}_{r_j,y}(v)\, d\mu_j^{T_k}(y). \tag{5}$$

Next, we choose the following partition $A_+ \cup A_- \cup A_0$ of $A \subset T_k$, illustrated by Figure 2 (note that the thick horizontal lines in the figure are the components of V_j in the box F_k and the dashed horizontal lines are the plaques):

$$\begin{aligned} A_+ &:= \left\{ y \in A \mid y \in \left(V_j \cap [W^{su}_{r_j,\epsilon}(v_k), y] \right) \neq [W^{su}_{r_j,\epsilon}(v_k), y] \right\}, \\ A_- &:= \left\{ y \in A \mid y \notin \left(V_j \cap [W^{su}_{r_j,\epsilon}(v_k), y] \right) \neq [W^{su}_{r_j,\epsilon}(v_k), y] \right\}, \\ A_0 &:= A \setminus (A_+ \cup A_-). \end{aligned}$$

If $y \in A_+ \cup A_-$ we have $d(y, \partial V_j) < \delta$ in the plaque and, consequently, $y \in \partial_\delta V_j$. This implies that

$$W^{su}_{r_j,\delta}(y) \subset \partial_{2\delta} V_j \subset \partial_{\epsilon_0} V_j$$

and

$$\operatorname{card}(A_+ \cup A_-) \inf\{ \lambda^{su}_{r_j,w}(W^{su}_{r_j,\delta}(w)) \mid w \in SM \} \leq \lambda(\partial_{\epsilon_0} V_j).$$

Consequently, we have with the constant C_0 from the proof of Lemma 4

$$\operatorname{card}(A_+ \cup A_-) \leq \frac{\lambda(\partial_{\epsilon_0} V_j)}{C_0},$$

for all j large enough. From (4) and (5) we conclude that

$$\begin{aligned} D_k(g) &\leq \int_{A_+ \cup A_-} \int_{W^{su}_{r_j,\delta}(y)} |g(v)|\, d\lambda^{su}_{r_j,y}\, d\mu_j^{T_k}(y) \\ &\leq \frac{1}{\lambda(V_j)} \sum_{y \in A_+ \cup A_-} \|g\|_\infty \lambda^{su}_{r_j,y}(W^{su}_{r_j,\delta}(y)) \\ &\leq \frac{1}{C_0} \frac{\lambda(\partial_{\epsilon_0} V_j)}{\lambda(V_j)} \|g\|_\infty \sup\{ \lambda^{su}_{r_j,w}(W^{su}_{r_j,\delta}(w)) \mid w \in SM \}, \end{aligned}$$

finishing the proof of Lemma 9. □

By choosing $\varphi \equiv 1$, Lemma 9 together with the isoperimetric property (1) implies, in particular, that every weak limit of the measures μ_{U_j} is a probability measure. Consequently, no weak limit of the family of measures μ_j^T can be zero for all transversals T.

Lemma 10. *For all $\varphi \in C(SM)$ we have*

$$\lim_{j \to \infty} \int_{SM} \varphi\, d\mu_{U_j} = \sum_{k=1}^{N} \int\limits_{T_k} \int\limits_{W^{su}_\delta(y)} \phi_k(v) \varphi(v)\, d\lambda^{su}_y(v)\, d\mu^s_{v_k}(y) = \int\limits_{SM} \varphi d\mu_{H^+},$$

where μ_v^s are the Margulis measures on the weak stable foliation, normalized in such a way that μ_{H^+} is a probability measure on SM.

Proof. The first equality follows from (3), uniform convergence of the foliations $W_{r_j}^{su}$ to the strong unstable foliation and the fact that $\mu_j^{T_k} \rightharpoonup \mu_{v_k}^s$, as $j \to \infty$. For the proof of the second equation in Lemma 10 we assume that supp $\varphi \subset F_{k_0}$. Then

$$\sum_{k=1}^{N} \int\limits_{T_k} \int\limits_{W_\delta^{su}(y)} \phi_k(v)\varphi(v)\, d\lambda_y^{su}(v)\, d\mu_{v_k}^s(y)$$
$$= \sum_{k:F_k \cap F_{k_0} \neq \emptyset} \int\limits_{T_k} \int\limits_{W_\delta^{su}(y)} \phi_k(v)\varphi(v)\, d\lambda_y^{su}(v)\, d\mu_{v_k}^s(y).$$

Since the cover $\{F_1, \dots, F_N\}$ is regular, there is a unique holonomy map $H_{k_1k_2} : D(H_{k_1k_2}) \to T_{k_2}$ defined by the intersections of the plaques in the two boxes F_{k_1} and F_{k_2}. Therefore, the integral over T_k on the right hand side of the above equality can be replaced by an integral over $D(H_{kk_0})$, and using holonomy invariance of the Margulis measures, we obtain

$$\sum_{k:F_k \cap F_{k_0} \neq \emptyset} \int\limits_{T_{k_0}} \int\limits_{W_\delta^{su}(y)} \phi_k(v)\varphi(v)\, d\lambda_y^{su}(v)\, d\mu_{v_{k_0}}^s(y) =$$
$$\int\limits_{T_{k_0}} \int\limits_{W_\delta^{su}(y)} \varphi(v)\, d\lambda_y^{su}(v)\, d\mu_{v_{k_0}}^s(y) = \int_{SM} \varphi\, d\mu_{H^+}.$$

This proves the second equation in Lemma 10. □

Lemmas 9 and 10 together with the isoperimetric property (1) immediately imply Theorem 1.

3. Horospherical means

Following the same arguments as in the previous section, we obtain an analogue of Theorem 1 for isoperimetric domains in horospheres. It is easily checked that Lemmata 3 - 10 also hold true for the choice $r_1 = r_2 = r_3 = \cdots = \infty$. In fact, many arguments are even simpler since we only have to deal with the strong unstable foliation W^{su} and do not need the uniform convergence of a sequence of foliations.

The result reads as follows:

Theorem 11. *Let (M, g) be a n-dimensional compact Riemannian manifold of negative curvature and $\tilde{M}$ be its universal covering. Let $H_v \subset \tilde{M}$ denote the horosphere centered at the geodesic ray in direction v. Let $H_v^{\pm}$ be the set of outward/inward unit normal vectors of H_v and*

λ the induced Riemannian volume on $H_v^{\pm}$. Let $U_j \subset H_{v_j}^+$ be a sequence of subsets satisfying the isoperimetric property

$$\frac{\lambda(\partial_{\epsilon_0} U_j)}{\lambda(U_j)} \to 0,$$

for some $\epsilon_0 > 0$. Then we have for all $\varphi \in C(SM)$:

$$\lim_{j\to\infty} \frac{1}{\lambda(U_j)} \int\limits_{U_j} \tilde{\varphi}\, d\lambda = \int\limits_{SM} \varphi\, d\mu_{H^+}. \tag{6}$$

An analogous statement holds for subsets $U_j \subset H_{v_j}^-$, where the limit measure μ_{H^+} in (6) *is replaced by μ_{H^-}.*

Remark The equidistribution result (6) in the case when all subsets U_j lie in the same horosphere H_v^+ coincides with the case $f = 0$ in Theorem 4.6 of [Sha].

The next proposition together with the following remark show that a particular choice of metric balls in horospheres fullfil the isoperimetric property needed in Theorem 11.

Proposition 12. *Let $\tilde{M}$ be a n-dimensional Hadamard manifold such that $-k_2^2 \le K \le -k_1^2 < 0$ where K denotes the sectional curvature and $0 < k_1 \le k_2$. Let $H \subset \tilde{M}$ be a horosphere and $B_r^H(p)$ be the ball of radius $r > 0$ about p in H with respect to the induced metric on H and* vol *denote the induced Riemannian volume. Then we have for all sufficiently large radii $r > 0$:*

$$\mathrm{vol}(B_r^H(p)) \le (4r)^{(n-1)\frac{k_2}{k_1}} \alpha,$$

where α is a constant larger than the volume of all balls of radius one in all horospheres of $\tilde{M}$.

Proof. Denote by $\pi : S\tilde{M} \to \tilde{M}$ the canonical projection and let $W^{su}(v)$ be the strong unstable manifold such that $\pi(W^{su}(v)) = H$. Then the restriction $\pi_0 : W^u(v) \to \tilde{M}$ of π is a diffeomorphism of the weak unstable manifold $W^u(v) = \bigcup_{t\in\mathbb{R}} W^{su}(\phi^t(v))$ onto $\tilde{M}$, where ϕ^t denotes the geodesic flow. We define $\varphi^t = \pi\phi^t\pi_0^{-1}$ the induced flow on $\tilde{M}$. Then $H_t = \varphi^t H$ defines a family of equidistant horospheres. If A is a measurable subset of H_{-t} of finite volume, then

$$\mathrm{vol}(\varphi^t A) \le e^{(n-1)k_2 t}\, \mathrm{vol}(A).$$

Furthermore, for $A = B_1^{H_{-t}}(\varphi^{-t}p) \subset H_{-t}$ we have for large enough $t > 0$:

$$B^H_{e^{k_1 t}/4}(p) \subset \varphi^t(A).$$

Hence,

$$\mathrm{vol}(B^H_{e^{k_1 t}/4}(p)) \le e^{(n-1)k_2 t}\alpha.$$

In particular for $r = e^{k_1 t}/4$, we obtain

$$\operatorname{vol}(B_r^H(p)) \leq (4r)^{(n-1)\frac{k_2}{k_1}} \alpha,$$

finishing the proof of the proposition. □

Corollary 13. *There exists a sequence of radii $R_j \to \infty$ with*

$$\frac{\operatorname{area} \partial B_{R_j}^H(p)}{\operatorname{vol} B_{R_j}^H(p)} \to 0$$

as $j \to \infty$.

Proof. Since $\int^R \operatorname{area} \partial B_r^H(p)\, dr = \operatorname{vol} B_R^H(p)$ we have

$$\frac{d}{dr} \operatorname{vol} B_r^H(p) = \operatorname{area} \partial B_r^H(p),$$

and therefore

$$\int^R \frac{\operatorname{area} \partial B_r^H(p)}{\operatorname{vol} B_r^H(p)} = \int^R \frac{d}{dr} \log \operatorname{vol} B_r^H(p) dr = \log \operatorname{vol} B_R^H(p). \tag{7}$$

Subexponential volume growth (Proposition 12) and (7) imply that there cannot be constants $C, R_0 > 0$ with

$$\frac{\operatorname{area} \partial B_r^H(p)}{\operatorname{vol} B_r^H(p)} \geq C > 0 \quad \text{for all } r \geq R_0,$$

finishing the proof. □

Remark Similarly, Proposition 12 implies that, for every $\epsilon_0 > 0$, there is a sequence $R_j \to \infty$ satisfying

$$\frac{\operatorname{vol} \partial_{\epsilon_0} B_{R_j}^H(p)}{\operatorname{vol} B_{R_j}^H(p)} \to 0.$$

Remark Note that there are particular horospheres in Hadamard manifolds which do not admit subdomains U_j with the isoperimetric property (1): Let $\tilde{M}$ be a symmetric space of non-compact type and G be the connected component of the identity of the isometry group and M be a compact quotient. For every $v \in \tilde{M}$ let H_v denote the horosphere centered at the geodesic ray in direction v and

$$(SM)_v := \operatorname{pr}(Gv) \subset SM,$$

where $\operatorname{pr} : S\tilde{M} \to SM$ is the canonical projection. Then we have (see [Pe, PS])

$$h_{top}^2(\phi^t) = h(H_v)^2 + h_{top}^2(\phi^t|_{(SM)_v}),$$

where $\phi^t : SM \to SM$ is the geodesic flow, h_{top} denotes the topological entropy and h denotes Cheeger's isoperimetric constant. Therefore, horospheres which are not orthogonal to a barycentric direction do not admit isoperimetric subdomains, in the higher rank case (since in this case we have $h_{top}(\phi^t|_{(SM)_v}) < h_{top}(\phi^t)$).

4. Spherical domains generated by the geodesic flow

In this section we apply Theorem 1 to particular subsets of the form $U_r := \phi^r(A)$, where $A \subset S_p\tilde{M}$ and $\phi^t : S\tilde{M} \to S\tilde{M}$ is the geodesic flow. Therefore it is necessary to know in which cases the sets U_r satisfy the isoperimetric condition (1). We study this condition in the more general situation of a non-positively curved rank one manifold.

Let us first fix some notation. $\phi^t : S\tilde{M} \to S\tilde{M} \cup \tilde{M}(\infty)$, $t \in \mathbb{R} \cup \{\pm\infty\}$ denotes the extended geodesic flow, defined by

$$\phi^t(v) = \begin{cases} \dot{c}_v(t) & \text{if } t \in \mathbb{R}, \\ c_v(\pm\infty) \in \tilde{M}(\infty) & \text{if } t = \pm\infty. \end{cases}$$

The Patterson-Sullivan measure on $\tilde{M}(\infty)$ with respect to $p \in \tilde{M}$ is denoted by μ_p. (For details about this measure see, e.g., [Kn1, Kn2]). Our main result reads as follows:

Proposition 14. *Let M be a n-dimensional, compact, non-positively curved rank one manifold and $\tilde{M}$ be its universal covering. Let $A \subset S_p\tilde{M}$ be a Borel set with $\mu_p(\phi^\infty(A)) \neq 0$ and*

$$\mu_p(\phi^\infty(\partial A)) = 0.$$

Then we have for all $\epsilon_0 > 0$:

$$\lim_{r\to\infty} \frac{\lambda(\partial_{\epsilon_0}\phi^r(A))}{\lambda(\phi^r(A))} = 0.$$

Remark The condition $\mu_p(\phi^\infty(A)) \neq 0$ in Proposition 14 is, e.g., satisfied if A has non-empty interior. This follows from Lemma 4.1 in [Kn2] and the fact that $\phi^\infty : S_p\tilde{M} \to \tilde{M}(\infty)$ is a homeomorphism.

Proof. To simplify notations in this proof, we identify the sets $\phi^r(A)$, $\partial_{\epsilon_0}\phi^r(A) \subset S_r(p)^+$ with their footpoint projections in $S_r(p)$. To avoid misunderstandings, we sometimes use λ_{n-1} (instead of the short version λ) to denote the induced Riemannian measure on $S_r(p) \subset \tilde{M}$.

Let

$$\mathrm{pr}_p : \tilde{M} \setminus \{p\} \to \tilde{M}(\infty)$$

be the projection onto $\tilde{M}(\infty)$ defined by

$$\mathrm{pr}_p(q) = c_{pq}(\infty),$$

where c_{pq} is the geodesic with $c_p(0) = p$, $c_p(d(p,q)) = q$. Let $B_r(x)$ denote the ball of radius $r > 0$ about x in $\tilde{M}$. Then there exists $R > 0$ and constants $C_\rho > 0$ for all $\rho \geq R$, such that

$$\frac{1}{C_\rho} e^{-hd(p,x)} \leq \mu_p(\mathrm{pr}_p(B_\rho(x) \cap S_{d(p,x)}(p))) \leq C_\rho e^{-hd(p,x)},$$

where h denotes the topological entropy of $\phi_r : SM \to SM$ (cf. Lemma 4.4 in [Kn2]).

Let $x_1, \dots, x_k \in \partial_{\epsilon_0}\phi_r(A)$ be a maximal R-separating set of $\partial_{\epsilon_0}\phi_r(A) \subset \tilde{M}$. This implies that

$$B_R(x_i) \cap B_R(x_j) = \emptyset \quad \text{for all } i \neq j,$$

and

$$\bigcup_{i=1}^{k} B_{3R}(x_i) \supset \partial_\epsilon \phi_r(A).$$

Let

$$f(r) := \mu_p(\mathrm{pr}_p(\bigcup_{i=1}^{k} B_R(x_i) \cap S_r(p))) = \sum_{i=1}^{k} \mu_p(\mathrm{pr}_p(B_R(x_i) \cap S_r(p))) \geq k\frac{e^{-hr}}{C_R}.$$

Therefore, we have $k \leq C_R f(r) e^{hr}$. Introducing

$$D_1 := \sup\{\lambda_{n-1}(B_{3R}(x) \cap S_r(p)) \mid r > 0, x \in S_r(p)\} < \infty,$$

we conclude that

$$\lambda(\partial_{\epsilon_0}\phi^r(A)) \leq \lambda_{n-1}(\bigcup_{i=1}^{k} B_{3R}(x_i) \cap S_r(p)) \leq D_1 k,$$

and we end up with

$$\lambda(\partial_{\epsilon_0}\phi^r(A)) \leq D_1 C_R f(r) e^{hr}. \tag{8}$$

Note that

$$f(r) \leq \mu_p(\partial_{\epsilon+R}\phi^r(A)) \to \mu_p(\phi^\infty(\partial A)) = 0. \tag{9}$$

Next, let $y_1, \dots, y_l \in \phi^r(A)$ be a maximal R-separating set of $\phi^r(A)$. Then

$$\mu_p(\mathrm{pr}_p(\bigcup_{i=1}^{l} B_{3R}(y_i) \cap S_r(p))) \leq \sum_{i=1}^{l} \mu_p(\mathrm{pr}_p(B_{3R}(y_i) \cap S_r(p))) \leq l C_{3R} e^{-hr},$$

and, consequently,

$$\mu_p(\phi^\infty(A)) \leq l C_{3R} e^{-hr}.$$

Since $\mu_p(\phi^\infty(A)) > 0$, this implies that

$$l \geq \frac{1}{C_{3R}} \mu_p(\phi^\infty(A)) e^{hr} =: \hat{C} e^{hr},$$

with $\hat{C} > 0$. Introducing

$$D_2 := \inf\{\lambda_{n-1}(B_R(x) \cap S_r(p)) \mid r \geq 1, x \in S_r(p)\} > 0,$$

we conclude that

$$\lambda(\phi^r(A)) \geq \sum_{i=1}^{l} \lambda_{n-1}(B_R(y_i) \cap S_r(p)) \geq D_2 l,$$

for all $r \geq 1$. Using this together with (8) and (9), we finally conclude that

$$\frac{\lambda(\partial_{\epsilon_0}\phi^r(A))}{\lambda(\phi^r(A))} \leq \frac{D_1 C_R e^{hr}}{D_2 \hat{C} e^{hr}} f(r) \to 0.$$

□

We conclude from Theorem 1 and Proposition 14:

Corollary 15. *Let M be a compact manifold of negative curvature, $\tilde{M}$ its universal covering and $A \subset S_p\tilde{M}$ be a subset with $\mu_p(\phi^\infty(A)) > 0$ and*

$$\mu_p(\phi^\infty(\partial A)) = 0. \tag{10}$$

Let $U_r = \phi^r(A) \subset S_r(p)^+$. Then we have

$$\lim_{r\to\infty} \frac{1}{\lambda(U_r)} \int_{U_r} \tilde{\varphi}\, d\lambda = \int_{SM} \varphi\, d\mu_{H^+},$$

for all $\varphi \in C(SM)$.

Assume that the boundary ∂A of the set $A \subset S_p\tilde{M}$ in Corollary 15 is a closed orientable hypersurface. The next result states that we can always satisfy the isoperimetric condition (10) via an arbitrarily small modification of A in terms of the Hausdorff distance d_H of $S_p\tilde{M}$.

Proposition 16. *For each continuous orientable closed hypersurface $S_1 \subset S_pM$ and every $\epsilon > 0$ there exists a closed smooth hypersurface $S_2 \subset S_pM$ with*

$$d_H(S_1, S_2) < \epsilon$$

and $\mu_p(\phi^\infty(S_2)) = 0$.

Proof. Let $S \subset S_pM$ be a closed smooth hypersurface satisfying

$$d_H(S_1, S) < \frac{\epsilon}{2},$$

and $N : S \to TS_pM$ be a global unit normal vector field of $S \subset S_pM$. Let ψ^t denote the geodesic flow on TS_pM and $S(\epsilon) \subset S_pM$ be the footpoint projection of the set $\psi^\epsilon(N(S)) \subset TS_pM$. For $\delta > 0$ small enough, the sets $S(\delta_1), S(\delta_2) \subset S_pM$ are, again, closed smooth hypersurfaces and disjoint for $0 \leq \delta_1 \leq \delta_2 \leq \delta$. Since $\mu_p(\tilde{M}(\infty)) < \infty$, we have

$$\mu_p(\phi^\infty(S(\delta_1))) = 0$$

for almost all $\delta_1 \in [0, \delta]$. Obviously, there exists a $\delta_1 \in [0, \delta]$ with

$$\mu_p(\phi^\infty(S(\delta_1))) = 0 \text{ and } d_H(S, S(\delta_1)) < \frac{\epsilon}{2}.$$

We choose $S_2 = S(\delta_1)$. □

REMARK Let $f : S_pM \to \mathbb{R}$ be a smooth function and $f^{-1}(c) \subset S_pM$ be a smooth hypersurface. It can be shown in a similar fashion that there are hypersurfaces $S_{c'} := f^{-1}(c') \subset S_pM$ satisfying

$$\mu_p(\phi^\infty(S_{c'})) = 0,$$

with c' arbitrarily close to c. This fact is important in the proof of Theorem 17 in the next section.

REMARK The condition (10) in Corollary 15 simplifies in the case when $\tilde{M}$ is a n-dimensional rank one symmetric space to

$$\theta_p(\partial A) = 0,$$

where θ_p is the normalized canonical Lebesgue measure of S_pM.

5. Spherical means with continuous density

The averaging result over isoperimetric subsets generated by the geodesic flow (Corollary 15) can be used to obtain a result about spherical means with respect to a continuous density f. It also provides a positive answer to a question which has been posed to the authors by Constantin Vernicos [Ve]. To state this result, let $\bar{\mu}_p$ denote the pullback of the Patterson-Sullivan measure μ_p via the homeomorphism $\phi^\infty : S_p\tilde{M} \to \tilde{M}(\infty)$.

THEOREM 17. *Let (M, g) be a compact Riemannian manifold of negative curvature and $\tilde{M}$ be its universal covering. Let $f \in C(S_p\tilde{M})$ and h be the topological entropy of the geodesic flow on SM. Then we have for all $\varphi \in C(SM)$:*

$$\lim_{r\to\infty} \frac{1}{e^{hr}} \int\limits_{S_r^+(p)} \tilde{\varphi}(v) f(\phi^{-r}(v))\, d\lambda(v) = \int\limits_{SM} \varphi \, d\mu_{H^+} \int\limits_{S_p\tilde{M}} f \, d\bar{\mu}_p.$$

Proof. We can assume, by uniform approximation, that $f \in C^\infty(S_pM)$. For a given $\epsilon > 0$ we choose regular values $c_0 < c_1 < \cdots < c_{N+1}$ of f with the following properties:

(i) $c_0 < -\|f\|_\infty$, $c_{N+1} > \|f\|_\infty$,
(ii) $c_{k+1} - c_k < \epsilon$ for all $k = 0, 1, \dots, N$,
(iii) $\mu_p(\phi^\infty(f^{-1}(c_k))) = 0$ for all $k = 0, 1, \dots, N+1$.

Let

$$f_\epsilon := \sum_{k=0}^{N} c_k \cdot \chi_{A_k},$$

with $A_k := f^{-1}([c_k, c_{k+1}))$. Note that

$$\partial A_k = f^{-1}(\{c_k\}) \cup f^{-1}(\{c_{k+1}\}),$$

and, consequently, $\mu_p(\phi^\infty(\partial A_k)) = 0$. We have $\|f - f_\epsilon\|_\infty < \epsilon$ by construction. Note that

$$\frac{1}{e^{hr}} \int\limits_{S_r^+(p)} \tilde\varphi(v) f_\epsilon(\phi^{-r}(v)\, d\lambda(v) = \sum_{k=0}^{N} \frac{c_k}{e^{hr}} \int\limits_{S_r^+(p)} \tilde\varphi(v) \chi_{A_k}(\phi^{-r}(v))\, d\lambda(v)$$

$$= \sum_{k=0}^{N} c_k \frac{\lambda(\phi^r(A_k))}{e^{hr}} \frac{1}{\lambda(\phi^r(A_k))} \int\limits_{\phi^r(A_k)} \tilde\varphi(v)\, d\lambda(v).$$

Using the following result about $\bar\mu_p$ in [Yue]

$$\int_{S_p\tilde M} F\, d\bar\mu_p = \lim_{r\to\infty} \frac{1}{e^{hr}} \int_{S_r^+(p)} F \circ \phi^{-r}\, d\lambda, \tag{11}$$

we conclude from $\mu_p(\partial\phi^\infty(A_k)) = 0$ that

$$\frac{\lambda(\phi^r(A_k))}{e^{hr}} \to \bar\mu_p(A_k).$$

This implies, together with Corollary 15,

$$\frac{1}{e^{hr}} \int\limits_{S_r^+(p)} \tilde\varphi(v) f_\epsilon(\phi^{-r}(v))\, d\lambda(v) \to \sum_{k=0}^{N} c_k\, \bar\mu_p(A_k) \cdot \int\limits_{SM} \varphi d\mu_{H^+}$$

$$= \int\limits_{S_p\tilde M} f_\epsilon\, d\bar\mu_p \cdot \int\limits_{SM} \varphi\, d\mu_{H^+}.$$

Since $f_\epsilon \to f$ uniformly, the convergence holds also for f. □

Introducing the *spherical mean* $M_{r,f}$ *with density* $f \in C(S_p\tilde M)$ as

$$M_{r,f}(\varphi) := \frac{1}{\lambda_f(S_r^+(p))} \int_{S_r^+(p)} \tilde\varphi(v) f(\phi^{-r}(v))\, d\lambda(v),$$

where

$$\lambda_f(S_r^+(p)) := \int_{S_r^+(p)} f(\phi^{-r}(v))\, d\lambda(v),$$

we can reformulate the above result in the following way:

Corollary 18. *Let* $f \in C(S_p\tilde M)$ *with*

$$\int\limits_{S_p\tilde M} f\, d\bar\mu_p \neq 0.$$

Then we have for all $\varphi \in C(SM)$*:*

$$\lim_{r\to\infty} M_{r,f}(\varphi) = \int\limits_{SM} \varphi d\mu_{H^+}.$$

Proof. Theorem 17 and (11) imply

$$\lim_{r\to\infty} \frac{1}{\lambda_f(S_r^+(p))} \int\limits_{S_r^+(p)} \tilde{\varphi}(v) f(\phi^{-r}(v))\, d\lambda(v) =$$

$$\lim_{r\to\infty} \frac{e^{hr}}{\lambda_f(S_r^+(p))} \frac{1}{e^{hr}} \int\limits_{S_r^+(p)} \tilde{\varphi}(v) f(\phi^{-r}(v))\, d\lambda(v) = \int\limits_{SM} \varphi \, d\mu_{H^+}.$$

□

Remark An analogue to Theorem 17 cannot hold for compact *flat* manifolds M: We use the notions introduced in Exampe a) of Section 2, in particular the disjoint sets $U, A \subset S^{n-1}$. Let $f \in C(S_p\mathbb{R}^n)$ be a non-negative, non-vanishing function with $\operatorname{supp} f \subset \{p\} \times U \subset S_p\mathbb{R}^n$ and let $\varphi \in C(SM)$ be a non-negative, non-vanishing function with $\operatorname{supp} \varphi \subset M \times A \subset SM$. Then we have

$$\int_{S_p\mathbb{R}^n} \tilde{\varphi}(\Phi^r(v)) f(v) d\theta_p(v) = 0,$$

for all $r > 0$, but

$$\int_{SM} \varphi d\mu_L \int_{S_p\mathbb{R}^n} f d\theta_p > 0,$$

where μ_L is the normalized Liouville measure on SM and θ_p is the normalized measure on $S_p\mathbb{R}^n$ which is invariant under rotations.

However, we have a corresponding result in the particular case when φ is constant in the fibers of SM (see Proposition 2.1 of [Pe]): If

$$C := \int\limits_{S_p\tilde{M}} f d\theta_p \neq 0,$$

then we have for all $\varphi_0 \in C(M)$ with $\varphi = \varphi_0 \circ \pi \in C(SM)$:

$$\lim_{r\to\infty} \frac{1}{C} \int\limits_{S_p\mathbb{R}^n} \tilde{\varphi}(\phi^r(v)) f(v)\, d\theta_p(v) = \int\limits_{SM} \varphi \, d\mu_L = \frac{1}{\operatorname{vol}(M)} \int\limits_M \varphi_0 \, dvol.$$

6. Mixing properties

In this section we prove Theorem 2. Let $F : SM \to SM$, $F(v) = -v$ denote the flip map. The following calculation, using convergence of spherical means, shows that $\mu_{H^-} = F^*\mu_{H^+}$:

$$\int_{SM} \varphi \, d(F^*\mu_{H^+}) = \int_{SM} \varphi \circ F \, d\mu_{H^+} = \lim_{r\to\infty} \frac{1}{\lambda(S_r^+(p))} \int_{S_r^+(p)} \tilde{\varphi} \circ F \, d\lambda$$

$$= \lim_{r\to\infty} \frac{1}{\lambda(S_r^-(p))} \int_{S_r^-(p)} \tilde{\varphi}(v) d\lambda(v) = \int_{SM} \varphi \, d\mu_{H^-}.$$

Next, we recall a useful relation between the horospherical measure and the Patterson-Sullivan measure. It is a reformulation of Theorem 6.1 in [Kn1] by using $\mu_{H^-} = F^*\mu_{H^+}$:

THEOREM 19. *Let $\varphi : SM \to \mathbb{R}$ be a continuous function. Then we have*

$$\int\limits_{SM} \varphi d\mu_{H^-} = \frac{\int\limits_M \int\limits_{S_pM} \varphi(v)\, d\bar{\mu}_p(v)\, dvol(p)}{\int\limits_M a(p)\, dvol(p)},$$

where $\bar{\mu}_p$ is the Patterson-Sullivan measure projected onto the fiber $S_p\tilde{M}$ via the identification $\phi^\infty : S_p\tilde{M} \to \tilde{M}(\infty)$.

Now we can start the proof of Theorem 2: First observe that

$$\int_{S_pM} \varphi(\phi^r v)\psi(v)\, d\theta^+_{r,p}(v) = \frac{1}{\lambda(S^+_r(p))} \int_{S^+_r(p)} \tilde{\varphi}(v)\, \tilde{\psi}(\phi^{-r}v)\, d\lambda.$$

Using $a(p) = \lim_{r\to\infty} \lambda(S_r(p))/e^{hr}$ and Theorem 17 we conclude that

$$\lim_{r\to\infty} \int_M a(p) \int_{S_pM} \varphi(\phi^r v)\psi(v)\, d\theta^+_{r,p}(v)\, dvol(p) =$$

$$\lim_{r\to\infty} \int_M \frac{1}{e^{hr}} \int_{S^+_r(p)} \tilde{\varphi}(v)\, \tilde{\psi}(\phi^{-r}v)\, d\lambda(v)\, dvol(p) =$$

$$\int_{SM} \varphi\, d\mu_{H^+} \int_M \int_{S_pM} \psi(v)\, d\bar{\mu}_p(v)\, dvol(p).$$

Theorem 2 follows now immediately with Theorem 19.

REMARK If M is a rank one locally symmetric space, Theorem 2 states nothing else that the geodesic flow $\phi^t : SM \to SM$ is mixing with respect to the Liouville measure, as explained in the Introduction.

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Fakultät für Mathematik, Ruhr-Universität Bochum, 44780 Bochum, Germany

Department of Mathematical Sciences, University of Durham, DUrham DH1 3LE, United Kingdom

E-mail address: gerhard.knieper@rub.de

E-mail address: norbert.peyerimhoff@durham.ac.uk