SECTIONAL CURVATURE OF POLYGONAL COMPLEXES WITH PLANAR SUBSTRUCTURES

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Abstract. In this paper we introduce a class of polygonal complexes for which we can define a notion of sectional combinatorial curvature. These complexes can be viewed as generalizations of 2-dimensional Euclidean and hyperbolic buildings. We focus on the case of non-positive and negative combinatorial curvature. As geometric results we obtain a Hadamard-Cartan type theorem, thinness of bigons, Gromov hyperbolicity and estimates for the Cheeger constant. We employ the latter to get spectral estimates, show discreteness of spectrum in the sense of a Donnelly-Li type theorem and corresponding eigenvalue asymptotics. Moreover, we prove a unique continuation theorem for eigenfunctions and the solvability of the Dirichlet problem at infinity.

Contents

1. Introduction

Since recent years there is an increasing interest in studying curvature notions on discrete spaces. First of all there are various approaches to Ricci curvature based on L^1-optimal transport on metric measure spaces starting with the work of Ollivier, [O1, O2]. These ideas were employed for graphs by various authors [BJL, JL, LLY, LY] to study geometric and spectral questions. A related and very effective definition using L^2-optimal transport was introduced in [EM]. Secondly, in [JL, LY] there is the approach of defining curvature bounds via curvature-dimension-inequalities using a calculus of Bakry-Emery based on Bochner's formula for Riemannian manifolds. Similar ideas were used [BHLLMY] to prove a Li-Yau inequality for graphs. Finally let us mention the work on so called Ricci-flat graphs [CY]. All these approaches have in common that they model some kind of Ricci curvature and that they are very useful to study lower curvature bounds.

Classically there is a curvature notion for planar tessellating graphs defined by an angular defect. These ides go back as far as to works of Descartes [F] and became mathematical folklore since then. Often there is no obvious relation of this curvature to the recent notions of Ricci curvature above. Despite the rather restrictive setting of planar graphs this curvature notion has proven to very effective to derive very strong spectral and geometric consequences of upper curvature bounds [BP1, BP2, Hi, K1, K2, KLPS, Woe] which often relate to results to upper bounds on sectional curvature of Riemannian manifolds. (For consequences on lower bounds see e.g. [DM, JHL, NS, S, Z] as well.) Thus, it seems desirable to identify a class of 'higher dimensional' graphs where on can define and introduce sectional curvature. This is the aim of this work.

The objects under investigation in this article are *polygonal complexes with planar substructures.* They are 2-dimensional CW-complexes equipped with a family of subcomplexes homeomorphic to the Euclidean plane, which we call *apartments*, since they have certain properties similar to the ones required for apartments in Euclidean and hyperbolic buildings. The 2-cells of a polygonal complex with planar substructurescan be viewed as polygons and they are called faces and their closures are called chambers. The geometry is based on this set of faces and their neighboring structures. In particular, there is a combinatorial distance function on the set of faces. Let us discuss the properties of apartments in more detail. First of all, we require that there are enough apartments, that is any two faces have to lie in at least one apartment (condition (PCPS1) in Definition 2.3 below). Sometimes, we require the stronger condition (PSPS1*) that every infinite geodesic ray of faces is contained in an apartment. The second crucial property is that all apartments are *convex* (see condition (PCPS2)). These properties are also similar to the ones satisfied by flats in symmetric spaces. The definition of polygonal complexes with planar substructurescomprises both planar tessellations and all Euclidean and hyperbolic buildings.

We use the apartments of a polygonal complex with planar substructuresto define combinatorial curvatures on them. Since these apartments could be seen in a vague sense as tangent planes of the polygonal complex with planar substructures, we call these curvatures *sectional curvatures*. We introduce sectional curvatures on the faces and on the corners of an apartment (see Definition 2.7), and they are invariants measuring the *local geometry* of the polygonal complex with planar substructures.

The definition of polygonal complexes with planar substructuresand basic notions are introduced in Section 2. The results in this article are then given in Sections 3 and 4. While most of these results are known for planar tessellations, it seems to us that several of these results were not known for Euclidean and hyperbolic buildings. Next, we explain our results in more detail.

In Section 3 we discuss implications of negative and non-positive curvature to the *global and asymptotic geometry* of a polygonal complex with planar substructures. Many of the presented results have well-known counterparts in the smooth setting of Riemannian manifolds. Amongst our results, we present a combinatorial Cartan-Hadamard theorem for non-positively curved polygonal complexes with planar substructures(see Theorem 3.1) and we conclude Gromov hyperbolicity and positivity of the Cheeger isoperimetric constant for negatively curved polygonal complexes with planar substructureswith certain bounds on the vertex and face degree (see Theorems 3.6 and 3.8). These results are based on negativity or non-positivity of the sectional corner curvature. We also state an analogue of Myers theorem in the case of strictly positive sectional face curvature (see Theorem 3.13).

Section 4 is devoted to spectral considerations of the Laplacian. We discuss combinatorial/geometric criteria to guarantee emptiness of the essential spectrum and to derive certain eigenvalue asymptotics on polygonal complexes with planar substructures(see Theorem 4.1).

We also show that non-positive sectional corner curvature on polygonal complexes with planar substructuresimplies absence of finitely supported eigenfunctions (see Theorem 4.3). Finally, we derive solvability of the Dirichlet problem at infinity for polygonal complexes with planar substructuresin the case of negative sectional corner curvature (see Theorem 4.6).

As mentioned before, Euclidean and hyperbolic buildings provide large classes of examples of polygonal complexes with planar substructures. While all these spaces have non-positive sectional face curvature, their corner curvature is not always necessarily non-positively curved. The main purpose of the final Section 5 is to provide a self-contained short survey over these important classes.

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2. Basic definitions

In this section we introduce *polygonal complexes with planar substructures* and define a notion of sectional curvature on theses spaces. In order to do so we introduce polygonal complexes and planar tessellations first. In the second subsection we explore some basic consequences of the convexity assumption we impose. In the third subsection we introduce a combinatorial sectional curvature notions for these spaces.

2.1. **Polygonal complexes with planar substructures.**

The following definition of polygonal complexes is found in [BB1].

Definition 2.1 (Polygonal complex)**.** A *polygonal complex* is a 2-dimensional CW-complex X with the following properties:

(PC1) The attaching maps of X are homeomorphisms.
(PC2) The intersection of any two closed cells of X is either empty or exactly one closed cell.

For a polygonal complex X we denote the 0-cells by V and call them *vertices*, we denote the 1-cells by E and call them the *edges* and we denote the 2-cells by F and call them the *faces*. We write $X = (V, E, F)$. Note that the closures of all edges and faces in X are necessarily compact (since they are images of compact sets under the continuous characteristic maps, see [Hat, Appendix]). We call two vertices v and w *adjacent* or *neighbors* if they are connected by an edge in which case we write $v \sim w$. We call two different faces f and g

adjacent or *neighbors* if their closures intersect in an edge and we write $f \sim g$. It is convenient to call the closure of a face a *chamber*.

The *degree* $|v| \in \mathbb{N}_0 \cup \{\infty\}$ of a vertex $v \in V$ is the number of vertices that are adjacent to v. The *degree* $|e| \in \mathbb{N}_0 \cup \{\infty\}$ of an edge $e \in E$ is the number of chambers containing e. The boundary ∂f of a face $f \in F$ is the set of all 1-cells $e \in E$ being contained in the closure $\overline{f}$. Since in CW-complexes every compact set can meet only finitely many cells (see [Hat, Prop. A.1]), we have $|\partial f| = \#\partial f < \infty$. The *degree* $|f|$ of a face $f \in F$ is the number of faces that are adjacent to f and, in contrast to $|\partial f|$, the face degree $|f|$ can be infinite.

We call a (finite, infinite or bi-infinite) sequence $\ldots, f_{i-1}, f_i, f_{i+1}, \ldots$ of pairwise distinct faces a *path* if successive faces are adjacent. The *length* of the path is one less than the number of components of the sequence. The *(combinatorial) distance* between two faces f and g is the length of the shortest path connecting f and g and the distance is denoted by $d(f, g)$. We call a (finite, infinite or bi-infinite) path (f_k) of faces a *geodesic* or a *gallery*, if we have for any two faces f_m and f_n in the path $d(f_m, f_n) = |m - n|$, i.e., the distance between f_m and f_n is realized by the path.

We say a polygonal complex X is *planar* if X is homeomorphic to $\mathbb{R}^2$. We also say that a polygonal complex X is *spherical* if X is homeomorphic to the two-sphere $\mathbb{S}^2$.

Next we introduce the notion of a planar tessellation following [BP1, BP2].

Definition 2.2 (Planar tessellation)**.** A polygonal complex $\Sigma = (V, E, F)$ is called a *(planar/spherical) tessellation* if Σ is planar/spherical and satisfies the following properties:

- (T1) Any edge is contained in precisely two different chambers.
- (T2) Any two different chambers are disjoint or have precisely either a vertex or a side in common.
- (T3) For any chamber the edges contained in it form a closed path without repeated vertices.
- (T4) Every vertex has finitely many neighbors.

Note that property (T3) is already implied by (PC1) and (PC2). The tessellations form the substructures which we will later need to define sectional curvature. Now, we are in a position to introduce polygonal complexes with planar substructures.

Definition 2.3. A *polygonal complex with planar substructures* is a polygonal complex $X = (V, E, F)$, together with a set $\mathcal{A}$ of subcomplexes whose elements $\Sigma = (V_\Sigma, E_\Sigma, F_\Sigma)$ are called *apartments*, with the following properties:

(PCPS1) For any two faces there is an apartment containing both of them.

(PCPS2) The apartments are convex (i.e., for any $\Sigma \in \mathcal{A}$ any finite gallery $f_1, \dots, f_n$ with end-faces f_1, f_n in Σ stays completely in Σ).

(PCPS3) The apartments are planar tessellations.

Similarly, we introduce *polygonal complexes with spherical substructures* by replacing property (PCPS3) in Definition 2.3 by

(PCSS3) The apartments are spherical tessellations.

Prominent examples of polygonal complexes with planar substructuresare Euclidean and hyperbolic buildings (see Section 5 for the definition of a building as well as several examples). Moreover, every planar tessellation is trivially a polygonal complex with planar substructures. For reasons of illustration, we like to introduce the following example of a Euclidean building.

Example 1. Let X_0 be the finite simplicial complex constructed from the seven equilateral Euclidean triangles illustrated in Figure 1 by identifying sides with the same labels x_i.

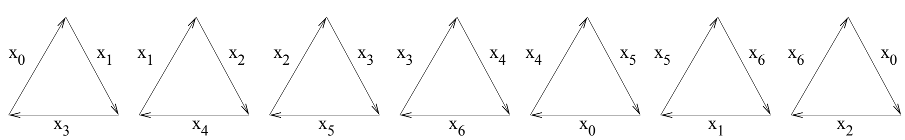

FIGURE 1. Labeling scheme for the simplicial complex X_0

Then X_0 has a single vertex which we denote by p_0, seven edges and seven faces. Its fundamental group $\Gamma = \pi_1(\Pi_0, p_0)$ has the following presentation

$$\Gamma = \langle x_0, \dots, x_6 \mid x_i x_{i+1} x_{i+3} = \mathrm{id} \text{ for } i = 0, 1, \dots, 6 \rangle$$

(where i is taken modulo 7). Let $X = (V, E, F)$ be the universal covering of X_0 together with the lifted triangulation. Then it follows from [BB2, Thm 6.5] that X is a *thick Euclidean building of type* $\widetilde{A}_2$ and every edge of X belongs to precisely 3 triangles. Therefore, X is a polygonal complex with planar substructures. The group of covering transformations is isomorphic to Γ and acts transitively on the vertices of this building (see [CMS]).

For some of our results we need the following slightly stronger assumption than (PCPS1):

(PCPS1*) Every (one-sided) infinite geodesic is contained in an apartment.

Condition (PCPS1*) is satisfied for all Euclidean and hyperbolic buildings with a *maximal apartment system* (see Theorem 5.5 below).

Finally, let us mention the following important fact. To a polygonal complex $X = (V, E, F)$ we can naturally associate a graph G_X by letting F be the vertex set of G_X and by defining the edges of that graph via the adjacency relation of the corresponding faces. This 'duality' becomes important when we use results for graphs in our context.

2.2. **Consequences of convexity.** The convexity assumption (PCPS2) is very important in our considerations. In this subsection we collect some of the immediate consequences.

Lemma 2.4. *Let X be a polygonal complex with planar substructures, Σ an apartment and let d_Σ the combinatorial distance within the apartment. Then, for any two faces $f, g \in F_\Sigma$*

$$d(f, g) = d_\Sigma(f, g).$$

Proof. The inequality '$\leq$' is clear. For the other direction '$\geq$' let $\gamma = (f_0, \ldots, f_n)$ be a path connecting f and g minimizing $d(f, g)$. As γ is a geodesic with end-faces in Σ it is completely contained in Σ by (PCPS2). Hence, the statement follows. □

We say a subset F_0 of F is connected if any two faces in F_0 can be joined by a path in F_0.

Lemma 2.5. *Let X be a polygonal complex with planar substructures. Let Σ_1 and Σ_2 be two apartments of X. Then the set $F_{\Sigma_1} \cap F_{\Sigma_2}$ is connected.*

Proof. Let f and g be two faces in $F_{\Sigma_1} \cap F_{\Sigma_2}$. Then, by (PCPS2), every geodesic connecting f and g is completely contained in Σ_1 and Σ_2. Thus, $F_{\Sigma_1} \cap F_{\Sigma_2}$ is connected. □

For a fixed face $o \in F$ (called *root*), we define the (combinatorial) spheres and balls about o by

$$\begin{aligned} S_n &= S_n(o) = \{f \in F \mid d(f, o) = n\} \quad \text{and} \\ B_n &= B_n(o) = \bigcup_{k=0}^{n} S_k, \end{aligned}$$

for $n \geq 0$. For $f \in F$, we let the *forward* and *backward degree* be given by

$$|f|_{\pm} = |\{g \in F \mid g \sim f,\, d(g,o) = d(f,o) \pm 1\}|,$$

and we call $g \in F$ with $g \sim f$ and $d(g,o) = d(f,o) + 1$ (respectively $d(g,o) = d(f,o) - 1$) a *forward* (respectively *backward*) *neighbor* of f. The next lemma shows that the convexity condition (PCPS2) imposes a lot of structure of the distance spheres.

Lemma 2.6. *Let X be a polygonal complex with planar substructure-sand $o \in F$ be a root. Let $f \in F$ with $f \in S_n$ for some $n \geq 0$ and $f_+ \in S_{n+1}$, $f_0 \in S_n$, $f_- \in S_{n-1}$ be neighbors of f. Then,*

(a) *Every face sharing the same edge with f and f_+ is in S_{n+1}.*
(b) *Every face sharing the same edge with f and f_0 is in $S_n \cup S_{n-1}$.*
(c) *Every face sharing the same edge with f and f_- is in S_n.*

Proof. (a) Let $g \in F$ be such that $\partial g \cap \partial f \cap \partial f^+ \neq \emptyset$. Since g is a neighbor of f_+, we have $d(o,f) \geq n$. Since g is a neighbor of f, we have $d(o,f) \leq n+1$. Therefore, we have $g \in S_n \cup S_{n+1}$. If g was in S_n, then there are geodesics from the root o over g to f_+ and from o over f to f_+. By (PCPS2) both of these geodesics lie together in one apartment. Hence, g lies in one apartment together with f, f_+ and o. Then, there is an edge contained in three faces f, f_+ and g within one apartment Σ. This contradicts (T1) in the definition of a planar tessellation. But Σ is a planar tessellation, by (PCPS3). Thus, $g \in S_{n+1}$.
(b) Let $g \in F$ be such that $\partial g \cap \partial f \cap \partial f_0 \neq \emptyset$. If g was in S_{n+1} then there were two geodesics from o to g, one via f and the other one via f_0. By a similar argument as in (a), the faces g, f, f_0 and o lie in the same apartment. Again this is impossible by (T1) and (PCPS3).
(c) Let $g \in F$ be such that $\partial g \cap \partial f \cap \partial f_- \neq \emptyset$. Clearly, g is in $S_n \cup S_{n-1}$. If g was in S_n then, by similar arguments as in (a) and (b), the faces g, f, f_- and o lie in the same apartment which is again impossible by (T1) and (PCPS3). □

2.3. **Sectional curvature.** For an apartment $\Sigma = (V_\Sigma, E_\Sigma, F_\Sigma)$, let $|v|_\Sigma$ be the *degree of v in Σ* which is the number of neighboring vertices in V_Σ. We notice that the degree of an edge in Σ, i.e., the number of faces in F_Σ bounded by the edge, is always equal to 2 by (T1). Moreover, the *degree $|f|_\Sigma$ of a face f in Σ* is equal to $|\partial f|$. Therefore, $|f|_{\Sigma_1} = |f|_{\Sigma_2}$ for any two apartments Σ_1, Σ_2 that contain f. Furthermore, for a polygonal complex with planar substructures$X$ and $\Sigma \in \mathcal{A}$ we let the set of corners of X and of Σ be given by

$$C = \{(v,f) \in V \times F \mid v \in f\}, \quad C_\Sigma = \{(v,f) \in V_\Sigma \times F_\Sigma \mid v \in f\}.$$

Definition 2.7 (Curvature)**.** Let Σ be an apartment of a polygonal complex with planar substructures X. The *sectional corner curvature* $\kappa_c^{(\Sigma)} : C_\Sigma \to \mathbb{R}$ with respect to Σ is given by

$$\kappa_c^{(\Sigma)}(v,f) = \frac{1}{|v|_\Sigma} - \frac{1}{2} + \frac{1}{|f|_\Sigma},$$

and the *sectional face curvature* $\kappa^{(\Sigma)} : F_\Sigma \to \mathbb{R}$ with respect to Σ is given as

$$\kappa^{(\Sigma)}(f) = \sum_{(v,f)\in C_\Sigma} \kappa_c^{(\Sigma)}(v,f) = 1 - \frac{|f|_\Sigma}{2} + \sum_{v\in V_\Sigma, v\in \overline{f}} \frac{1}{|v|_\Sigma}.$$

The above combinatorial curvature notions are motivated by a combinatorial version of the Gauss-Bonnet Theorem for closed surfaces. We have for polygonal tessellations $\Sigma = (V,E,F)$ of a closed surface S (see [BP1, Thm 1.4])

$$\chi(S) = \sum_{f\in F} \kappa^{(\Sigma)}(f) = \sum_{(v,f)\in C_\Sigma} \kappa^{(\Sigma)}(v,f), \tag{1}$$

where $\chi(S)$ is the Euler characteristic of S. The sectional curvatures in Definition 2.7 are then the intrinsic curvatures in the apartments Σ, and the apartments Σ can be understood as discrete analogues of specific tangent planes. Note that curvature is a local concept and, for a given corner or face, only information of the nearest neighboring faces in the apartment are needed for its calculation.

Example 2. Let us return to Example 1 and consider its sectional curvatures. Each apartment Σ of X_0 is isomorphic to a tessellation of a Euclidean plane by equilateral triangles. This means that we have for every corner $(v,f) \in C_\Sigma$:

$$\kappa_c^{(\Sigma)}(v,f) = \frac{1}{|v|_\Sigma} - \frac{1}{2} + \frac{1}{|f|_\Sigma} = \frac{1}{6} - \frac{1}{2} + \frac{1}{3} = 0,$$

i.e., X_0 has vanishing sectional corner curvature. Consequently, its sectional face curvature is also zero.

3. Geometry

In this section we discuss implications of the curvature sign to the global geometry of polygonal complexes with planar substructures-like emptiness of cut-locus, Gromov hyperbolicity or positivity of the Cheeger constant. Before we enter into these topics, we first introduce some more useful combinatorial notions.

We say X is *locally finite* if for all $v \in V$ and $e \in E$

$$|v| < \infty \quad \text{and} \quad |e| < \infty.$$

Since $|f| = \sum_{e \in \partial f} |e|$, we also have $|f| < \infty$ for locally finite polygonal complexes. We define for a face $f \in F$

$$m_E(f) = \min_{e \in \partial f}(|e| - 1), \qquad M_E(f) = \max_{e \in \partial f}(|e| - 1)$$

the minimal and maximal number of neighbors over one edge of f. The *minimal* and *maximal thickness* of X is then given by

$$m_E = \min_{f \in F} m_E(f), \quad M_E = \max_{f \in F} M_E(f).$$

The *maximal vertex* and *face degree* are defined by

$$M_V = \sup_{v \in V} |v|, \qquad M_F = \sup_{f \in F} |f|.$$

Note that we always have $M_E \leq M_F$.

3.1. **Absence of cut-locus.** We first present a theorem which is an analogue of the Hadamard-Cartan theorem from Riemannian manifolds. It is a rather immediate consequence of convexity and [BP2, Thm. 1] for plane tessellating graphs.

For a face $f \in F$ in a polygonal complex $X = (V, E, F)$ the *cut locus* of f is defined as

$$\mathrm{Cut}(f) = \{g \in F \mid d(f, \cdot) \text{ attains a local maximum in } g\}.$$

Absence of cut locus means that $\mathrm{Cut}(f) = \emptyset$ for all $f \in F$ which means that every finite geodesic starting in f can be continued to a infinite geodesic.

Theorem 3.1. *Let $X = (V, E, F)$ be a polygonal complex with planar substructuressuch that $\kappa_c^{(\Sigma)} \leq 0$ for all apartments $\Sigma \in \mathcal{A}$. Then,* $\mathrm{Cut}(f) = \emptyset$ *for all $f \in F$. Moreover, every geodesic within an apartment Σ can be continued to an infinite geodesic within Σ.*

We conclude from Theorem 3.1 that emptiness of of cut-locus holds, e.g., for our Example 1 and Examples 8-11. Note also that the condition of non-positive sectional *corner curvature* in Theorem 3.1 cannot be weakened to non-positive sectional *face curvature* as Figure 2 in [BP2] shows.

Proof. Let $f \in F$. Choose $g \in F$ and let Σ be an apartment which contains both f and g (which exists by (PCPS1)). By [BP2, Thm. 1] the cut locus of f within Σ is empty that is there a face $h \in F_\Sigma$ with $g \sim h$ such that $d_\Sigma(f, h) = d_\Sigma(f, g) + 1$. (Note that [BP2, Thm. 1] is formulated in the dual setting which, however, can be carried over directly.) As $d = d_\Sigma$, by Lemma 2.4, we conclude $g \notin \mathrm{Cut}(f)$. Since this holds for all $g \in F$, we have $\mathrm{Cut}(f) = \emptyset$. The second statement is an immediate consequence of [BP2, Thm. 1] and Lemma 2.4. $\square$

Corollary 3.2. *Let $X = (V, E, F)$ be a polygonal complex with planar substructuressuch that $\kappa_c^{(\Sigma)} \leq 0$ for all $\Sigma \in \mathcal{A}$. Then, every face has at least one forward neighbor.*

3.2. **Thinness of bigons.** In this subsection we show a useful hyperbolicity criterion.

Let $X = (V, E, F)$ be a polygonal complex. A *bigon* is a pair of geodesics $(f_0, \dots, f_n)$ and $(g_0, \dots, g_n)$ such that $f_0 = g_0$ and $f_n = g_n$. We say a bigon is δ-*thin* for $\delta \geq 0$, if $d(f_k, g_k) \leq \delta$ for all $k = 0, \dots, n$.

Theorem 3.3. *Let $X = (V, E, F)$ be a polygonal complex with planar substructuressuch that $\kappa_c^{(\Sigma)} < 0$ for all apartments $\Sigma \in \mathcal{A}$. Then, every bigon is* 1*-thin.*

Proof. Let $\gamma_1 = (f_0, \dots, f_n)$ and $\gamma_2 = (g_0, \dots, g_n)$ be a bigon and $\Sigma \in \mathcal{A}$ be an apartment that contains $f_0 = g_0$ and $f_n = g_n$. By the convexity assumption (PCPS2) the apartment Σ contains both geodesics γ_1 and γ_2 and, therefore, the pair (γ_1, γ_2) is a bigon within Σ. By [BP2, Thm. 2] it follows that $d_\Sigma(f_k, g_k) \leq 1$ for $k = 0, \dots, n$, and by Lemma 2.4 we conclude that $d(f_k, g_k) \leq 1$ for $k = 0, \dots, n$. $\square$

We have an immediate consequence.

Corollary 3.4. *Let $X = (V, E, F)$ be a polygonal complex with planar substructuressuch that $\kappa_c^{(\Sigma)} < 0$ for all $\Sigma \in \mathcal{A}$. Let $f_1, f_2 \in F$ with $d(f_1, f_2) = n$. Then we have for all $0 \leq k \leq n$:*

$$|B_k(f_1) \cap B_{n-k}(f_2)| \leq 2.$$

In particular, if f_1 is considered as a root, f_2 has at most two backward neighbors.

Proof. By convexity we can restrict our considerations on any apartment $\Sigma \in \mathcal{A}$ containing f_1 and f_2. Every $f \in B_k(f_1) \cap B_{n-k})(f_2)$ must obviously satisfy $d(f_1, f) = k$ and $d(f, f_2) = n - k$. If there were 3 faces in the intersection $B_k(f_1) \cap B_{n-k}(f_2) \subset F_\Sigma$, then there are 3 geodesics from f_1 to f_2 in Σ. Then, one of the three geodesics is enclosed by the other two in Σ and the other two geodesics form a bigon. Then this bigon in not 1-thin which contradicts the previous theorem. $\square$

In fact, the last statement of Corollary 3.4 holds even for non-positive sectional corner curvature.

Proposition 3.5. *Let $X = (V, E, F)$ be a polygonal complex with planar substructuressuch that $\kappa_c^{(\Sigma)} \leq 0$ for all $\Sigma \in \mathcal{A}$ and $o \in F$ be a root. Then every face has at most two backward neighbors.*

Proof. This is a consequence of the results in [BP2]. Let $f \in F$. Let $\Sigma \in \mathcal{A}$ be an apartment containing o and f. Then the ball $B_n \cap \Sigma$ is an admissible polygon in Σ in the sense of [BP2, Def. 2.2] and ∂f and ∂B_n is a connected path of length ≤ 2, by [BP2, Prop. 2.5]. This shows that f can have at most two backward neighbors. □

3.3. **Gromov hyperbolicity.** Recall from the end of Subsection 2.1 that every polygonal complex $X = (V, E, F)$ can also be viewed as a metric space via the associated graph G_X and its natural combinatorial distance function. Geodesics $(f_i)_i \subset F$ in X correspond then to (vertex) geodesics in G_X. With this understanding, we call the polygonal complex (X, d) Gromov hyperbolic if there exists $\delta > 0$ such that any side of any geodesic triangle in G_X lies in the δ-neighborhood of the union of the two other sides of the triangle. We show Gromov hyperbolicity of a polygonal complex with planar substructures(X, d) with negative sectional corner curvature as well as properties of the Gromov boundary $X(\infty)$ under the additional boundedness assumption of the vertex and face degree. For details on the Gromov boundary (and the Gromov product used to define it) we refer to [BH, Chpt. III.H].

Theorem 3.6. *Let X be a polygonal complex with planar substructureswith $M_V, M_F < \infty$ and $\kappa_c^{(\Sigma)} < 0$ for all $\Sigma \in \mathcal{A}$. Then, (X, d) and all its apartments are Gromov hyperbolic spaces. If additionally (*PCPS1**) is satisfied then every connected component of the Gromov boundary $X(\infty)$ contains the Gromov boundary of an apartment which is homeomorphic to S^1.*

It is easy to see that the Euclidean buildings in Example 1 and 8 are not Gromov hyperbolic. Theorem 3.6 is not applicable since these examples have vanishing sectional corner curvature.

The main ingredient in the first part of the proof of Theorem 3.6 is the fact that all bigons in (X, d) are 1-thin (Theorem 3.3). The same holds true within all apartments. [Pa, Theorem 1.4] tells us that the statement of the theorem would then be true in the case of Cayley graphs. For general G_X, we need the following generalization given in the unpublished Masters dissertation of Pomroy (a proof of it can be found in [ChN, Appendix]):

Theorem 3.7 (Pomroy)**.** *If for a geodesic metric space there are $\epsilon, \rho > 0$ such that ρ-bigons are uniformly ϵ-thin, then the space is Gromov hyperbolic.*

For G_X to satisfy the requirement of a *geodesic* metric space, we must view it as a *metric graph* with all its edge lengths equal to one. Let us clarify the other notions in Pomroy's theorem: A ρ-bigon is a pair of $(1,\rho)$ quasi-geodesics γ_1, γ_2 with the same end points, i.e.,

$$|t-t'| - \rho \leq d(\gamma_i(t), \gamma_i(t')) \leq |t-t'| + \rho \quad \forall\, t, t'.$$

Choosing $\delta < 1/2$, we can then conclude from Theorems 3.3 and 3.7 that (X,d) and all its apartments are Gromov hyperbolic.

Next we prove the rest of the theorem assuming (PCPS1*). From $M_F < \infty$ we conclude that G_X is a *proper* hyperbolic geodesic space and, therefore, the geodesic boundary (defined via equivalence classes of geodesic rays, where rays are equivalent iff they stay in bounded distance to each other) and the Gromov boundary coincide (see, e.g., [BH, Lm. III.H.3.1]) and we can think of any boundary point $\xi \in X(\infty)$ as being represented by a geodesic ray $(f_i)_i \subset F$. Using (PCPS1*), there is an apartment $\Sigma \in \mathcal{A}$ such that $(f_i)_i \subset F_\Sigma$ and $\xi \in \Sigma(\infty) \subset X(\infty)$. We also know from [BP2, Cor. 5] that $\Sigma(\infty)$ is homeomorphic to S^1, finishing the proof. $\square$

3.4. **Cheeger isoperimetric constants.** In this subsection we prove how negative curvature and other geometric quantities effect positivity of the Cheeger isoperimetric constant.

Let $X = (V,E,F)$ be a *locally finite* polygonal complex. We consider the following Cheeger constant as it is very for useful for spectral estimates. For $G \subseteq F$ we define

$$\alpha_G = \inf_{H \subseteq G} \frac{|\partial H|}{\mathrm{vol}(H)}$$

with

$$\partial H = \{(f,g) \in F \times F \mid \text{where } f \in H,\, g \in F \setminus H \text{ with } f \sim g\},$$

and

$$\mathrm{vol}(H) = \sum_{f \in H} |f|.$$

Note that $\alpha_G \leq 1$. We set $\alpha = \alpha_F$.

Firstly, we present a result that shows positivity of the Cheeger isoperimetric constant for negative sectional corner curvature under the additional assumption of bounded geometry. This result is a consequence of a general result of Cao [C], which also holds in the smooth setting of Riemannian manifolds. Secondly, we give more explicit estimates for the Cheeger constant.

Theorem 3.8. *Let $X = (V, E, F)$ be a polygonal complex with planar substructuressuch that $\kappa_c^{(\Sigma)} < 0$ for all $\Sigma \in \mathcal{A}$. Assume that X additionally satisfies (PCPS1*) and $M_V, M_F < \infty$. Then, $\alpha > 0$.*

A straightforward consequence of Theorem 3.8 is

Corollary 3.9. *Every locally finite hyperbolic building with regular hyperbolic polygons as faces has a positive Cheeger constant $\alpha > 0$.*

In particular, all buildings in Examples 9-11 have positive Cheeger constant (we can assume without loss of generality that these buildings are equipped with a maximal system of apartments).

Proof of Theorem 3.8. Note that by the comment at the end of Subsection 2.1 we can associate to every polygonal complex with planar substructures$X = (V, E, F)$ a graph G_X by considering the faces of X as vertices in G_X and the edge relation given by the adjacency relation of the faces. In this light [C, Thm. 1] tells us that a polygonal complex (X, d) has positive Cheeger isoperimetric constant if the following four assumptions are satisfied

(1) (X, d) has bounded face degree $M_F < \infty$,
(2) (X, d) admits a quasi-pole,
(3) (X, d) is Gromov hyperbolic,
(4) every connected component of the Gromov boundary $X(\infty)$ has positive diameter (with respect to a fixed Gromov metric),

where (2) means that there is a finite set $\Omega \subset F$ of faces and a $\delta > 0$ such that every face $f \in F$ is found in a δ-neighborhood of a geodesic ray emanating from this finite set. Moreover, for (4) we follow [C] and define for two geodesic rays $(f_i), (f_i') \subset F$ with the same initial face $f_0 = f_0'$ representing the points $\xi, \eta \in X(\infty)$:

$$d_{f_0,\epsilon}(\xi, \eta) = \liminf_{n\to\infty} \exp(-\varepsilon(n - \tfrac{1}{2}d(f_n, f_n')),$$

Then there is an $\varepsilon > 0$ such that $d_{f_0,\epsilon}$ is a metric which is called a Gromov metric. Note that the Cheeger constant considered in [C] is defined as

$$h = \inf_{H\subseteq F} \frac{|\partial_F H|}{|H|},$$

where $\partial_F H = \{f \in F \mid d(f, H) = 1\}$. As every face in $\partial_F H$ is connected with H via at least one edge we have $|\partial H| \geq |\partial_F H|$. Also $\mathrm{vol}(H) \leq M_F|H|$ and, therefore,

$$\alpha \geq \frac{h}{M_F}.$$

Hence, by the assumption $M_F < \infty$ the constant α is positive whenever h is. Thus, it remains to check the conditions (1)-(4).

Let $X = (V, E, F)$ be a polygonal complex with planar substructureswhich satisfies the assumptions of the theorem. Then, (1) is obviously satisfied. Secondly, by absence of cut-locus, Theorem 3.1, condition (2) is satisfied and by Theorem 3.6 condition (3) is satisfied. Finally, let us turn to (4). By Theorem 3.6 and the assumption (PCPS1*) we know that every connected component of the Gromov boundary of X contains the Gromov boundary of an apartment. Therefore, it suffices to show (4) for the Gromov boundary of an apartment. We observe that we find in every apartment a bi-infinite geodesic. This can be seen as follows: Let $(f_{-n}, \ldots, f_n)$ be a geodesic in an apartment $\Sigma \in \mathcal{A}$. By [BP2, Thm. 1] the face f_n is not in $\mathrm{Cut}_\Sigma(f_{-n})$ and, therefore, there is $f_{n+1} \in \Sigma$ such that $(f_{-n}, \ldots, f_{n+1})$ is a geodesic. Simultaneously, f_{-n} is not in $\mathrm{Cut}_\Sigma(f_{n+1})$ and therefore there is $f_{-(n+1)} \in \Sigma$ such that $(f_{-(n+1)}, \ldots, f_{n+1})$ is a geodesic in Σ. In this way , we construct a bi-infinite geodesic $(f_n)_{n\in\mathbb{Z}}$. Let $\xi, \eta \in X(\infty)$ be the end points of the geodesics $(f_n)_{n\geq 0} \subset F_\Sigma$. Since $(f_n)_{n\in\mathbb{Z}}$ is a bi-infinite geodesic, we have $d(f_n, f_{-n}))) = 2n$. So, we obtain for any $\varepsilon > 0$

$$d_{f_0,\varepsilon}(\xi, \eta) = \liminf_{n\to\infty} \exp(-\varepsilon(n - \tfrac{1}{2}d(f_n, f_{-n}))) = 1.$$

Hence, (4) is satisfied and we finished the proof. $\square$

Remark 3.10. The question whether a Gromov hyperbolic space has positive Cheeger constant is very subtle. Note that every infinite tree T is Gromov hyperbolic. But if we attach to one of its vertices the ray $[0, \infty)$ with integer vertices then the new tree $\widetilde{T}_1$ is still Gromov hyperbolic but it has vanishing Cheeger constant. This new ray adds an isolated point to the Gromov boundary of T and therefore assumption (4) is violated for $\widetilde{T}_1$. On the other hand, if we attach to a sequence of vertices $(v_n)_{n\in\mathbb{N}}$ in T the segments $[0, n]$ with integer vertices and denote the new tree by $\widetilde{T}_2$, then this new tree has again vanishing Cheeger constant. In this case both trees T and $\widetilde{T}_2$ even have the same boundaries, but $\widetilde{T}_2$ cannot have a quasi-pole since the newly added vertices do not lie in geodesic rays and, therefore, assumption (2) is violated (see end of Subsection 1.1 in [C]).

The next result provides explicit lower bounds for the Cheeger constant in terms of the face degrees and minimal and maximal thickness.

Theorem 3.11. *Let X be a* locally finite *polygonal complex with planar substructures. Then,*

$$\alpha \geq \inf_{f\in F}\Big(\frac{m_E(f)}{M_E(f)}\Big(1-\frac{6}{|\partial f|}\Big)\Big) \geq \frac{m_E}{M_E}\Big(1-\frac{6}{\min_{f\in F}|\partial f|}\Big).$$

In particular, $\alpha > 0$ if $|\partial f| \geq 7$ and $M_E < \infty$. Secondly,

$$\alpha \geq \inf_{f\in F}\frac{m_E(f)-2}{|f|} \geq \frac{m_E-2}{M_F}.$$

In particular, $\alpha > 0$ if $m_E \geq 3$ and $M_F < \infty$.

The theorem implies in particular that all locally finite Euclidean buildings with minimal thickness $m_E \geq 3$ (i.e., every edge is contained in at least 4 chambers) have positive Cheeger constant. Since the minimal thickness of Example 1 is $m_E = 2$, we do not know whether this Euclidean building has positive Cheeger constant. Moreover, we see that all locally finite hyperbolic buildings with generating polygon P at least a 7-gon have also a positive Cheeger constant.

Proof. Translating [DKa, Lemma 1.15] into the 'dual' language (as the comment at the end of Section 2.1 indicates) tells us that if there is a root $o \in V$ and $C \geq 0$ such that

$$|f|_+ - |f|_- \geq C|f|$$

for all $f \in F$, then $\alpha \geq C$. Thus, it suffices to estimate $\inf_{f\in F}(|f|_+ - |f|_-)/|f|$ to get a lower bound on α. For $f \in F$, let $n \geq 0$ be such that $f \in S_n$ and let Σ be an apartment that contains f. By Proposition 3.5 we immediately have $|f|_- \leq 2$. Moreover, by [BP1, Theorem 3.2] (combined with Theorem 3.1) there are at most two neighbors of f in $F_\Sigma \cap S_n$ and, therefore, $|f|_+ \geq m_E(f)|f|_{\Sigma,+} \geq m_E(f)(|\partial f|-4)$. Here $|f|_{\Sigma,+}$ denotes the number of forward neighbors of f within Σ, which is $|\partial f|$ minus the number (≤ 2) of backward neighbors of f minus the number (≤ 2) of neighbors of f in $F_\Sigma \cap S_n$. Moreover, $|f| \leq M_E(f)|\partial f|$. Hence,

$$\frac{|f|_+ - |f|_-}{|f|} \geq \frac{m_E(f)}{M_E(f)}\left(1-\frac{6}{|\partial f|}\right)$$

which yields the first inequality. On the other hand, we have by Theorem 3.1 and Lemma 2.6 (a) $|f|_+ \geq m_E(f)$. Hence, by $|f|_- \leq 2$

$$\frac{|f|_+ - |f|_-}{|f|} \geq \frac{m_E(f)-2}{|f|}$$

This finishes the proof. $\square$

From the proof we may easily extract the following statement which turns out to be useful for studying the essential spectrum of the Laplacian. Define for a locally finite polygonal complex $X = (V, E, F)$ the *Cheeger constant at infinity* by

$$\alpha_\infty = \sup_{K \subseteq F \,\text{finite}} \alpha_{F \setminus K}.$$

Corollary 3.12. *Let X be a locally finite polygonal complex with planar substructures. Then*

$$\alpha_\infty \geq \sup_{K \subseteq F \,\text{finite}} \inf_{f \in F \setminus K} \frac{m_E(f)}{M_E(f)} \Big(1 - \frac{6}{|\partial f|} \Big)$$

3.5. **Finiteness and infiniteness.** In this subsection we show that positivity or non-positivity of *sectional face curvature* determines whether a locally finite polygonal complex with planar/spherical substructures is finite or infinite. The statement that positive curvature implies finiteness is an analogue of a theorem of Myers for Riemannian manifolds [M].

Theorem 3.13. *Let $X = (V, E, F)$ be a* locally finite *polygonal complex with planar/spherical substructureswith apartment system $\mathcal{A}$.*

(a) *If we have $\kappa^{(\Sigma)}(f) > 0$ for all $\Sigma \in \mathcal{A}$ and all $f \in F_\Sigma$, then F is finite and X is a polygonal complex with spherical substructures.*
(b) *If we have $\kappa^{(\Sigma)}(f) \leq 0$ for all $\Sigma \in \mathcal{A}$ and all $f \in F_\Sigma$, then F is infinite and X is a polygonal complex with planar substructures.*

Proof. Note first that every planar tessellation has infinitely many faces (since the closure of every face is compact) while every spherical tessellation has finitely many faces. Therefore F_Σ ($\Sigma \in \mathcal{A}$) is infinite if X is a polygonal complex with planar substructuresand finite if X is a polygonal complex with spherical substructures.

We first assume that X is a polygonal complex with planar/spherical substructureswith F a finite set. We will show that there a faces with positive sectional face curvature. Choose an apartment $\Sigma \in \mathcal{A}$. By the Gauß-Bonnet theorem (see (1)), we have

$$\sum_{f \in F_\Sigma} \kappa^{(\Sigma)}(f) = \chi(\mathbb{S}^2) = 2,$$

where χ denotes the Euler characteristic. Hence, $\kappa^{(\Sigma)}$ must be positive on some faces. This shows (b).

Now, we assume that $\kappa^{(\Sigma)}(f) > 0$ for all $\Sigma \in \mathcal{A}$ and all $f \in F_\Sigma$. By DeVos-Mohar's proof of Higuchi's conjecture [DM, Theorem 1.7] (which is again stated in the dual formulation) every apartment must be finite. Moreover, the number of faces (in their case vertices) in an apartment is uniformly bounded by 3444[1] except for prisms and antiprisms. A prism in our dual setting are two wheels of triangles glued together along their boundaries and an antiprism are two wheels of squares glued together along their boundaries (see Figure 2). We can think of these two wheels as representing the lower and upper hemisphere of $\mathbb{S}^2$ and the boundaries as agreeing with the equator of the sphere $\mathbb{S}^2$.

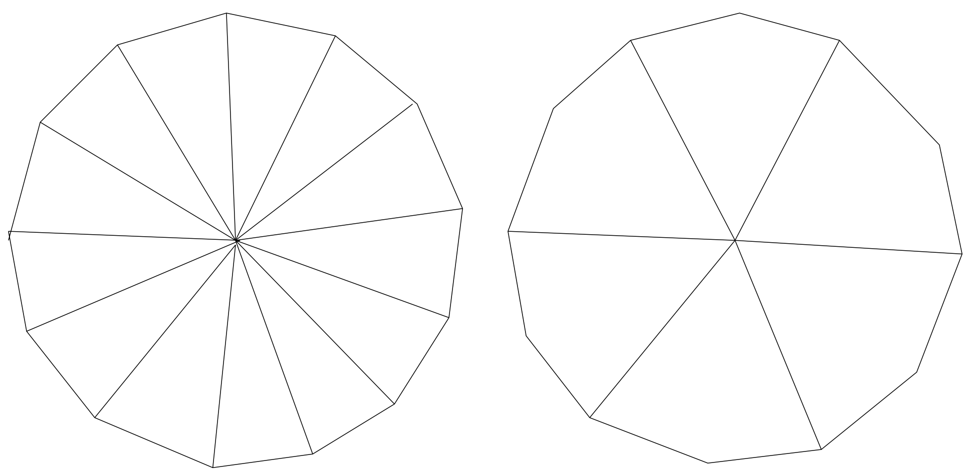

Figure 2. A wheel of triangles and a wheel of squares

If F is infinite, then there exists a face $f_0 \in F$ and a sequence of faces $f_n \in F$ with $d(f_0, f_n) \to \infty$ because of the local finiteness. Then f_0 must lie in a sequence Σ_k of spherical apartments $\mathbb{S}^2$ tessellated by pairs of wheels with number of faces going to infinity, glued together along the equator. Assuming that f_0 lies always in the lower hemisphere of $\Sigma_k \cong \mathbb{S}^2$, then the south pole of all these apartments would be one and the same vertex $v_0 \in \overline{f_0}$. But this would imply that $|v_0| = \infty$, which contradicts to (T4). Therefore, F must be finite which implies that X is a polygonal complex with spherical substructures. $\square$

4. Spectral theory

In this section we turn to the spectral theory of the Laplacian on polygonal complexes. As the geometric structure is determined by assumptions on the faces it is only natural to consider the Laplacian

[1]Note that in the meantime the bound has been improved by Zhang [Z] to 580 vertices while the largest known graphs with positive curvature has 208 vertices and was constructed by Nicholson and Sneddon [NS].

for functions on the faces. The reader who prefers to think about the Laplacian as an operator on functions on the vertices is referred to comment at the end of Section 2.1. That is we can associate a graph G_X to each polygonal complex $X = (V, E, F)$ in a natural way.

Let $X = (V, E, F)$ be a locally finite polygonal complex and

$$\ell^2(F) = \{\varphi : F \to \mathbb{C} \mid \sum_{f \in F} |\varphi(f)|^2 < \infty\}.$$

For functions $\varphi, \psi \in \ell^2(F)$ the standard scalar product is given by

$$\langle \varphi, \psi \rangle = \sum_{f \in F} \overline{\varphi(f)} \psi(f),$$

and the norm is given by $\|\varphi\| = \sqrt{\langle \varphi, \varphi \rangle}$. Define the Laplacian Δ by

$$\Delta\varphi(f) = \sum_{g \in F, g \sim f} (\varphi(f) - \varphi(g))$$

for functions in the domain

$$D(\Delta) = \{\psi \in \ell^2(F) \mid \Delta\psi \in \ell^2(F)\}.$$

It can be checked directly that the operator is positive and, moreover, it is selfadjoint by [Woj, Theorem 1.3.1].

By standard Cheeger estimates [K1] based on [DKe, Fu] we have

$$\lambda_0(\Delta) \geq m_F(1 - \sqrt{1 - \alpha^2}),$$

where $\lambda_0(\Delta)$ denotes the bottom of the spectrum of Δ and

$$m_F = \min_{f \in F} |f|.$$

Applying Theorem 3.8 gives a criterion when the bottom of the spectrum is positive and Theorem 3.11 even gives explicit estimates.

4.1. **Discreteness of spectrum and eigenvalue asymptotics.** In this subsection we address the question under which circumstances the spectrum of Δ is purely discrete. We prove an analogue of a theorem of Donnelly-Li, [DL], for Riemannian manifolds that curvature tending to $-\infty$ outside increasing compacta implies emptiness of the essential spectrum.

For a selfadjoint operator T we denote the eigenvalues below the essential spectrum in increasing order counted with multiplicity by $\lambda_n(T)$, $n \geq 0$. For two sequences (a_n), (b_n) we write $a_n \sim b_n$ if there is $c > 0$ such that $c^{-1}a_n \leq b_n \leq ca_n$. We denote the maximal operator of multiplication by the face degree by D_F. That is D_F is an operator from

$\{\varphi \in \ell^2(F) \mid |\cdot|\varphi \in \ell^2(F)\}$ to $\ell^2(F)$ acting as

$$D_F\varphi(f) = |f|\varphi(f).$$

We call X *balanced* if there is $C > 0$ such that $Cm_E(f) \geq M_E(f)$ and *strongly balanced* if

$$\sup_{K\subseteq F \text{ finite}} \inf_{f\in F\setminus K} \frac{m_E(f)}{M_E(f)} = 1.$$

That means that C in the definition of balanced equals 1 asymptotically. An analogue of the Donnelly-Li result reads as follows.

Theorem 4.1. *Let $X = (V, E, F)$ be a* locally finite *polygonal complex with planar substructuresthat is balanced. If*

$$\kappa_\infty := \inf_{K\subseteq F \text{ finite}} \sup_{f\in F\setminus K, \Sigma\in\mathcal{A}, f\in\Sigma} \kappa^{(\Sigma)}(f) = -\infty,$$

then the spectrum of Δ is purely discrete and $\lambda_n(\Delta) \sim \lambda_n(D_F)$. If, additionally, X is strongly balanced, then $\lambda_n(\Delta)/\lambda_n(D_F) \to 1$, as $n \to \infty$. Finally, under the additional assumption $M_E < \infty$, purely discrete spectrum of Δ implies $\kappa_\infty = -\infty$.

We like to mention that the result here holds for the *generally unbounded* discrete Laplacian. The first result on the essential spectrum of graphs analogous to Donnelly-Li was proved by Fujiwara [Fu] and he considered the *geometric or normalized* Laplacian. The very different spectral behavior of these two operators is discussed in [K1].

The proof of Theorem 4.1 is based on the following proposition.

Proposition 4.2. *Let $X = (V, E, F)$ be a locally finite polygonal complex with planar substructures. If*

$$a := \sup_{K\subseteq F \text{ finite}} \inf_{f\in F\setminus K} \frac{m_E(f)}{M_E(f)}\Big(1 - \frac{6}{|\partial f|}\Big) > 0,$$

then the spectrum of Δ is discrete if and only if

$$\sup_{K\subseteq F \text{ finite}} \inf_{f\in F\setminus K} |f| = \infty. \tag{2}$$

In this case,

$$(1 - \sqrt{1-a^2}) \leq \liminf_{n\to\infty} \frac{\lambda_n(\Delta)}{\lambda_n(D_F)} \leq \limsup_{n\to\infty} \frac{\lambda_n(\Delta)}{\lambda_n(D_F)} \leq (1 + \sqrt{1-a^2}).$$

Proof. The characterization of discreteness of spectrum follows from Corollary 3.12 and [K1, Thm. 2]. The asymptotics of eigenvalues follow combining Corollary 3.12 and [BGK, Thms. 2.2. and 5.3.]. □

Proof of Theorem 4.1. We observe that for all $\Sigma \in \mathcal{A}$ and $f \in F_\Sigma$

$$-\frac{|f|_\Sigma}{2} \leq \kappa^{(\Sigma)}(f).$$

Hence, $\kappa_\infty = -\infty$ implies $\sup_{K \subseteq F \text{ finite}} \inf_{f \in F \setminus K} |f| = \infty$. Combining this with the assumption that X is balanced with constant C implies that $a \geq 1/C$, where a is taken from Proposition 4.2. In the case of X being strongly balanced we have $a = 1$. Thus, the first part of the theorem follows from Proposition 4.2. Conversely, if $\kappa_\infty \geq -c > -\infty$ then there is a sequence of faces f_n with $d(f, f_n) \to \infty$ for any fixed face $f \in F$ and apartments Σ_n, $n \geq 0$, such that

$$-c < \kappa^{(\Sigma_n)}(f_n) \leq 1 - \frac{|f|_\Sigma}{6} \leq 1 - \frac{|f|}{6M_E},$$

where we used $|v|_\Sigma \geq 3$ which holds by (T2). We conclude that $|f_n|$ is uniformly bounded by some constant $c' > 0$. Thus, the essential spectrum of Δ starts below c' (confer [K1, Thm. 1]) and Δ does not have purely discrete spectrum. □

Example 3. The simplest example of a polygonal complex with planar substructuressatisfying the conditions of Theorem 4.1 is a planar tessellation $X = (V, E, F)$ with one apartment $\Sigma = X$ and root $o \in F$ such that $\lim_{n\to\infty} \inf_{f \in S_n} |\partial f| = \infty$. In this case we have

$$\kappa^{(\Sigma)}(f) \leq 1 - \frac{|\partial f|}{6},$$

and we see that $\kappa_\infty = -\infty$. Moreover, X is strongly balanced since we have $m_E(f) = M_E(f) = 1$. Therefore, the spectrum of Δ is purely discrete and $\lambda_n(\Delta)/\lambda_n(D_F) \to 1$.

Note that purely discrete spectrum can also be established by increasing $m_E(f)$ instead of $|\partial f|$ for all faces outside compact sets (by keeping the polygonal complex balanced) and applying Proposition 4.2 directly. The condition (2) follows then directly from $|f| \geq m_E(f)$.

4.2. **Unique continuation of eigenfunctions.** While unique continuation results hold in great generality for continuum models with very mild assumptions, there are very natural examples for graphs with finitely supported eigenfunctions, see [DLMSY] and various other references. In this subsection we prove that for non-positive curvature there are no finitely supported eigenfunctions.

Theorem 4.3. *Let* $X = (V, E, F)$ *be a* locally finite *polygonal complex with planar substructuressuch that* $\kappa_c^{(\Sigma)} \leq 0$ *for all* $\Sigma \in \mathcal{A}$*. Then,* Δ *admits no finitely supported eigenfunctions.*

Cases where we do not have finite supported eigenfunctions are therefore Example 1 and Examples 8-11.

In [KLPS, K2] results like Theorem 4.3 are found for the planar case and more general operators. Indeed, we consider here also nearest neighbor operators, where we even do not need local finiteness.

Definition 4.4. Let $X = (V, E, F)$ be a polygonal complex. We call A a *nearest neighbor operator* on X if there is $a : F \times F \to \mathbb{C}$

(NNO1) $a(f, g) \neq 0$ if $f \sim g$.
(NNO2) $a(f, g) = 0$ if $f \not\sim g$.
(NNO3) $\sum_{g \in F} |a(f, g)| < \infty$ for all $f \in F$.

and A acts as

$$A\varphi(f) = \sum_{g \in F} a(f, g)\varphi(g),$$

on functions φ in

$$\widetilde{D}(A) = \{\varphi : F \to \mathbb{C} \mid \sum_{g \in F} |a(f, g)\varphi(g)| < \infty \quad \forall f \in F\}.$$

The summability assumption (NNO3) guarantees that the functions of finite support are included in $\widetilde{D}(A)$. Clearly, the Laplacian introduced at the beginning of this section is a nearest neighbor operator, where we can also add an arbitrary potential to be in the general setting of Schrödinger operators. Theorem 4.3 is an immediate consequence of the following theorem.

Theorem 4.5. *Let $X = (V, E, F)$ be a polygonal complex with planar substructuressuch that $\kappa_c^{(\Sigma)} \leq 0$ for all $\Sigma \in \mathcal{A}$ and A be a nearest neighbor operator on X. Then A does not admits eigenfunctions supported within a distance ball B_n around a root $o \in F$.*

Proof. Let $\varphi \in \widetilde{D}(A)$ be an eigenfunction of A to the eigenvalue λ. Let k be t such that φ vanishes completely on all distance spheres at levels larger or equal than k from a root $o \in F$. Let $f_0 \in F$ be a face at distance $k - 1$. We want to show that then $\varphi(f_0) = 0$. Let Σ be an apartment containing o and f_0. Since we do not have cut-locus in any of the apartments due to non-positive sectional corner curvature, Theorem 3.1, there exists a face $g_0 \in F_\Sigma$ adjacent to f_0 with $d(o, g_0) = k$. By assumption, we have $\varphi(g_0) = 0$. Now, by convexity, all faces $f \in F$ with $d(f, o) = k - 1$ adjacent to g_0 lie within Σ. By Proposition 3.5 there can be at most two such faces, one of them equal to f_0. If there is only one such face, namely f_0, we conclude from the eigenfunction identity evaluated at g_0 that we have $\varphi(f_0) = 0$. If there are two such faces, say f_0, f_1, then we conclude from the eigenfunction identity evaluated at g_0 that $a(g_0, f_0)\varphi(f_0) = -a(g_0, f_1)\varphi(f_1)$. With the notation of [BP2, Section 2.2] the vertex v_0 in the intersection of $\overline{f_0}, \overline{f_1}$ and $\overline{g_0}$ has label b (with respect to the tessellation Σ). By [BP2, Cor. 2.7.] v_0 has two neighbors of label a^+ such that one is in intersection of $\overline{f_0} \cap \overline{g_0}$ and the other which we denote by v_1 is in the intersection of $\overline{f_1} \cap \overline{g_0}$. The label a^+ of v_1 implies that the face f_1 has another neighbor g_1 in S_k. By assumption $\varphi(g_1) = 0$ and applying the same arguments to g_1 we find $f_2 \in S_{k-1} \cap F_\Sigma$, $f_2 \sim g_1$ such that $a(g_1, f_1)\varphi(f_1) = -a(g_1, f_2)\varphi(f_2)$. Proceeding inductively we find the sequences $(f_0, \dots, f_n)$, $f_0 = f_n$, and $(g_0, \dots, g_n)$, $g_0 = g_n$ of faces in Σ that form a closed boundary walk and boundary vertices $(v_0, \dots, v_{2n})$, $v_0 = v_{2n}$, with labels $b, a^+, b, a^+, b, \dots$. However, this is geometrically impossible [KLPS, Prop. 13]. Hence, we conclude $\varphi(f_0) = 0$. As this argument applies for all faces in S_{k-1} we deduce that φ vanishes on S_{k-1}. Repeating this argument for S_{k-j}, $j = 2, \dots, k$ yields that φ vanishes on B_k and thus by assumption on F. We finished the proof. $\square$

We conclude this subsection by giving examples of tessellations with negative sectional face curvature that admit finitely supported eigenfunctions. This shows the assumption in the theorem cannot be modified to negative sectional face curvature instead of non-positive sectional corner curvature.

Example 4. Let Σ_n, $n \geq 3$, be a bipartite tessellation of the plane $\mathbb{R}^2$ with squares as follows. Specifically, there are two infinite sets of vertices V_1 and V_2, where the vertices in V_1 have degree $2n$ and the vertices in V_2 have degree 3. The tessellation Σ_n is now given such that vertices in V_1 are only connected to vertices in V_2 and vice versa. Hence, each face contains two vertices of V_1 and two of V_2. See Figure 3 for the tessellation Σ_4, realized in the hyperbolic Poincaré unit disk.

The face curvature is then given by

$$\kappa(f) = 1 - \frac{|f|}{2} + \sum_{v \in f} \frac{1}{|v|} = 1 - 2 + \frac{2}{3} + \frac{2}{2n} = -\frac{n-3}{3n}.$$

For $n > 3$ the face curvature is negative and in the interval $(-1/3, 0)$. On the other hand, we have for the corner curvatures

$$\kappa_c(v_1, f) = -\frac{n-2}{4n}, \quad \kappa_c(v_2, f) = \frac{1}{12} > 0,$$

with $v_1 \in V_1$ and $v_2 \in V_2$ and $v_1, v_2 \in \overline{f}$. Moreover, for a vertex with degree $2n$ let $F_0 = \{f_1, \dots, f_{2n}\}$ be the faces around it in cyclic order.

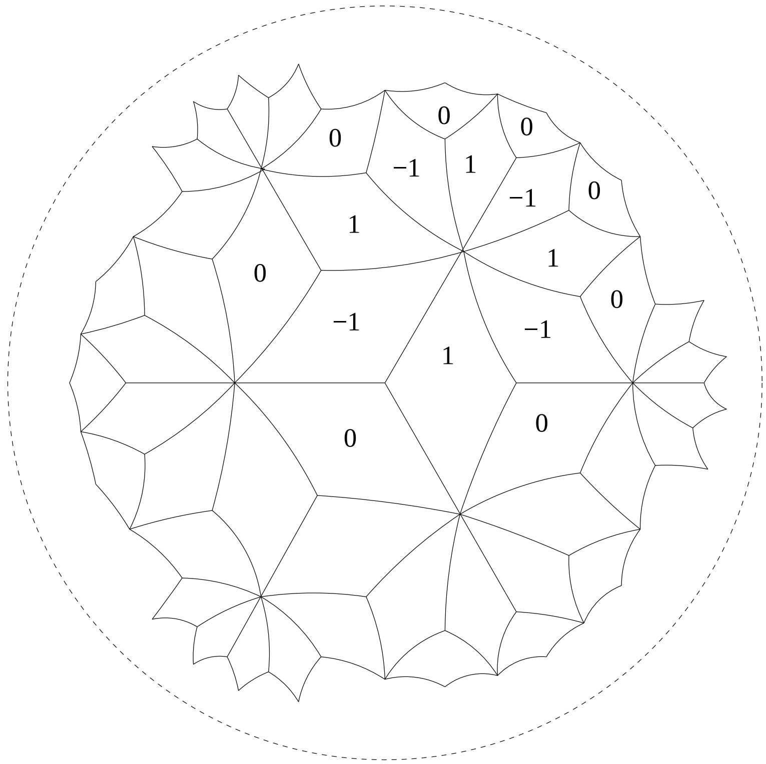

FIGURE 3. Part of the tessellation Σ_4 with a finitely supported eigenfunction. Faces with no entry are chosen to take the value 0.

Let a function φ with support in F_0 be given such that $\varphi(f_{2j}) = 1$ and $\varphi(f_{2j-1}) = -1$ for $j = 1, \ldots, n$. Then, φ is a finitely supported eigenfunction of Δ to the eigenvalue 6. Looking at the dual regular graph ${\Sigma_n}^*$ with constant vertex degree 4, we see that the Δ-eigenfunction φ of Σ_n corresponds to an eigenvector of the adjacency matrix of ${\Sigma_n}^*$ to the eigenvalue -2.

4.3. **The Dirichlet problem at infinity.** We assume that $X = (V, E, F)$ is a polygonal complex with planar substructureswith strictly negative sectional corner curvature and that (PCPS1*) holds. Moreover, we assume $M_V, M_F < \infty$. Then we know from Theorem 3.6 that (X, d) is Gromov hyperbolic and that the boundary $X(\infty)$ carries a natural topological structure. Moreover, $\overline{X} = X \cup X(\infty)$ is compact (see [BH, Prop. III.H.3.7(4)]). Given a function $F \in C(X(\infty))$, the *Dirichlet problem at infinity* asks whether there is a unique continuous function $f \in C(\overline{X})$ which agrees with F on $X(\infty)$ and such that the restriction $f_0 = f|_X$ is harmonic (i.e., $\Delta f = 0$). The existence of such a function f is the main problem since uniqueness of the solution follows from the maximum principle. Applying the general theory of [Anc] to Theorem 3.8 answers this question positively.

Theorem 4.6. *Let $X = (V, E, F)$ be a polygonal complex with planar substructuressuch that $\kappa_c^{(\Sigma)} < 0$ for all $\Sigma \in \mathcal{A}$. Assume that X additionally satisfies (PCPS1*) and $M_V, M_F < \infty$. Then (X, d) is Gromov hyperbolic and the* Dirichlet problem at infinity is solvable *on X. In particular, there are infinitely many linearly independent bounded non-constant harmonic functions on X.*

Spaces where the theorem is applicable and the Dirichlet problem at infinity can be solved are all locally finite hyperbolic buildings with regular hyperbolic polygons as faces.

Proof. Gromov hyperbolicity of (X, d) follows from Theorem 3.6. Let P denote the averaging operator $P\varphi(f) = \frac{1}{|f|}\sum_{g\sim f}\varphi(g)$. It is easy to see that P satisfies the properties of [Anc, Assumptions 1.1]. Note further that a function φ of F satisfies $\Delta\varphi = 0$ if and only if $P\varphi = \varphi$. We know from Theorem 3.8 that the Cheeger constant α of X is positive. We conclude from [Anc, Prop. 4.4] that the crucial condition (*) in [Anc] is therefore satisfied. Moreover, we deduce from Gromov hyperbolicity of (X, d) and [Anc, Cor. 6.10] that the assumptions (G.A) in [Anc, Thm. 5.2] are satisfied and the Gromov compactification agrees with the P-Martin compactification of X. Then the statement follows from [Anc, Cor. 5.4]. $\square$

Remark 4.7. We already mentioned in the proof of Theorem 4.6 that Ancona's theory also implies that the Gromov and the geodesic boundary of (X, d) agrees with the *P-Martin boundary* boundary. The P-Martin boundary is an analytically defined boundary based on asymptotic properties of Green's functions $G : F \times F \to [0, \infty)$ (see [Anc, Section V]).

5. Examples

In this section, we will mainly focus on non-positively curved polygonal complexes with planar substructures. Rich classes of examples are provided by 2-dimensional Euclidean and hyperbolic buildings. Before we consider these classes more closely, let us start with particularly simple examples.

5.1. **Simple examples and basic notions.** As mentioned earlier, every planar tessellation $\Sigma = (V, E, F)$ is trivially a polygonal complex with planar substructureswith just one apartment, i.e., $\mathcal{A} = \{\Sigma\}$.

Next, let us introduce *morphisms* between two complexes X_1 and X_2: These are continuous maps from X_1 to X_2 mapping k-cells of X_1 homeomorphically to k-cells of X_2, for all k. A homeomorphism $f : X_1 \to X_2$ is an *isomorphism* if both f and f^{-1} are morphisms. In this case we call X_1 and X_2 isomorphic complexes.

Example 5 ("Book")**.** Let $\mathcal{H} = (V, E, F)$ be the tessellation of the upper half space $\{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$ where

$$V = \{(x, y) \in \mathbb{Z}^2 \mid y \geq 0\},$$

E is the set of horizontal and vertical straight Euclidean line segments of length 1 connecting two vertices of V, and F is the set of all Euclidean unit squares with vertices in V. Let k be an integer ≥ 2 and X_k be the polygonal complex obtained by taking k copies of $\mathcal{H}$ and identifying them along their boundaries $\mathbb{R} \times \{0\} \subset \mathcal{H}$. We can think of X_k as a book with the copies of $\mathcal{H}$ as its pages. Note that the union of any two pages can be understood as a tessellation of the plane by squares. Every such choice represents an apartment of the polygonal complex with planar substructuresX_k. It is straightforward to see that X_k has non-positive sectional corner curvature. Books can also be obtained by combining pages with more general and different polygonal structures by using isomorphisms between their boundaries (considered as 1-dimensional cell complexes). Moreover, it is also possible to consider books with *infinitely many* pages, they are obviously non-locally finite polygonal complexes with planar substructures.

Example 6. Let us present a **non-example** of a polygonal complex with planar substructures. Let $X = (V, E, F)$ be given by $V = \mathbb{Z}^3$, E be the set of straight Euclidean line segments of length 1 connecting two vertices of V, and F be the set of all unit squares with vertices in V. X is obviously a polygonal complex, but there does not exist a choice of apartments (planes tessellated by squares) satisfying both conditions (PCPS1) and (PCPS2). The set of all planes parallel to the coordinate planes does not satisfy (PCPS1). Thus we also need to declare certain topological planes which are bent to be apartments. But it is easy to see that the convexity property (PCPS2) is violated for any such bent plane.

A useful notion for the local combinatorial description of a a polygonal complexes is the link of a vertex, defined as follows:

Definition 5.1 (Link)**.** Let $X = (V, E, F)$ be a polygonal complex. The *(edge) link* $L(v)$ of a vertex $v \in V$ is a graph defined as follows: Every edge adjacent to v is represented by a vertex in $L(v)$, and two vertices w_1, w_2 in $L(v)$ are connected by an edge in $L(v)$ if the edges in X corresponding to w_1, w_2 are edges of a face f in F.

Polygonal complexes are often described via the type of their faces and the graphs appearing as links. It is known that for any $k \geq 6$ and $n \geq 3$ there is a continuum of non-isomorphic simply connected polygonal complexes $X = (V, E, F)$ such that $|\partial f| = k$ for all $f \in F$ and that the links $L(v)$ of all vertices $v \in V$ are the 1-skeletons of an n-simplex (see, e.g, [BB1, Thm. 1]). The link $L(v)$ of every vertex v in Example 1 is the classical Heawood graph which is a regular graph with 14 vertices of degree 3. The Heawood graph is a generalized 3-gon and the flag-graph of a finite projective plane. Generalized m-gons appear as links of Euclidean and hyperbolic buildings.

Definition 5.2 (Generalized m-gon)**.** Let $m \geq 2$ be an integer. A *generalized m-gon* is a connected bipartite graph of diameter m and of girth $2m$ such that each vertex has degree ≥ 2.

The adjacency matrices of regular generalized m-gons have interesting spectral properties. In particular, they are Ramanujan graphs (see [Lub, Section 8.3]). Spectral properties of the links $L(v)$ of vertices of 2-dimensional simplicial complexes were also very useful to obtain Kazdhan property (T) for groups acting cocompactly in these complexes (see [BaSw]). Closely related to the links are the chamber links:

Definition 5.3 (Chamber link)**.** Let $X = (V, E, F)$ be a polygonal complex. The *chamber link* $L_{Ch}(v)$ of a vertex $v \in V$ is a graph defined as follows: Every chamber adjacent to v is represented by a vertex in $L_{Ch}(v)$ and two vertices w_1, w_2 in $L_{Ch}(v)$ are connected by an edge in $L_{Ch}(v)$ if the faces in X corresponding to w_1, w_2 are adjacent to each other via an edge in X adjacent to v.

The chamber links $L_{Ch}(v)$ are the line graphs of the links $L(v)$. Therefore, if $L(v)$ is a k-regular graph then the spectra of both graphs are closely related via the identity $\chi_{L_{ch}(v)}(\lambda) = (\lambda + 2)^r \chi_{L(v)}(\lambda + 2 - k)$ between the characteristic polynomials, where r is the difference $|E(L(v))| - |V(L(v))|$ (see, e..g, [Bi, Thm 3.8]).

Example 7. In the Euclidean building $X = (V, E, F)$ in Example 1, there are 14 edges emanating from every vertex $v \in V$ and 21 faces adjacent to v. Let us denote these 21 faces by $a_i, b_i, c_i, d_i, e_i, f_i, g_i$, $i = 1, 2, 3$, where $a_i, \dots, g_i$ are triangles with boundary labels $\{x_0, x_1, x_3\}$, $\dots$, $\{x_6, x_0, x_2\}$, respectively. Looking at one of the apartments containing v, we encounter the configuration illustrated in Figure 4.

The chamber link $L_{Ch}(v)$ is a graph with 21 vertices and can be realized as a tessellation of a flat torus. It has the structure illustrated

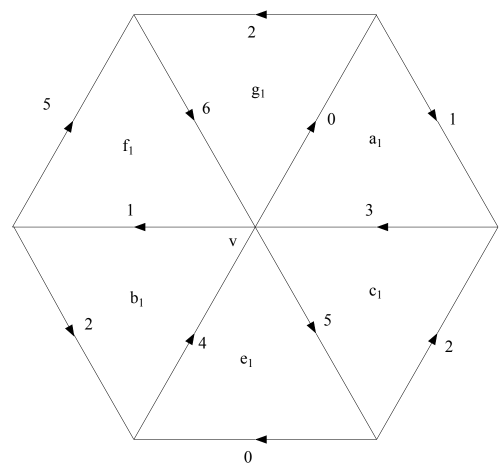

FIGURE 4. adjacent chambers around v within an apartment

in Figure 5 , where we identify the thick upper and lower side and thick left and right hand side.

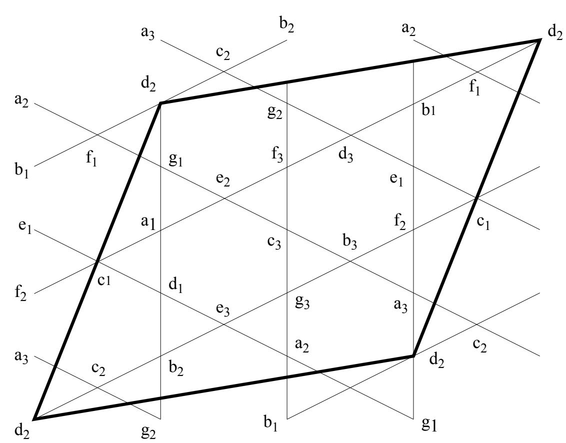

FIGURE 5. The graph $L_{Ch}(v)$

This tessellation of the flat torus has interesting properties: The set of faces consists of 14 triangles and 7 hexagons. For every apartment Σ with $v \in V_\Sigma$ there exists precisely one of the hexagons such that Σ contains also all six triangles represented by the vertex labels of this hexagon. Moreover, each of these hexagons shares precisely one joint vertex with every other hexagon. Moreover, $L_{Ch}(v)$ is Ramanujan and its adjacency matrix has the eigenvalues 4 (with multiplicity 1), $1 \pm \sqrt{2}$ (each with multiplicity 6), and -2 (with multiplicity 8). Each of the

7 hexagons contributes an eigenvector of the eigenspace $\mathbb{E}_{-2}$ to the eigenvalue -2: choose alternatively the values ± 1 along the vertices of this hexagon and the values 0 at all other vertices. These eigenvectors span a 7-dimensional subspace of $\mathbb{E}_{-2}$.

5.2. **Euclidean and hyperbolic buildings.** Let us give a quick introduction into 2-dimensional Euclidean and hyperbolic buildings, following essentially [GP]. In contrast to our Definition 2.1, the cells in the polygonal complexes used for Coxeter complexes and buildings have an additional metric structure, namely, the 1-cells are open Euclidean or hyperbolic geodesic segments and the 2-cells interiors are Euclidean or hyperbolic polygons (we restrict our considerations to compact ones), and the attaching maps are isometries (see also [BH, Sct. I.7.37]). We call an isometric isomorphism between two polygonal complexes an *isometry*, for simplicity. The closures of the 2-cells are called *chambers* of the polygonal complex.

Important *planar* polygonal complexes are *Coxeter complexes*, which we introduce first. Let $\mathbb{X}$ stand for either the Euclidean plane $\mathbb{R}^2$ or the hyperbolic plane $\mathbb{H}^2$. Let $P \subset \mathbb{X}$ be a compact polygon with k vertices such that the interior angle at vertex i is of the form π/m_i with $m_i \geq 2$. We call such a polygon P a *Coxeter polygon*. Each side of P is contained in a bi-infinite geodesic $g \subset \mathbb{X}$ and the corresponding reflection s_g along g is an element in the group if Euclidean/hyperbolic isometries $\mathrm{Iso}(\mathbb{X})$. Let $S = \{s_1, \dots, s_k\}$ be the set of reflections along all sides of P and let W be the group generated by these isometries. Then it is a well known fact due to H. Poincaré that W is a discrete subgroup of $\mathrm{Iso}(\mathbb{X})$ and that (W, S) has the presentation

$$W = \langle S \mid (s_i s_j)^{m_{ij}} = \mathrm{id} \rangle, \tag{3}$$

where $m_{ii} = 1$ for all i, $2 \leq m_{ij} < \infty$ if the sides $i \neq j$ meet in a vertex v, and $\frac{\pi}{m_{ij}}$ is the interior angle of P at v, and $m_{ij} = \infty$ if the sides $i \neq j$ do not meet in a vertex (i.e., no relation between s_i and s_j). Note that $m_{ij} = 2$ for $i \neq j$ means that the corresponding reflections s_i and s_j commute. A group with such a presentation (3) is called a *Coxeter group*. Moreover, P is a fundamental domain of W, i.e., the translates $\{gP \mid g \in W\}$ form an tessellation of $\mathbb{X}$, which is a planar polygonal complex in the above sense. We refer to it as $C(W, S)$ and call the polygon P the *generating polygon* of the Coxeter group. Moreover, every edge e in $\mathcal{C}(W, S)$ carries a label $i \in \{1, 2, \dots, k\}$, defined as follows: Let $g_1 P, g_2 P$ be the two translates of P such that $e \in \overline{g_1 P} \cap \overline{g_2 P}$ (these translates are uniquely determined by property (T1)). Then $g_2 = g_1 s_i$. Another way of understanding the label i is that the reflection along the geodesic containing the edge e in $\mathcal{X}$ is conjugate to s_i within the Coxeter group W.

Definition 5.4 (Building)**.** Let $\mathbb{X} \in \{\mathbb{R}^2, \mathbb{H}^2\}$, $P \subset \mathbb{X}$ be a Coxeter polygon and (W, S) be the associated Coxeter group. A (2-dimensional) *building of type* $(W, S = \{s_1, \ldots, s_k\})$ is a polygonal complex $X = (V, E, F)$, together with a labeling $E \to \{1, \ldots, k\}$ and a set $\mathcal{A}$ of subcomplexes whose elements $\Sigma = (V_\Sigma, E_\Sigma, F_\Sigma)$ are called *apartments*, with the following properties:

(B1) For any two cells of X there is an apartment containing both of them.
(B2) If Σ_1 and Σ_2 are two apartments containing two cells c_1, c_2 of X, then there exists a label-preserving isometry $f : \Sigma_1 \to \Sigma_2$ which fixes c_1 and c_2 pointwise.
(B3) Each apartment Σ is label-preserving isometry to the planar tessellation $\mathcal{C}(W, S)$.

The building X is called *Euclidean* or *hyperbolic* if $\mathbb{X} = \mathbb{R}^2$ or $\mathbb{X} = \mathbb{H}^2$. A building is called *thick* if every edge is contained in at least three chambers. A building which is not thick is called a *thin* building.

Disregarding the additional Euclidean or hyperbolic structure of the cells of a building, we can view it and its apartments as polygonal complexes in the sense of Definitions 2.1 and 2.2. Since the apartments of buildings are always convex (see [GP, p. 164(l. -5)] and also [Br, Prop. on p. 88] or [Ga, Prop. on p. 59] for simplicial buildings), we see that every buildings is a polygonal complex with planar substructures.

Let us first mention examples of Euclidean buildings. Example 1 in Subsection 2.1 was an example of a thick Euclidean building based on an equilateral Euclidean triangle $P \subset \mathbb{R}^2$. Note that there are only three choices of Euclidean Coxeter triangles with interior angles $\{\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\}$, $\{\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}\}$ or $\{\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}\}$. Each of these choices leads to a unique Coxeter group and a class of Euclidean buildings, and the corresponding Coxeter group and buildings are said to be of type $\widetilde{A}_2$, $\widetilde{C}_2$ and $\widetilde{G}_2$, respectively. Example 1 is therefore a Euclidean building of type $\widetilde{A}_2$. Even though there are only three types, a classification of all buildings of one of these types is impossible because of their abundance (see [Ro, p. 157]). Note that while Euclidean buildings of all three types have vanishing sectional face curvature, only Euclidean buildings of type $\widetilde{A}_2$ have non-positive sectional corner curvature amongst these three types. Next, we consider a natural class of Euclidean buildings based on a square.

Example 8 (Product of trees)**.** Let $r, s \geq 2$ and T_r and T_s be infinite regular metric trees of vertex degrees r and s, respectively. All edge lengths are chosen to be 1. We can think of one of the trees, say T_r, to be *horizontal* and the other one to be *vertical*. Then the product $T_r \times T_s$ carries a natural structure of a thick Euclidean building $X = (V, E, F)$ with $P = [0,1]^2 \subset \mathbb{R}^2$. The set V consists of all pairs (x, y) where x and y are vertices in T_r and T_s respectively. Two vertices $(x_1, y_1), (x_2, y_2) \in V$ are connected by an edge in E, if either ($x_1 = x_2$ and $y_1 \sim_{T_s} y_2$) or ($y_1 = y_2$ and $x_1 \sim_{T_r} x_2$). In the first case we call the edge in E horizontal and in the second case we call the edge in E vertical. The chambers are the unit squares with boundary vertices $(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2)$ for any choice $x_1 \sim_{T_r} x_2$ and $y_1 \sim_{T_s} y_2$. All vertices in $T_r \times T_s$ have degree $r+s$ (with r emanating horizontal and s emanating vertices edges). Moreover, a vertical edge is contained in precisely r chambers while a horizontal edge is contained in precisely s chambers. Two bi-infinite combinatorial geodesics $g_1 \subset T_r$ and $g_2 \subset T_s$ can be viewed as infinite regular trees of vertex degrees 2 and the corresponding subcomplex $\Sigma = \Sigma_{g_1, g_2} = g_1 \times g_2$ is isomorphic to a regular tessellation of $\mathbb{R}^2$ by unit squares. We choose $\mathcal{A}$ to be the set of all those subcomplexes. We then have for every corner $(v, f) \in C_\Sigma$:

$$\kappa_c^{(\Sigma)}(v, f) = \frac{1}{|v|_\Sigma} - \frac{1}{2} + \frac{1}{|f|_\Sigma} = \frac{1}{4} - \frac{1}{2} + \frac{1}{4} = 0,$$

i.e., X has vanishing sectional corner and face curvature.

Finally, let us consider some examples of hyperbolic buildings. Note first that while all hyperbolic buildings have negative sectional face curvature they do not always have also non-positive sectional corner curvature: Consider a tessellation of the hyperbolic plane by triangles with interior angles $\frac{\pi}{r}, \frac{\pi}{s}, \frac{\pi}{t}$ with $r, s, t \geq 2$ and

$$\frac{1}{r} + \frac{1}{s} + \frac{1}{t} < 1.$$

This tessellation is a thin hyperbolic building and it has non-positive corner curvature if and only if $r, s, t \geq 3$. Henceforth, we only consider hyperbolic buildings with *regular polygons* as faces. These hyperbolic buildings have always negative sectional corner curvature. We start with hyperbolic buildings whose faces are right-angled polygons.

Example 9 ("Bourdon buildings")**.** Let $p \geq 5$ and $q \geq 3$. Then there is a *unique* hyperbolic building $X_{p,q}$ with the following properties (see [Bou]): All chambers are regular right-angled hyperbolic p-gons and the link $L(v)$ of every vertex is the complete bipartite graph $K_{q,q}$. Since every edge of $X_{p,q}$ lies in q chambers, $X_{p,q}$ is a thick building.

This polygonal complex with planar substructureshas constant negative sectional corner curvature:

$$\kappa_c^{(\Sigma)}(v,f) = \frac{1}{|v|_\Sigma} - \frac{1}{2} + \frac{1}{|f|_\Sigma} = \frac{1}{p} - \frac{1}{4} < 0.$$

A way to obtain compact polygonal complexes is to label the oriented edges of finitely many Euclidean or hyperbolic polygons and to identify edges with the same labels (these edges must obviously have the same length). We call such a compact polygonal complex a *polyhedron.* Its universal covering is then a polygonal complex with a cocompact group action. The links of its vertices provide useful information in the decision whether its universal covering is a Euclidean or hyperbolic building. Example 1 is based on this approach.

Generalized m-gons coincide exactly with spherical buildings of rank 2 (see [GP, Section 3.1.1]). Combining this fact with [GP, Cor. 2.4 and Thme. 2.5] shows the following fact: Let $k \geq 3$ and $m \geq 2$ satisfying $mp > 2m + p$. Assume that X_0 is obtained by identifying the sides of several regular hyperbolic k-gons with angles $\frac{\pi}{m}$ such that the links of all vertices agree and are a generalized m-gon. Then the universal covering $X = (V, E, F)$ of X_0 is a hyperbolic building. Moreover, the sectional corner curvature at every corner $(v, f) \in V \times F$, $v \in \overline{F}$, is given by

$$\kappa_c^{(\Sigma)}(v,f) = \frac{1}{|v|_\Sigma} - \frac{1}{2} + \frac{1}{|f|_\Sigma} = \frac{1}{2m} - \frac{1}{2} + \frac{1}{p} = \frac{2m+p-mp}{2mp} < 0.$$

Example 10 (see [Vd, VdK])**.** Let $\mathcal{K}$ be a *polygonal presentation* associated to the disjoint connected bipartite graphs $G_1, \dots, G_n$ in the sense of [VdK, Def. 1.2]. Assume that all G_i are copies of the same generalized m-gon. Every cyclic k-tuple $(x_1, \dots, x_k) \in \mathcal{K}$ provides a clockwise labeling of the oriented edges of a regular hyperbolic k-gon with angles $\frac{\pi}{m}$. If $mp > 2m+p$ then the universal covering of the polyhedron corresponding to $\mathcal{K}$ is a hyperbolic building. This approach provides examples of hyperbolic buildings with k-sided chambers for arbitrary $k \geq 3$ *with a cocompact group action.* The triangle presentations given in [VdK] lead to explicit hyperbolic buildings with regular triangles as faces.

Techniques of Haglund [Hag] provide us with the following result.

Example 11 (see [GP, Thme. 3.6])**.** Let $P \subset \mathbb{H}^2$ be a regular hyperbolic polygon with angles $\frac{\pi}{m}$, $m \geq 3$ and an even number of sides. Let

(W, S) be the associated Coxeter group. Let L be an algebraic generalized m-gon over a field with large enough cardinality[2]. Then there are *uncountable many* hyperbolic buildings of type (W, S) with faces isometric to P such that all links are isomorphic to L.

5.3. **Buildings with maximal apartment systems.** Every building comes with a choice of apartment system. This choice is not unique as the following one-dimensional example shows which has easy analogues in higher dimensions.

Example 12. Let $T_r = (V, E)$ be a regular metric tree of edge length 1 and vertex degree $r \geq 3$, and let $\phi : E \to \{1, 2, \ldots, r\}$ be a labeling of the edges such that the r edges emanating from every vertex carry pairwise different labels. Let $\mathcal{A}$ be the set of all bi-infinite paths $(f_k)_k$ such that the bi-infinite sequence $x_k = \phi(f_k)$ has no doublings (i.e., $x_k = x_{k+1}$ for some $k \in \mathbb{Z}$) and is periodic (i.e., there exists $t \geq 1$ such that $x_{k+t} = x_k$ for all $k \in \mathbb{Z}$). Then it is easy to see that T_r together with $\mathcal{A}$ as its system of apartments forms a one-dimensional Euclidean building. Another choice $\mathcal{A}'$ of an apartment system is the set of all bi-infinite paths $(f_k)_k$ such that the bi-infinite sequence $x_k = \phi(f_k)$ has no doublings and becomes eventually periodic (i.e., there is a $t \in \mathbb{N}$ such that $x_{k+t} = x_k$ for all $k \geq t$). It is obvious that $\mathcal{A}'$ is a strictly bigger apartment system than $\mathcal{A}$.

Since any union of apartment systems of a building $X = (V, E, F)$ forms again an apartment system, there exists a unique maximal system of apartments (this maximal apartment system is always chosen in [GP, Dfn. 1.4]). In order to have examples of polygonal complexes with planar substructuressatisfying the stronger axiom (PCPS1*), we need to choose buildings with maximal apartment systems. Then we have the following fact.

Theorem 5.5. *Every Euclidean or hyperbolic building with a maximal apartment system satisfies the axioms (PCPS1*), (PCPS2), (PCPS3).*

Let $X = (V, E, F)$ be a Euclidean or hyperbolic building with a maximal apartment system $\mathcal{A}$ and associated Coxeter group (W, S). For the proof of Theorem 5.5 we introduce the following W-valued (non-symmetric) distance function $\delta : F \times F \to W$: Let $f, f' \in F$ and $\Sigma \in \mathcal{A}$ an apartment such that $f, f' \in F_\Sigma$. We can then identify Σ with the Coxeter complex $\mathcal{C}(W, S)$ via a label-preserving isometry ψ and can then think of the group (W, S) acting on the faces of Σ. Let

[2]The term "algebraic" refers to the fact that the m-gon is based on a Chevalley quadruple, see [GP, Dfn. 3.3]

$f_0 \in F_\Sigma$ be the polygon corresponding to the generating polygon of the Coxeter group via the identification ψ. Then there exist $g, g' \in W$ such that $f = gf_0$ and $f' = g'f_0$ and we set $\delta(f, f') = g^{-1}g'$. It can be checked that this definition is independent of the choice of apartment and the choice of ψ (for the arguments see, e.g., [Ga, Lemma on p. 243]). Writing $g^{-1}g' \in G$ as a product of reflections $s_{i_1} \cdots s_{i_k}$ with $s_{i_j} \in S$ of minimal word length k yields a geodesic $\widetilde{f_j} = (gs_{i_1} \cdots s_{i_j})f_0$ in Σ such that $f = \widetilde{f_0}$ and $f' = \widetilde{f_k}$. Given two subsets $F_1, F_2 \subset F$, a *strong isometry* from F_1 to F_2 is a map $\alpha : F_1 \to F_2$ satisfying $\delta(\alpha(f), \alpha(f')) = \delta(f, f')$ for all $f, f' \in F_1$. To finish the proof, we need the following result (see [Ga, Thm. on p. 247], the proof given there is for simplicial buildings but the result carries over verbatim to the polygonal case).

Theorem 5.6. *Let $X = (V, E, F)$ be a Euclidean or hyperbolic building and $F_0 \subset F$. If F_0 is strongly isometric to a subset of an apartment, then F_0 is contained in an apartment in the maximal apartment system of X.*

To finish the proof we consider a one-sided infinite geodesic $F_0 = (f_k)_{k \geq 0} \subset F$ and choose a generating polyon $f'_0 \in F_\Sigma$ of the Coxeter group of an apartment $\Sigma \in \mathcal{A}$ via a fixed isometric isomorphism. Then we find a one-sided infinite sequence $(s_{i_k})_{k \geq 0}$ such that

$$\delta(f_0, f_k) = s_{i_0} s_{i_1} \cdots s_{i_k} \in W, \tag{4}$$

since f_0, f_k lie in a joint apartment and the description (4) is independent of the choice of apartment. It is then easy to check that the map $\alpha : F_0 \to F_\Sigma$,

$$\alpha(f_k) = s_{i_0} s_{i_1} \cdots s_{i_k} f'_0$$

is a strong isometry onto its image. Applying Theorem 5.6 finishes then the proof of Theorem 5.5. $\square$

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