1. \( x_n \) is a Cauchy sequence: Note that \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < \infty \). For every \( \epsilon > 0 \) there is an \( n_0 \) such that \( \sum_{n=n_0}^{\infty} \frac{1}{n^2} < \epsilon \), and therefore, for \( n, m \geq n_0, m \geq n \) :

\[
d(x_n, x_m) \leq d(x_n, x_{n+1}) + \cdots + d(x_{m-1}, x_m) \leq \sum_{i=n}^{m} \frac{1}{i^2} < \epsilon.
\]

2. Assume that \( f : M \rightarrow M' \) is continuous and \( U \subset M' \) is open. Let \( x \in f^{-1}(U) \). Since \( U \) is open, there exists an \( \epsilon > 0 \) such that \( B(f(x), \epsilon) \subset U \). Since \( f \) is continuous, there exists a \( \delta > 0 \) such that \( f(z) \in B(f(x), \epsilon) \) for all \( z \in M \) with \( d(z, x) < \delta \). But this means that \( B(x, \delta) \subset f^{-1}(U) \). Therefore, \( f^{-1}(U) \) is open.

Assume \( f : M \rightarrow M' \) satisfies \( f^{-1}(U) \) open in \( M \) for all open \( u \subset M' \). Let \( x \in M \). We want to prove continuity of \( f \) at \( x \). Given an \( \epsilon > 0 \), \( B := B(f(x), \epsilon) \subset M' \) is open. Then \( f^{-1}(B) \) is open in \( M \) and contains \( x \). Therefore, there exists a \( \delta > 0 \) such that \( B(x, \delta) \subset f^{-1}(B) \). But this means that \( d'(f(y), f(x)) < \epsilon \) for all \( y \in M \) with \( d(y, x) < \delta \).

3. (a) Look at \( g(x) = f(x) - x \). Then \( g(a) \geq 0 \) and \( g(b) \leq 0 \), so there must be a \( x \in [a, b] \) with \( g(x) = 0 \). This implies \( f(x) = x \).

(b) Since \( f'(x) < 1 \) for all \( x \in [a, b] \) and \( |f'(x)| \) is continuous on \( [a, b] \), it attains its maximum \( M \) on \( [a, b] \), which must satisfy \( M < 1 \). Using the Mean Value Theorem, we obtain

\[
|f(x) - f(y)| \leq |f'(\xi)| \cdot |x - y| \leq M \cdot |x - y|,
\]

for some \( \xi \) between \( x \) and \( y \). This means that \( f : [a, b] \rightarrow [a, b] \) is a contraction on the metric space \( (M, d) = ([a, b], d(x, y) = |x - y|) \). The statement of the exercise is then just an application of the Contraction Mapping Principle.

(c) Choose \( f(x) = a + b - x \). Then \( f'(x) = -1 \). Choose, e.g. \( x_0 = a \), then we have \( x_n = b \) for all odd \( n \) and \( x_n = a \) for all even \( n \).

4. We have \( F(x, t) = 2tx \) and

\[
|F(x, t) - F(y, t)| = 2|t| \cdot |x - y|,
\]

and if we restrict \( t \) to a finite interval \( (-C, C) \), we have Lipschitz continuity of \( F \) in the \( x \) variable with constant \( L = 2C \). Let \( \delta_0 \equiv c \). We
obtain

\[
\begin{align*}
\beta_1(t) &= c + \int_0^t 2scds = c + t^2c, \\
\beta_2(t) &= c + \int_0^t 2s(c + s^2c)ds = c + t^2c + \frac{t^4}{2}c, \\
\beta_3(t) &= c + t^2c + \frac{t^4}{2}c + \frac{t^6}{3!}c.
\end{align*}
\]

This suggests that the (unique) solution might be \( x(t) = ce^{t^2} \). A check shows: \( \dot{x}(t) = 2tce^{t^2} = 2tx(t) \) and \( x(0) = c \).