1. Obviously, \( \frac{\partial \varphi}{\partial u_1} \times \frac{\partial \varphi}{\partial u_2}(u) \in \mathbb{R}^3 \) is perpendicular to the vectors \( v_1 := \frac{\partial \varphi}{\partial u_1}(u) \in \mathbb{R}^3 \) and \( v_2 := \frac{\partial \varphi}{\partial u_2}(u) \in \mathbb{R}^3 \). So it remains to show that \( c'(0) \) is a linear combination of these vectors \( v_1, v_2 \). For this we observe that \( c : (a, b) \rightarrow S \) with \( c(0) = \varphi(u) \) is the image of a curve \( \tilde{c} : (a, b) \rightarrow U \) under the map \( \varphi \). (We simply choose \( \tilde{c} := \varphi^{-1} \circ c \).) Moreover, we have \( \tilde{c}(0) = u \). Denote the components of \( \tilde{c} \) by \( \tilde{c}_1, \tilde{c}_2 \). By the chain rule, we have

\[
c'(0) = (\varphi \circ \tilde{c})'(0) = \frac{\partial \varphi}{\partial u_1}(\tilde{c}(0))\tilde{c}'_1(0) + \frac{\partial \varphi}{\partial u_2}(\tilde{c}(0))\tilde{c}'_2(0) = c'_1(0)v_1 + c'_2(0)v_2.
\]

Thus \( c'(0) \in \text{span}\{v_1, v_2\} \), which we wanted to show.

2. We have \( c'(t) = (1 - \cos t, \sin t) \) and

\[
\|c'(t)\|_2^2 = 2 - 2\cos t = 2(1 - \cos(t/2 + t/2)) = 2(1 + \sin^2(t/2) - \cos^2(t/2)) = 4\sin^2(t/2).
\]

This implies that \( \|c'(t)\|_2 = 2\sin(t/2) \) and

\[
L(c) = \int_0^{2\pi} \|c'(t)\|_2 dt = \int_0^{2\pi} 2\sin(t/2) dt = 4 \int_0^{\pi} \sin(s) ds = 8.
\]

3. The coefficient functions of \( \omega \) are \( f_1(x, y) = -\frac{y}{x^2 + y^2} \) and \( f_2(x, y) = \frac{x}{x^2 + y^2} \). We obtain

\[
\frac{\partial f_1}{\partial y}(x, y) = -\frac{(x^2 + y^2) - 2y^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2},
\]

\[
\frac{\partial f_2}{\partial x}(x, y) = \frac{(x^2 + y^2) - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}.
\]

This implies closedness of \( \omega \).

Since \( c'(t) = (-r \sin t, r \cos t) \), we obtain

\[
dx(c'(t)) = -r \sin t, \quad dy(c'(t)) = r \cos t.
\]

This implies that

\[
\int_c \omega = \int_0^{2\pi} \omega_{c(t)}(c'(t)) dt = \int_0^{2\pi} -\frac{r \sin t}{r^2} dx(c'(t)) + \frac{r \cos t}{r^2} dy(c'(t)) dt = \int_0^{2\pi} \sin^2 t + \cos^2 t dt = 2\pi.
\]

If we had \( \omega = df \), then Lemma 6.10 would tell us that

\[
\int_c \omega = f(c(2\pi)) - f(c(0)) = f(r, 0) - f(r, 0) = 0.
\]

This contradicts to \( \int_c \omega = 2\pi \).
4. The coefficient functions of $\omega$ are $f_1(x, y) = 2xy^3$ and $f_2(x, y) = 3x^2y^2$. We obtain

$$\frac{\partial f_1}{\partial y}(x, y) = 6xy^2,$$
$$\frac{\partial f_2}{\partial x}(x, y) = 6xy^2.$$

This implies closedness of $\omega$. Using Poincaré’s Lemma, we conclude that $\omega$ is exact. Obviously, we have $\omega = df$ with $f(x, y) = x^2y^3$. Using Lemma 6.10, we conclude that

$$\int_c \omega = \int_c df = f(x, y) - f(0, 0) = x^2y^3 = x^8 = y^4.$$