1. Determine the critical points of
   (a) \( f : \mathbb{R}^2 \to \mathbb{R}^3 \) given by \( f(x, y) = (x^2, 2x + e^x \cos(y), xy \sin(xy)) \).
   (b) \( g : \mathbb{R}^3 \to \mathbb{R}^2 \) given by \( g(x, y, z) = (2x^2 + (y - 1)^2, z(\cos(y) - 1)) \).

2. Let \( M = \{(x, y, z) \in \mathbb{R}^3 \mid x^4 + y^2 + 2z^2 = 4\} \).
   (a) Show that \( M \) is a manifold.
   (b) For \( p = (-1, 1, 1) \), determine the tangent space \( T_pM \).

3. Show that
   \[
   M = \{(x, y, z) \in \mathbb{R}^3 \mid (x - 1)^2 + y^2 = 5, y = z\}
   \]
   is a compact manifold and the extremal values of \( f(x, y, z) = x^2 + y^2 + z \)
on \( M \) are 11 and 1.

4. (a) Find the point of the sphere \( x^2 + y^2 + z^2 = 1 \) which is at the greatest
distance from the point \((1, 2, 3) \in \mathbb{R}^3 \).
   (b) Find the rectangle of greatest perimeter inscribed in the ellipse
   \[
   \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.
   \]

5. Let \( p, q > 1 \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \).
   (a) Show that
   \[
   1 \leq \frac{1}{p} u^p + \frac{1}{q} v^q
   \]
   for all positive numbers \( u, v \) with \( u \cdot v = 1 \).
   **Hint:** Lagrange multipliers.
   (b) Show that
   \[
   uv \leq \frac{1}{p} u^p + \frac{1}{q} v^q
   \]
   for all \( u, v \geq 0 \).
   (c) (Hölder’s Inequality) Let \( x, y \in \mathbb{R}^n \). Show that
   \[
   \sum_{i=1}^{n} |x_i y_i| \leq \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \cdot \left( \sum_{i=1}^{n} |y_i|^q \right)^{\frac{1}{q}}.
   \]
   **Hint:** Use \( u = \frac{|x_i|}{(\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}} \).
(d) Let $p > 1$. Show that

$$\|x\|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

defines a norm on $\mathbb{R}^n$.

**Hint:** Write

$$|x_i + y_i|^p = |x_i + y_i|^{p-1}|x_i + y_i| \leq |x_i + y_i|^{p-1}|x_i| + |x_i + y_i|^{p-1}|y_i|$$

and note that $p + q = pq$. 