1. Let $\omega \in \Omega^1(\mathbb{R}^2\setminus\{0\})$ be given by

$$\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy,$$

and $\gamma : [0, 8] \to \mathbb{R}^2$ be the piecewise smooth curve defined by

$$\gamma(t) = \begin{cases} 
  (1, t) & \text{for } 0 \leq t < 1, \\
  (2-t, 1) & \text{for } 1 \leq t < 3, \\
  (-1, 4-t) & \text{for } 3 \leq t < 5, \\
  (t-6, -1) & \text{for } 5 \leq t < 7, \\
  (1, t-8) & \text{for } 7 \leq t \leq 8.
\end{cases}$$

(i) Draw the curve $\gamma$.

(ii) Calculate directly $\int_\gamma \omega$. You can use (without proof) the fact that

$$\int dx = \frac{2}{\sqrt{\Delta}} \arctan \frac{2ax+b}{\sqrt{\Delta}},$$

if $\Delta = 4ac - b^2 > 0$.

(iii) Find a free homotopy between the curves $\gamma$ and $c$ (from Exercise 3 of Exercise Sheet 10) in $\mathbb{R}^2\setminus\{0\}$.

2. (i) Show that if $\omega \in \Omega^1(U)$ and $c : [a, b] \to U$ is a smooth curve with $\|F_\omega(c(t))\| \leq M$ for all $t \in [a, b]$, then

$$|\int_c \omega| \leq M \cdot L(c).$$

(ii) Let $\omega \in \Omega^1(\mathbb{R}^2\setminus\{0\})$ be a closed form. Assume that $\|F_\omega\|$ is bounded in a disk of centre 0. Use Corollary 6.19 to show that $\omega$ is exact in $\mathbb{R}^2\setminus\{0\}$.

(iii) Why is this not a contradiction to the non-exactness of the form

$$\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

in Exercise 3 of Exercise Sheet 10?