Algebraic Geometry III/IV

Solutions, set 9.

Exercise 12.

(a) The polynomial \( F(X,Y,Z) \) is given by

\[
F(X,Y,Z) = aX^2 + bY^2 + cZ^2 + 2dXY + 2eXZ + 2fYZ.
\]

It is easy to see that the condition \((x,y,z) \neq 0\) and \(F_X(x,y,z) = F_Y(x,y,z) = F_Z(x,y,z) = 0\) is equivalent to

\[
\begin{align*}
2ax + 2dy + 2ez &= 0, \\
2by + 2dx + 2fz &= 0, \\
2cz + 2ex + 2fy &= 0,
\end{align*}
\]

which, in turn, is equivalent to

\[
\begin{pmatrix}
a & d & e \\
d & b & f \\
e & f & c
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = 0.
\]

This means, we have a nontrivial simultaneous solution \(F_X(x,y,z) = F_Y(x,y,z) = F_Z(x,y,z) = 0\) if and only if \(A\) has a nontrivial kernel, i.e., if and only if \(\det A = 0\). But any such solution satisfies obviously also \(F(x,y,z) = 0\), i.e., is a singular point of \(C_F\), and vice versa.

(b) The tangent line of \(C_F\) at \([\alpha,\beta,\gamma] \in C_F\) is given by the equation

\[
F_X(\alpha,\beta,\gamma)X + F_Y(\alpha,\beta,\gamma)Y + F_Z(\alpha,\beta,\gamma)Z = 0,
\]

i.e.,

\[
(2\alpha + 2d\beta + 2e\gamma)X + (2b\beta + 2d\alpha + 2f\gamma)Y + (2c\gamma + 2e\alpha + 2f\beta)Z = 0,
\]

i.e.,

\[
(\alpha \quad \beta \quad \gamma) A \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = 0.
\]
(c) We conclude from (b) that
\[
\mathcal{T}(C) = \left\{ C_h \mid H(X, Y, Z) = (\alpha \, \beta \, \gamma) A \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \text{ and } [\alpha, \beta, \gamma] \in C \right\}.
\]
This implies that
\[
C^* = \Phi(\mathcal{T}(C)) = \{ [x, y, z] \in \mathbb{P}^2_C \mid (x \, y \, z) = (\alpha \, \beta \, \gamma) A \text{ for some } [\alpha, \beta, \gamma] \in C \},
\]
i.e.,
\[
C^* = \{ [x, y, z] \in \mathbb{P}^2_C \mid [x \, y \, z] A^{-1} \in C \}.
\]
Now we have for every \((x \, y \, z) \neq 0\), using \(A^\top = A\),
\[
[x \, y \, z] A^{-1} \in C \iff (x \, y \, z) A^{-1} A ((x \, y \, z) A^{-1})^\top \\
\iff (x \, y \, z) A^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix},
\]
\[
\iff [x, y, z] \in C_G
\]
with
\[
G(X, Y, Z) = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} A^{-1} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}.
\]
This shows that \(C^* = C_G\).

**Exercise 13.** Recall that \(F(X, Y, Z) = 3Y^4 + 4Y^3Z + X^4\).

(a) We have \(F(0, 1, 0) = 3 \neq 0\), i.e., \([0, 1, 0] \notin C_F\). This guarantees that the map \(\pi : C_F \to \mathbb{P}^1_C\), \(\pi([a, b, c]) = [a, c]\) is well defined.

(b) We have \(F_Y(X, Y, Z) = 12Y^2(Y + Z) = 0\). So the solutions of \(F(P) = F_Y(P) = 0\) are given by
\[
R = \{[0, 0, 1], [\pm 1, 1, -1], [\pm i, 1, -1]\} \subset C_F \cap C_{F_Y},
\]
and
\[
B = \pi(R) = \{[0, 1], [\pm 1, -1], [\pm i, -1]\}.
\]

2
We see that $B$ contains 5 points. The $y$-coordinate of the each of the points in $\pi^{-1}([0, 1])$ are given by the equation $y^3(3y+4) = 0$, so we have $|\pi^{-1}([0, 1])| = 2$. The $y$-coordinate of each of the points in $\pi^{-1}([x, -1])$ with $x \in \{\pm 1, \pm i\}$ satisfies the equation $3y^4 + 4y^3 + 1 = (y+1)^2(3y^2 - 2y + 1) = 0$, so we have $|\pi^{-1}([x, -1])| = 3$ for all $x \in \{\pm 1, \pm i\}$.

(c) Since the singularities are a subset of $R$, we conclude from $F_X(X, Y, Z) = 3X^3$ and $F_Z(X, Y, Z) = 4Y^3$ that the only point in Sing($C_F$) is $P = [0, 0, 1]$, which is one of the two points in $\pi^{-1}([0, 1])$.

(d) We only need to carry out the blow-up procedure in the singular point $P$. We first choose affine coordinates via the identification $(x, y) \mapsto [x, y, 1]$ and obtain the affine polynomial

$$f(x, y) = F(x, y, 1) = 3y^4 + 4y^3 + x^4.$$ 

We see that we have a triple tangent line given by $y = 0$. So we can blow-up in $U_0$. We set $(x, y) = (x_1, x_1y_1)$ and obtain

$$f(x_1, x_1y_1) = x_1^3(3x_1y_1^4 + 4y_1^3 + x_1),$$

so the strict transform of $f$ in $U_0$ is

$$f^{(1)}(x_1, y_1) = 3x_1y_1^4 + 4y_1^3 + x_1.$$ 

The preimages of $(x, y) = (0, 0)$ under the strict transform are given by $x_1 = x = 0$ and $f^{(1)}(0, y_1) = 4y_1^3 = 0$, i.e., only $(x_1, y_1) = (0, 0)$. This is a non-singular point of $C_{f^{(1)}}$ since

$$f_{x_1}^{(1)}(0, 0) = 1.$$ 

So the blow-up process stops after one blow-up with a non-singular model $\psi : \tilde{C} \to C_F$.

(e) Since $B$ contains 5 points, we know from a result in the lectures that there exists a triangulation $T$ of $\mathbb{P}_C^1$ with the five points of $B$, and $3 \cdot 5 - 6 = 9$ edges and $2 \cdot 5 - 4 = 6$ triangles. The preimage $\pi^{-1}(B) \subset C_F$ contains $1 \cdot 2 + 4 \cdot 3 = 14$ points, and the preimage of $P$ under the blow-up procedure $\psi : \tilde{C} \to C_F$ consists of only one point. Since $\deg F = 4$, we end up with an induced triangulation of $\tilde{C}$ with $V = 14$ vertices,
\[ E = 4 \cdot 9 = 36 \text{ edges and } F = 4 \cdot 6 = 24 \text{ triangles. This implies that } \tilde{C} \text{ has the Euler number} \]
\[ \chi(\tilde{C}) = V - E + F = 14 - 36 + 24 = 2. \]

(f) Using the relation \[ \chi(\tilde{C}) = 2 - 2g(\tilde{C}), \] we conclude that the genus of the non-singular model \( \tilde{C} \) is
\[ g(\tilde{C}) = 1 - \frac{\chi(\tilde{C})}{2} = 1 - \frac{2}{2} = 0. \]

Exercise 14. Recall that \( F(X, Y, Z) = Y^4 - 2X^2Y^2 + XZ^3 \).

(a) We have \( F(0, 1, 0) = 1 \neq 0 \), i.e., \([0, 1, 0] \notin C_F\). This guarantees that the map \( \pi : C_F \to \mathbb{P}^1_C, \pi([a, b, c]) = [a, c] \) is well defined.

(b) We have \( F_Y(X, Y, Z) = 4Y(Y + X)(Y - X) = 0. \) So the solutions of \( F(P) = F_Y(P) = 0 \) are given by
\[ R = \{[0, 0, 1], [1, 0, 0], [\xi, \pm\xi, 1] \text{ with } \xi^3 = 1 \} \subset C_F \cap C_{F_Y}, \]
eight points in total, and
\[ B = \pi(R) = \{[0, 1], [1, 0], [\xi, 1] \text{ with } \xi^3 = 1 \}, \]
five points in total. The \( y \)-coordinate of the each of the points in \( \pi^{-1}([0, 1]) \) are given by the equation \( y^4 = 0 \), so we have \( |\pi^{-1}([0, 1])| = 1. \) The \( y \)-coordinate of the each of the points in \( \pi^{-1}([1, 0]) \) are given by the equation \( y^2(y^2 - 2) = 0 \), so we have \( |\pi^{-1}([0, 1])| = 3. \) The \( y \)-coordinate of each of the points in \( \pi^{-1}([\xi, 1]) \) with \( \xi^3 = 1 \) satisfies the equation \( y^4 - 2\xi^2y^2 + \xi = (y - \xi)^2(y + \xi)^2 = 0 \), so we have \( |\pi^{-1}([\xi, 1])| = 2. \) So there are in total \( 1 + 3 + 3 \cdot 2 = 10 \) points in \( \pi^{-1}(B) \).

(c) Since the singularities are a subset of \( R \), we conclude from \( F_X(X, Y, Z) = -4XY^2 + Z^3 \) and \( F_Z(X, Y, Z) = 3XZ^2 \) that the only point in \( \text{Sing}(C_F) \) is \( P = [1, 0, 0] \), since \( F_X(0, 0, 1) = 1 \neq 0 \) and \( F_Z(\xi, \pm\xi, 1) = 3\xi \neq 0 \).
(d) We only need to carry out the blow-up procedure in the singular point \( P \). We first choose affine coordinates via the identification \((x, y) \mapsto [1, x, y]\) and obtain the affine polynomial

\[
f(x, y) = F(1, x, y) = x^4 - 2x^2 + y^3.
\]

We see that we have a triple tangent line given by \( x = 0 \). So we need to blow-up in \( U_1 \). We set \((x, y) = (x_1 y_1, y_1)\) and obtain

\[
f(x_1 y_1, y_1) = y_1^2 (x_1^4 y_1^2 - 2x_1^2 + y_1),
\]

so the strict transform of \( f \) in \( U_1 \) is

\[
f^{(1)}(x_1, y_1) = x_1^4 y_1^2 - 2x_1^2 + y_1.
\]

The preimages of \((x, y) = (0, 0)\) under the strict transform are given by \( y_1 = y = 0 \) and \( f^{(1)}(x_1, 0) = -2x_1^2 = 0 \), i.e., only \((x_1, y_1) = (0, 0)\). This is a non-singular point of \( C_{f^{(1)}} \) since

\[
f^{(1)}_{y_1}(0, 0) = 1.
\]

So the blow-up process stops after one blow-up with a non-singular model \( \psi : \tilde{C} \to C_F \).

(e) Since \( B \) contains 5 points, we know from a result in the lectures that there exists a triangulation \( \mathcal{T} \) of \( \mathbb{P}^1_C \) with the five points of \( B \), and \( 3 \cdot 5 - 6 = 9 \) edges and \( 2 \cdot 5 - 4 = 6 \) triangles. The preimage \( \pi^{-1}(B) \subset C_F \) contains 10 points, and the preimage of \( P \) under the blow-up procedure \( \psi : \tilde{C} \to C_F \) consists of only one point. Since \( \deg F = 4 \), we end up with an induced triangulation of \( C \) with \( V = 10 \) vertices, \( E = 4 \cdot 9 = 36 \) edges and \( F = 4 \cdot 6 = 24 \) triangles. This implies that \( \tilde{C} \) has the Euler number

\[
\chi(\tilde{C}) = V - E + F = 10 - 36 + 24 = -2.
\]

(f) Using the relation \( \chi(\tilde{C}) = 2 - 2g(\tilde{C}) \), we conclude that the genus of the non-singular model \( \tilde{C} \) is

\[
g(\tilde{C}) = 1 - \frac{\chi(\tilde{C})}{2} = 1 - \frac{-2}{2} = 2.
\]
**Exercise 15.** Recall that $F(X,Y,Z) = X^5 + 3Y^5 - 5Y^3Z^2$.

(a) We have $F(0,1,0) = 3 \neq 0$, i.e., $[0,1,0] \notin C_F$. This guarantees that the map $\pi : C_F \to \mathbb{P}^1_C$, $\pi([a,b,c]) = [a,c]$ is well defined.

(b) We have $F_Y(X,Y,Z) = 15Y^2(Y - Z)(Y + Z) = 0$. So the solutions of $F(P) = F_Y(P) = 0$ are given by $R = \{[0,0,1], [\alpha,1,1], [-\alpha,-1,1] \text{ with } \alpha^5 = 2\} \subset C_F \cap C_{F_Y}$, and $B = \pi(R) = \{[0,1], [\pm\alpha,1] \text{ with } \alpha^5 = 2\}$. We see that $B$ contains 11 points and so does $R$. The $y$-coordinate of each of the points in $\pi^{-1}([0,1])$ are given by the equation $y^3(3y^2 - 5) = 0$, so we have $|\pi^{-1}([0,1])| = 3$. The $y$-coordinate of each of the points in $\pi^{-1}([\alpha,1])$ with $\alpha^5 = 2$ satisfies the equation $3y^5 - 5y^3 + 2 = (y - 1)^2(3y^3 + 6y^2 + 4y + 2) = 0$. Note that $y = 1$ is not a solution of $g(y) = 3y^3 + 6y^2 + 4y + 2$. Moreover, we have for the discriminant $D(g) = R(g,g')$, where $R(g,h)$ is the resultant of $g,h$,

$$D(g) = R(g,g') = \det \begin{pmatrix} 2 & 4 & 6 & 3 & 0 \\ 0 & 2 & 4 & 6 & 3 \\ 4 & 12 & 9 & 0 & 0 \\ 0 & 4 & 12 & 9 & 0 \end{pmatrix} = 900 \neq 0,$$

so $g(y)$ does not have multiple roots and we have $|\pi^{-1}([\alpha,1])]| = 4$. A similar argument leads also to $|\pi^{-1}([-\alpha,1])| = 4$. So we have in total $1 \cdot 3 + 10 \cdot 4 = 43$ points in $\pi^{-1}(B) \subset C_F$.

(c) Since the singularities are a subset of $R$, we conclude from $F_X(X,Y,Z) = 5X^4$ and $F_Z(X,Y,Z) = -10Y^3Z$ that the only point in $\text{Sing}(C_F)$ is $P = [0,0,1]$, which is one of the two points in $\pi^{-1}([0,1])$.

(d) We only need to carry out the blow-up procedure in the singular point $P$. We first choose affine coordinates via the identification $(x,y) \mapsto [x,y,1]$ and obtain the affine polynomial

$$f(x,y) = F(x,y,1) = x^5 + 3y^5 - 5y^3.$$
We see that we have a triple tangent line given by \( y = 0 \). So we can blow-up in \( U_0 \). We set \((x,y) = (x_1,x_1y_1)\) and obtain
\[
f(x_1,x_1y_1) = x_1^3(x_1^2 + 3x_1^2y_1^5 - 5y_1^3),
\]
so the strict transform of \( f \) in \( U_0 \) is
\[
f^{(1)}(x_1,y_1) = x_1^2 + 3x_1^2y_1^5 - 5y_1^3.
\]
The preimages of \((x,y) = (0,0)\) under the strict transform are given by \( x_1 = x = 0 \) and \( f^{(1)}(0,y_1) = -5y_1^3 = 0 \), i.e., only \((x_1,y_1) = (0,0)\). This is still a singular point of \( C_{f^{(1)}} \) since
\[
f^{(1)}_{x_1}(x_1,y_1) = 2x_1 + 6x_1y_1^5, \quad f^{(1)}_{y_1}(x_1,y_1) = 15x_1^2y_1^4 - 15y_1^2.
\]
At \((0,0) \in C_{f^{(1)}}\), we have again a double tangent line given by \( x_1 = 0 \). So we need to carry out the next blow-up again in \( U_1 \). We obtain
\[
f^{(1)}(x_2y_2,y_2) = y_2^2(x_2^2 + 3x_2^2y_2^5 - 5y_2),
\]
so the strict transform of \( f^{(1)} \) in \( U_1 \) is
\[
f^{(2)}(x_2,y_2) = x_2^2 + 3x_2^2y_2^5 - 5y_2.
\]
The preimages of \((x_1,y_1) = (0,0)\) under the strict transform are given by \( y_2 = y_1 = 0 \) and \( f^{(2)}(x_2,0) = x_2^2 = 0 \), i.e., only \((x_2,y_2) = (0,0)\). Since \( f^{(2)}_{y_2}(x_2,y_2) = -5 \neq 0 \), the point \((0,0) \in C_{f^{(2)}}\) is no longer singular and the blow-up process stops with a non-singular model \( \psi : \tilde{C} \to C_F \).

(e) Since \( B \) contains 11 points, we know from a result in the lectures that there exists a triangulation \( T \) of \( \mathbb{P}^1_C \) with the 11 points of \( B \), and \( 3 \cdot 11 - 6 = 27 \) edges and \( 2 \cdot 11 - 4 = 18 \) triangles. The preimage \( \pi^{-1}(B) \subset C_F \) contains 43 points, and the preimage of \( P \) under the blow-up procedure \( \psi : \tilde{C} \to C_F \) consists of only one point. Since \( \deg F = 5 \), we end up with an induced triangulation of \( \tilde{C} \) with \( V = 43 \) vertices, \( E = 5 \cdot 27 = 135 \) edges and \( F = 5 \cdot 18 = 90 \) triangles. This implies that \( \tilde{C} \) has the Euler number
\[
\chi(\tilde{C}) = V - E + F = 43 - 135 + 90 = -2.
\]

(f) Using the relation \( \chi(\tilde{C}) = 2 - 2g(\tilde{C}) \), we conclude that the genus of the non-singular model \( \tilde{C} \) is
\[
g(\tilde{C}) = 1 - \frac{\chi(\tilde{C})}{2} = 1 - \frac{-2}{2} = 2.
\]