Algebraic Geometry III/IV

Solutions, set 7.

Exercise 10.

(a) Let \( f(x, y) = x^2 - x^4 - y^4 \). We have

\[
\begin{align*}
  f_x(x, y) &= 2x(1 - \sqrt{2}x)(1 + \sqrt{2}x), \\
  f_y(x, y) &= -4y^3.
\end{align*}
\]

The condition \( f_y(x, y) = 0 \) implies \( y = 0 \) and, because of \( f(x, 0) = x^2(1 - x)(1 + x) \), we see that \( f(x, y) = f_x(x, y) = f_y(x, y) = 0 \) has the only solution \((x, y) = (0, 0)\). Next we calculate the tangent lines of \( C_f \) at \((0, 0)\). We have

\[
  f(x, y) = f_2(x, y) + f_4(x, y),
\]

with \( f_2(x, y) = x^2 \) and \( f_4(x, y) = -x^4 - y^4 \). Therefore, we have a double tangent line given by \( x = 0 \). So we need to consider the blow-up in \( U_1 \).

We set \((x, y) = (x_1y_1, y_1)\) and obtain

\[
  f(x_1y_1, y_1) = y_1^2(x_1^2 - x_1^4y_1^2 - y_1^2),
\]

so the strict transform of \( f \) in \( U_1 \) is

\[
  f^{(1)}(x_1, y_1) = x_1^2 - x_1^4y_1^2 - y_1^2.
\]

We now have

\[
\begin{align*}
  f^{(1)}_{x_1}(x_1, y_1) &= 2x_1(1 - 2x_1^2y_1^2), \\
  f^{(1)}_{y_1}(x_1, y_1) &= -2y_1(x_1^4 + 1).
\end{align*}
\]

The preimages of \((x, y) = (0, 0)\) under the strict transform are given by \( y_1 = y = 0 \) and \( f^{(1)}(x_1, 0) = x_1^2 = 0 \), i.e., only \((x_1, y_1) = (0, 0)\), which
is still a singular point of $C_{f(1)}$. Next we calculate the tangent lines of $C_{f(1)}$ at $(0, 0)$. We have

$$f^{(1)}(x_1, y_1) = f_2^{(1)}(x_1, y_1) + f_6^{(1)}(x_1, y_1)$$

with $f_2^{(1)}(x_1, y_1) = (x_1 - y_1)(x_1 + y_1)$ and $f_6^{(1)}(x_1, y_1) = -x_1^4y_1^2$. Therefore, we have the two tangent lines $x_1 = y_1$ and $x_1 = -y_1$. So we can consider the next blow-up in $U_0$. We set $(x_1, y_1) = (x_2, x_2y_2)$ and obtain

$$f^{(1)}(x_2, x_2y_2) = x_2^2(1 - x_2^4y_2^2 - y_2^2),$$

so the strict transform of $f^{(1)}$ in $U_0$ is

$$f^{(2)}(x_2, y_2) = 1 - x_2^4y_2^2 - y_2^2.$$ 

The preimages of $(x_1, y_1) = (0, 0)$ under the strict transform are given by $x_2 = x_1 = 0$ and $f^{(2)}(0, y_2) = 1 - y_2^2 = 0$, i.e., $(x_2, y_2) = (0, \pm 1)$. We now have

$$f_{x_2}^{(2)}(x_2, y_2) = -4x_2^3y_2^2,$$

$$f_{y_2}^{(2)}(x_2, y_2) = -2y_2(x_2^4 + 1).$$

Since $f_{y_2}^{(2)}(0, \pm 1) = \mp 2 \neq 0$, all singularities are now resolved and the blow-up process stops.

(b) Let $g(x, y) = y^3 - x^5$. We have

$$g_x(x, y) = -5x^4,$$

$$g_y(x, y) = 3y^2,$$

so $g_x(x, y) = g_y(x, y) = 0$ implies that $(x, y) = (0, 0)$. Therefore, the only singularity of $C_g$ is $(0, 0)$. The tangent lines of $C_g$ at $(0, 0)$ are $y = 0$ (trice) and we can consider the blow-up in $U_0$. We set $(x, y) = (x_1, x_1y_1)$ and obtain

$$g(x_1, x_1y_1) = x_1^3(y_1^3 - x_1^2),$$

i.e., the strict transform of $g$ in $U_0$ is

$$g^{(1)}(x_1, y_1) = y_1^3 - x_1^2.$$
The preimages of \((x, y) = (0, 0)\) under the strict transform are given by \(x_1 = x = 0\) and \(g^{(1)}(0, y_1) = y_1^3 = 0\), i.e., \((x_1, y_1) = (0, 0)\). Since

\[
\begin{align*}
g^{(1)}_{x_1}(x_1, y_1) &= -2x_1, \\
g^{(1)}_{y_1}(x_1, y_1) &= -3y_1^2,
\end{align*}
\]

the point \((0, 0)\) is still a singularity of \(C_{g^{(1)}}\). The tangent lines of \(C_{g^{(1)}}\) at \((0, 0)\) are \(x = 0\) (twice) and we need to carry out the blow-up in \(U_1\).

We set \((x_1, y_1) = (x_2y_2, y_2)\) and obtain

\[
g^{(1)}(x_2y_2, y_2) = y_2^2(y_2 - x_2^2).
\]

The strict transform of \(g^{(1)}\) in \(U_1\) is therefore

\[
g^{(2)}(x_2, y_2) = y_2 - x_2^2.
\]

The preimages of \((x_1, y_1) = (0, 0)\) under the strict transform are given by \(y_2 = y_1 = 0\) and \(g^{(2)}(x_2, 0) = -x_2^2 = 0\), i.e., \((x_2, y_2) = (0, 0)\). Since \(g^{(2)}_{y_2}(x_2, y_2) = 1 \neq 0\), all singularities are resolved and the blow-up process stops.