Algebraic Geometry III/IV

Solutions, set 2.

Exercise 3. Using the chain rule when differentiating $k(t)$ we obtain

$$
k'(t) = F_X(f(t), g(t), h(t))f'(t) + F_Y(f(t), g(t), h(t))g'(t) + F_Z(f(t), g(t), h(t))h'(t).$$

Setting $t = 0$ and recalling that $P = [f(0), g(0), h(0)]$ we conclude that

$$
k'(0) = F_X(P)f'(0) + F_Y(P)g'(0) + F_Z(P)h'(0).
$$

Now, if $k'(0) \neq 0$, then at least one of $F_X(P), F_Y(P), F_Z(P)$ is not vanishing and, therefore, $P$ is a nonsingular point of $C_F$. Recall that the tangent line $L'$ of $C_F$ at the nonsingular point $P$ is given by

$$
F_X(P)X + F_Y(P)Y + F_Z(P)Z = 0.
$$

If we would have $L' = L$ then we could conclude $[f(t), g(t), h(t)] \in L'$ for all $t \in (-T, T)$ and, therefore, by differentiation

$$
0 = \left. \frac{d}{dt} \right|_{t=0} (F_X(P)f(t) + F_Y(P)g(t) + F_Z(P)h(t)) = F_X(P)f'(0) + F_Y(P)g'(0) + F_Z(P)h'(0) = k'(0).
$$

But this would be in contradiction to $k'(0) \neq 0$.

Exercise 4. Let $C_F \subset \mathbb{P}^2_C$ be a nonsingular projective cubic and

$$
\mathcal{H}_F = \det \begin{pmatrix}
F_{XX} & F_{XZ} & F_{XZ} \\
F_{YX} & F_{YY} & F_{YZ} \\
F_{ZX} & F_{ZY} & F_{ZZ}
\end{pmatrix}
$$

be its Hessian.
(a) First of all, $C_F$ and $C_{H_F}$ do not share a common factor for, otherwise, we would have $C_F \subset C_{H_F}$, because $C_F$ is nonsingular and, therefore, irreducible. But then every point of $C_F$ is a flex. A theorem in last term’s lecture states that this implies that $\deg C_F = 1$, which is a contradiction. Recall that $\deg H_G = 3(d-2)$ where $d$ is the degree of $G$. Since $\deg F = 3$, we have $\deg H_G = 3$, and we can now apply Bezout’s Theorem and conclude that $C_F \cap C_{H_F}$ is finite and

$$\sum_{P \in C_F \cap C_{H_F}} \text{ind}_P(F, H_F) = 3 \cdot 3 = 9.$$  

(b) Using the comments in the exercise, we can assume that $P = [0, 1, 0]$ and

$$F(X, Y, Z) = Y^2Z - X^3 + (1 + \lambda)X^2Z - \lambda XZ^2$$

for some $\lambda \in \mathbb{C} - \{0, 1\}$. Then we have

$$F_X = -3X^2 + 2(1 + \lambda)XZ - \lambda Z^2, \quad F_Y = 2YZ,$$

$$F_Z = Y^2 + (1 + \lambda)X^2 - 2\lambda XZ,$$

and the tangent line $L$ is given by the equation

$$F_X(0, 1, 0)X + F_Y(0, 1, 0)Y + F_Z(0, 1, 0)Z = 0.$$  

The statement follows now from $F_X(0, 1, 0) = 0$, $F_Y(0, 1, 0) = 1$ and $F_Z(0, 1, 0) = 1$.

(c) We have

$$F_{XX} = -6X + 2(1 + \lambda)Z, \quad F_{XY} = 0, \quad F_{XZ} = 2(1 + \lambda)X - 2\lambda Z$$

and

$$F_{YX} = 0, \quad F_{YY} = 2Z, \quad F_{YZ} = 2Y,$$

and

$$F_{ZZ} = 2(1 + \lambda)X - 2\lambda Z, \quad F_{ZV} = 2Y, \quad F_{ZZ} = -2\lambda X,$$

and therefore

$$k(t) = H_F(t, 1, 0) = \det \begin{pmatrix} -6t & 0 & 2(1 + \lambda)t \\ 0 & 0 & 2 \\ 2(1 + \lambda)t & 2 & 2\lambda t \end{pmatrix} = 24t.$$  

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(d) Choosing \( f(t) = t, g(t) = 1, h(t) = 0 \), we have
\[
k(t) = \mathcal{H}_F(f(t), g(t), h(t)) = 24t
\]
and \( k'(0) = 24 \neq 0 \). Moreover, if \( L \) denotes the line \( Z = 0 \), which is the tangent line of \( C_F \) at the nonsingular point \( P = [0, 1, 0] \), then \( [f(t), g(t), h(t)] \in L \) for all \( t \in \mathbb{R} \), and we can apply Exercise 3 to conclude that \( P \) is also a nonsingular point of the curve \( \mathcal{H}_F = 0 \) and that \( L \) is not the tangent line of \( \mathcal{H}_F = 0 \) at \( P \).

(e) Recall that we started with an arbitrary flex \( P \) of \( C_F \). The result in (d) shows that
\[
\text{ind}_F(F, \mathcal{H}_F) = 1.
\]
Therefore all intersection indices in the formula in (a) are equal to 1 and there must be nine summands. But the flexes of \( C_F \) agree precisely with the points of the intersection \( C_F \cap C_{\mathcal{H}_F} \), so the curve \( C_F \) must have precisely 9 flexes.