# Notes on Complex Hyperbolic Geometry

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July 11, 2003

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1 Introduction

The unit ball in $\mathbb{C}^2$ has a natural metric of constant negative holomorphic sectional curvature (which we normalise to be $-1$), called the Bergman metric. As such it forms a model for complex hyperbolic 2-space $H^2_C$ analogous to the ball model of (real) hyperbolic space $H^2_R$. The main difference is that the (real) sectional curvature is no longer constant, but is pinched between $-1$ and $-1/4$. Another standard model for complex hyperbolic space is a paraboloid in $\mathbb{C}^2$ called the Siegel domain. This is analogous to the half space model of $H^2_R$. As complex hyperbolic 1-space is just the unit disc in $\mathbb{C}$ with the Poincaré metric, $H^2_C$ is a natural generalisation of plane hyperbolic geometry which is different from the more familiar generalisation of higher dimensional real hyperbolic space.

An alternative description of $H^2_C$ is given by the projective model. Here we take a Hermitian form of signature $(2,1)$ on $\mathbb{C}^3$. Projectivising the set of complex lines on which this form is negative gives another model for complex hyperbolic space. By taking a suitable form and making a choice of section we can recover the ball model and the Siegel domain model. The Bergman metric is given by a simple distance formula in terms of the Hermitian form which is closely related to the Cauchy-Schwarz inequality. From this description we can show that all holomorphic isometries of complex hyperbolic space are given by the projectivisation of unitary matrices preserving the Hermitian form. All antiholomorphic isometries are given applying such a matrix followed by complex conjugation. This means that we can use complex linear algebra to study the geometry of complex hyperbolic space.

As well as studying isometries, we want to consider certain special classes of submanifolds of complex hyperbolic space. We will see that the totally geodesic submanifolds have dimension at most 2. (In fact, for $n$ dimensional complex hyperbolic space, totally geodesic subspaces are are either embedded copies of $H^m_C$ or $H^m_R$ for $1 \leq m \leq n$. Thus, the real dimension of a totally geodesic submanifold is either at most $n$, for embedded copies of $H^m_R$, or else is even, for embedded copies of $H^m_C$.) In particular, there are no totally geodesic real hypersurfaces in $H^2_C$. This increases the difficulty of constructing polyhedra (for example fundamental polyhedra for discrete groups of complex hyperbolic isometries). In a later chapter we will describe some classes of real hypersurfaces that can be used to build polyhedra.

The boundary of complex hyperbolic 2-space is the one point compactification of the Heisenberg group in the same way that the boundary of real hyperbolic space is the one point compactification of Euclidean space of one dimension lower. Just as the internal geometry of real hyperbolic space may be studied using conformal geometry on the boundary, so the internal geometry of complex hyperbolic space may be studied using CR-geometry.
on the Heisenberg group. Moreover, the Heisenberg group is 3 dimensional and so it is easy to illustrate geometrical objects.

In order to make things as concrete as possible, we have chosen restrict our attention to $H^2$. Many of the results we develop will hold for complex hyperbolic space in all dimensions. There will often be analogues for other rank 1 symmetric spaces of non-compact type, quaternionic hyperbolic space $H^n_{\mathbb{H}}$ and the octonionic hyperbolic plane $H^2_{\mathbb{O}}$. We will not discuss these here.

2 Complex hyperbolic 2-space

2.1 Hermitian forms on $\mathbb{C}^{2,1}$

Let $A = (a_{ij})$ be a $k \times l$ complex matrix. The Hermitian transpose of $A$ is the $l \times k$ complex matrix $A^* = (\bar{a}_{ji})$ formed by complex conjugating each entry of $A$ and then taking the transpose. As with ordinary transpose, the Hermitian transpose of a product is the product of the Hermitian transposes in the reverse order. That is $(AB)^* = B^*A^*$. Clearly $(A^{*})^* = A$.

A $k \times k$ complex matrix $A$ is said to be Hermitian if it equals its own Hermitian transpose $A = A^*$. Let $A$ be a Hermitian matrix and $\mu$ an eigenvalue of $A$ with eigenvector $x$. We claim that $\mu$ is real. In order to see this, observe that

$$\mu x^*x = x^*(\mu x) = x^*Ax = x^*A^*x = (Ax)^*x = (\mu x)^*x = \overline{\mu} x^*x.$$  

Since $x^*x$ is real and non-zero we see that $\mu$ is real.

To each $k \times k$ Hermitian matrix $A$ we can naturally associate an Hermitian form $\langle \cdot, \cdot \rangle : \mathbb{C}^k \times \mathbb{C}^k \rightarrow \mathbb{C}$ given by $\langle z, w \rangle = w^*Az$ (note that we change the order) where $w$ and $z$ are column vectors in $\mathbb{C}^k$. Hermitian forms are sesquilinear, that is they are linear in the first factor and conjugate linear in the second factor. In other words, for $z$, $z_1$, $z_2$, $w$ column vectors in $\mathbb{C}^k$ and $\lambda$ a complex scalar, we have

$$\langle z_1 + z_2, w \rangle = w^*A(z_1 + z_2) = w^*Az_1 + w^*Az_2 = \langle z_1, w \rangle + \langle z_2, w \rangle,$$

$$\langle \lambda z, w \rangle = w^*A(\lambda z) = \lambda w^*Az = \lambda \langle z, w \rangle,$$

$$\langle w, z \rangle = z^*Aw = z^*A^*w = (w^*Az)^* = \overline{\langle z, w \rangle}.$$  

From these we see that

$$\langle z, z \rangle \in \mathbb{R},$$

$$\langle z, \lambda w \rangle = \overline{\langle z, w \rangle},$$

$$\langle \lambda z, \lambda w \rangle = |\lambda|^2 \langle z, w \rangle.$$  

Let $\mathbb{C}^{2,1}$ be the complex vector space of (complex) dimension 3 equipped with a non-degenerate, indefinite Hermitian form $\langle \cdot, \cdot \rangle$ of signature $(2, 1)$. This means that $\langle \cdot, \cdot \rangle$ is given by a non-singular $3 \times 3$ Hermitian matrix $J$ with 2 positive eigenvalues and 1 negative eigenvalue. There are two standard matrices $J$ which give different Hermitian forms on $\mathbb{C}^{2,1}$. Following Epstein [6] we call these the first and second Hermitian forms. Let $z$, $w$ be the column vectors $(z_1, z_2, z_3)^t$ and $(w_1, w_2, w_3)^t$ respectively. The first Hermitian form is defined to be

$$\langle z, w \rangle_1 = z_1\overline{w_1} + z_2\overline{w_2} - z_3\overline{w_3}.$$
It is given by the Hermitian matrix $J_1$:

$$J_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$ 

The second Hermitian form is defined to be:

$$\langle z, w \rangle_2 = z_1\overline{w}_3 + z_2\overline{w}_2 + z_3\overline{w}_1.$$ 

It is given by the Hermitian matrix $J_2$:

$$J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$ 

Sometimes we want to specify which of these two Hermitian forms to use. When there is no subscript then you can use either of these (or your favourite Hermitian form on $\mathbb{C}^3$ of signature (2, 1)).

There are other Hermitian forms which are widely used in the literature. In particular, Chen and Greenberg (page 67 of [3]) give a close relative of the second Hermitian form. We will refer to this as the third Hermitian form. It is given by

$$\langle z, w \rangle_3 = -z_1\overline{w}_2 - z_2\overline{w}_1 + z_3\overline{w}_3.$$ 

It is given by the Hermitian matrix $J_3$:

$$J_3 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

The third Hermitian form has been used extensively by Kamiya, Hersonsky and Paulin.

### 2.2 Three models of complex hyperbolic space

If $z \in \mathbb{C}^{2,1}$ then we know that $\langle z, z \rangle$ is real. Thus we may define subsets $V_-$, $V_0$ and $V_+$ of $\mathbb{C}^{2,1}$ by

$$V_- = \{ z \in \mathbb{C}^{2,1} \mid \langle z, z \rangle < 0 \},$$
$$V_0 = \{ z \in \mathbb{C}^{2,1} \mid \langle z, z \rangle = 0 \},$$
$$V_+ = \{ z \in \mathbb{C}^{2,1} \mid \langle z, z \rangle > 0 \}.$$ 

We say that $z \in \mathbb{C}^{2,1}$ is negative, null or positive if $z$ is in $V_-$, $V_0$ or $V_+$ respectively.

Motivated by special relativity, these are sometimes called time-like, light-like and space-like. Because $\langle \lambda z, \lambda z \rangle = |\lambda|^2 \langle z, z \rangle$ we see that for any non-zero complex scalar $\lambda$ the point $\lambda z$ is negative, null or positive if and only if $z$ is. Therefore we define a projection map $P$ on those points of $\mathbb{C}^{2,1}$ with $z_3 \neq 0$. This projection map is defined by

$$P : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} z_1/z_3 \\ z_2/z_3 \end{bmatrix} \in \mathbb{C}^2.$$
In fact this projection map is the restriction to one chart of the usual projection map \( \mathbb{P} : \mathbb{C}^{2,1} \rightarrow \mathbb{C} \mathbb{P}^2 \).

The projective model of complex hyperbolic space is defined to be the collection of negative lines in \( \mathbb{C}^{2,1} \) and its boundary is defined to be the collection of null lines. In other words \( \mathbb{H}^2_\mathbb{C} \) is \( \mathbb{P} \mathbb{V}_- \) and \( \partial \mathbb{H}^2_\mathbb{C} \) is \( \mathbb{P} \mathbb{V}_0 \).

We define the other two standard models of complex hyperbolic space by taking the section defined by \( z_3 = 1 \) for the first and second Hermitian forms. In other words, if we take column vectors \( z = (z_1, z_2, 1)^t \) in \( \mathbb{C}^{2,1} \), then consider what it means for \( \langle z, z \rangle \) to be negative.

For the first Hermitian form we obtain \( z \in \mathbb{H}^2_\mathbb{C} \) provided:

\[
\langle z, z \rangle_1 = z_1 \bar{z}_1 + z_2 \bar{z}_2 - 1 < 0.
\]

In other words

\[
|z_1|^2 + |z_2|^2 < 1.
\]

Thus \( z = (z_1, z_2) \) is in the unit ball in \( \mathbb{C}^2 \). This forms the unit ball model of complex hyperbolic space. The boundary of the unit ball model is the sphere \( S^3 \) given by

\[
|z_1|^2 + |z_2|^2 = 1.
\]

We say that the standard lift of a point \( z = (z_1, z_2) \) in the unit ball (or its boundary) to \( \mathbb{C}^{2,1} \) is the column vector \( z = (z_1, z_2, 1)^t \) of \( \mathbb{C}^{2,1} \) whose first 2 coordinates are those of \( z \) and whose last coordinate is 1. Thus if \( z \) and \( w \) are the standard lifts of points \( z \) and \( w \) we have \( \langle z, w \rangle_1 = w^* z - 1 = z_1 \bar{w}_1 + z_2 \bar{w}_2 - 1 \).

For the second Hermitian form we obtain \( z \in \mathbb{H}^2_\mathbb{C} \) provided:

\[
\langle z, z \rangle_2 = z_1 + z_2 \bar{z}_2 + \bar{z}_1 < 0.
\]

In other words

\[
2 \Re(z_1) + |z_2|^2 < 0.
\]

Thus \( z = (z_1, z_2) \) is in a domain in \( \mathbb{C}^2 \) whose boundary is the paraboloid defined by

\[
2 \Re(z_1) + |z_2|^2 = 0.
\]

This domain is called the Siegel domain and forms the Siegel domain model of \( \mathbb{H}^2_\mathbb{C} \). We again say that the standard lift of a point \( z \) in the Siegel domain to \( \mathbb{C}^{2,1} \) is the column vector \( z = (z_1, z_2, 1)^t \) of \( \mathbb{C}^{2,1} \). We usually compactify the Siegel domain by adding a point at infinity, denoted \( \infty \). The standard lift of \( \infty \) is the column vector \( (1, 0, 0)^t \in \mathbb{C}^{2,1} \).

For the projective model the metric on \( \mathbb{H}^2_\mathbb{C} \), called the Bergman metric is given by the distance function \( \rho(\cdot, \cdot) \) defined by the formula

\[
\cosh^2 \left( \frac{\rho(z, w)}{2} \right) = \frac{\langle z, w \rangle \langle w, z \rangle}{\langle z, z \rangle \langle w, w \rangle}.
\]

For the ball model and Siegel domain model one can find the distance between points \( z \) and \( w \) by plugging their standard lifts \( z \) and \( w \) into the above formula. However, as may easily be seen, this formula is independent of which lifts \( z \) and \( w \) in \( \mathbb{C}^{2,1} \) of \( z \) and \( w \) we choose.
Alternatively, the Bergman metric is given by
\[ ds^2 = \frac{-4}{(z, z)^2} \det \begin{pmatrix} (z, z) & (dz, z) \\ (z, dz) & (dz, dz) \end{pmatrix}. \tag{1} \]

**Proposition 2.1** In the ball model of \( \mathbb{H}^2 \mathbb{C} \) the volume form is given by
\[ d\text{Vol} = \frac{16}{(1 - |z_1|^2 - |z_2|^2)^3} d\text{vol} \]
where \( d\text{vol} \) is the volume element
\[(1/2i)^2 dz_1 \wedge dz_1 \wedge dz_2 \wedge dz_2 = dx_1 dy_1 dx_2 dy_2.\]

**Proof:** Substituting for the first Hermitian form in (1) we have
\[ ds^2 = \frac{-4}{(|z_1|^2 + |z_2|^2 - 1)^2} \det \begin{pmatrix} |z_1|^2 + |z_2|^2 - 1 & z_1 d z_1 + z_2 d z_2 \\ z_1 d z_1 + z_2 d z_2 & |d z_1|^2 + |d z_2|^2 \end{pmatrix} \]
\[ = \frac{4(1 - |z_1|^2 - |z_2|^2)(|d z_1|^2 + |d z_2|^2) + 4|z_1 d z_1 + z_2 d z_2|^2}{(|z_1|^2 + |z_2|^2 - 1)^2}. \]

Converting to real coordinates, \( x_1 + iy_1 = z_1, x_2 + iy_2 = z_2 \), and denoting
\[ r^2 = x_1^2 + y_1^2 + x_2^2 + y_2^2 = |z_1|^2 + |z_2|^2 < 1, \]
we have
\[ ds^2 = (dx_1 \ dy_1 \ dx_2 \ dy_2) g_{\mathbb{R}} \begin{pmatrix} dx_1 \\ dy_1 \\ dx_2 \\ dy_2 \end{pmatrix} \]
where
\[ g_{\mathbb{R}} = \frac{4}{(1 - r^2)^3} \begin{pmatrix} 1 - x_2^2 - y_2^2 & 0 & x_1 x_2 + y_1 y_2 & x_1 y_2 - y_1 x_2 \\ 0 & 1 - x_1^2 - y_1^2 & -x_1 y_2 + y_1 x_2 & x_1 x_2 + y_1 y_2 \\ x_1 x_2 + y_1 y_2 & -x_1 y_2 + y_1 x_2 & 1 - x_1^2 - y_1^2 & 0 \\ x_1 y_2 + y_1 x_2 & x_1 x_2 + y_1 y_2 & 0 & 1 - x_1^2 - y_1^2 \end{pmatrix}. \]

Now
\[ \det(g_{\mathbb{R}}) = \frac{256}{(1 - r^2)^6}. \]

Thus the volume form is
\[ d\text{Vol} = \sqrt{\det(g_{\mathbb{R}})} dx_1 dy_1 dx_2 dy_2 = \frac{16}{(1 - r^2)^3} dx_1 dy_1 dx_2 dy_2 \]
as claimed. \( \square \)
Proposition 2.2 In the Siegel domain model of $H^2_C$ the volume form is given by

$$d\text{Vol} = \frac{16}{(-z_1 - \bar{z}_1 - |z_2|^2)^3} d\text{vol}$$

where $d\text{vol}$ is the volume element

$$(1/2i)^2 dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 = dx_1 dy_1 dx_2 dy_2.$$

Proof: Substituting for the second Hermitian form in (1) we have

$$ds^2 = \frac{-4}{(z_1 + |z_2|^2 + \bar{z}_1)^2} \det \begin{pmatrix} z_1 + |z_2|^2 + \bar{z}_1 & dz_1 + \bar{z}_2 dz_2 \\ d\bar{z}_1 + z_2 d\bar{z}_2 & |dz_2|^2 \end{pmatrix}$$

$$= \frac{-4(z_1 + |z_2|^2 + \bar{z}_1)|dz_2|^2 + 4|dz_1 + \bar{z}_2 dz_2|^2}{(z_1 + |z_2|^2 + \bar{z}_1)^2}.$$

Converting to real coordinates, $x_1 + iy_1 = z_1$, $x_2 + iy_2 = z_2$, we have

$$ds^2 = (dx_1 \ dy_1 \ dx_2 \ dy_2) \mathbf{g}_R \begin{pmatrix} dx_1 \\ dy_1 \\ dx_2 \\ dy_2 \end{pmatrix}$$

where

$$\mathbf{g}_R = \frac{4}{(2x_1 + x_2^2 + y_2^2)^2} \begin{pmatrix} 1 & 0 & x_2 & y_2 \\ 0 & 1 & -y_2 & x_2 \\ x_2 & -y_2 & -2x_1 & 0 \\ y_2 & x_2 & 0 & -2x_1 \end{pmatrix}.$$ 

Now

$$\det(\mathbf{g}_R) = \frac{256}{(2x_1 + x_2^2 + y_2^2)^6}.$$ 

Thus the volume form is

$$d\text{Vol} = \sqrt{\det(\mathbf{g}_R)} dx_1 dy_1 dx_2 dy_2 = \frac{16}{(-2x_1 - x_2^2 - y_2^2)^3} dx_1 dy_1 dx_2 dy_2$$

as claimed (since $2x_1 + x_2^2 + y_2^2 < 0$ we take the negative sign in the square root). □

2.3 Cayley transforms

Given two Hermitian forms of signature (2,1) we can pass between them using a Cayley transform. This is not unique for we may precompose and postcompose by any unitary matrix preserving the relevant Hermitian form. The following Cayley transform interchanges the first and second Hermitian forms

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$ (2)
Observe that \( C^{-1} = C \). In order to see that \( C \) is a Cayley transform, we calculate
\[
\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\]

When we are dealing with groups of matrices whose entries lie in a ring \( \mathcal{O} \) (for example the Picard modular groups) it will be necessary to choose a Cayley transform \( C \) so that the entries of \( C \) and \( C^{-1} \) are all integers. This will show that group of matrices in \( \text{GL}(3, \mathcal{O}) \) preserving the Hermitian forms \( \langle \cdot, \cdot \rangle_1 \) and \( \langle \cdot, \cdot \rangle_2 \) are conjugate in \( \text{GL}(3, \mathcal{O}) \). In this case we may choose the Cayley transform sending the ball model to the Siegel domain to be:
\[
C_0 = \begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix}
\]
which has inverse:
\[
C_0^{-1} = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix}.
\]

In order to see this, we calculate that
\[
\begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix}
\]
\[
= \begin{pmatrix} -1 & 2 & 2 \\ -2 & 3 & 2 \\ 2 & -2 & -1 \end{pmatrix}
\]
\[
= \begin{pmatrix} \frac{1}{\sqrt{2}} & 1 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & -\sqrt{2} \\ 0 & 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2} & -\sqrt{2} & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 1 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.
\]

The result follows since
\[
\begin{pmatrix} \frac{1}{\sqrt{2}} & 1 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & -\sqrt{2} \\ 0 & 0 & \sqrt{2} \end{pmatrix}
\]
is unitary with respect to the second Hermitian form with inverse
\[
\begin{pmatrix} \sqrt{2} & -\sqrt{2} & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 1 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.
\]
3 Isometries

3.1 The unitary groups of the first Hermitian form

Let $A$ be a matrix which preserves the first Hermitian form, that is a unitary matrix. In other words for all $v$ and $w$ in $\mathbb{C}^2$ we have

$$w^* A^* J_1 A v = \langle Av, Aw \rangle_1 = \langle v, w \rangle_1 = w^* J_1 v.$$

By letting $v$ and $w$ run through a basis of of $\mathbb{C}^2$, we see that this means

$$A^* J_1 A = J_1.$$

In other words,

$$J_1 = J_1 A^* J_1 A.$$

Writing $A$ in terms of its entries gives

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}, \quad A^{-1} = J_1^{-1} A^* J_1 = \begin{bmatrix} \overline{a} & \overline{d} & -\overline{g} \\ \overline{b} & \overline{e} & -\overline{h} \\ -\overline{c} & -\overline{f} & \overline{j} \end{bmatrix}. \quad (3)$$

We now use this expression to find relationships between the entries of $A$. The resulting identities will be used many times in later sections.

From elementary linear algebra, we know that

$$A^{-1} = \frac{\text{adj}(A)}{\det(A)}$$

where $\text{adj}(A)$ is the adjugate matrix:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} e_j - f h & ch - b j & bf - c e \\ f g - d j & a j - c g & cd - a f \\ d h - e g & b g - a h & a e - b d \end{bmatrix}.$$

Writing $\Delta = \det(A)$ and comparing these two expressions for $A^{-1}$ gives

$$\pi \Delta = e_j - f h \quad (4)$$

$$\bar{b} \Delta = f g - d j \quad (5)$$

$$\tau \Delta = e g - d h \quad (6)$$

$$\bar{d} \Delta = c h - b j \quad (7)$$

$$\bar{e} \Delta = a j - c g \quad (8)$$

$$\bar{f} \Delta = a h - b g \quad (9)$$

$$\bar{g} \Delta = c e - b f \quad (10)$$

$$\bar{h} \Delta = a f - c d \quad (11)$$

$$\bar{f} \Delta = a e - b d \quad (12)$$

As $A$ is unitary we have

$$|\Delta|^2 = \det(A^*) \det(A) = \det(J_1^{-1} A^* J_1 A) = \det(A^{-1} A) = 1.$$

From the equations $AA^{-1} = I$ and (3) we have the following identities relating the entries of $A$:

$$1 = |a|^2 + |b|^2 - |c|^2, \quad (13)$$

$$1 = |d|^2 + |e|^2 - |f|^2, \quad (14)$$

$$1 = -|g|^2 - |h|^2 + |j|^2, \quad (15)$$

$$0 = a \overline{d} + b \overline{e} - c \overline{f}, \quad (16)$$

$$0 = a \overline{g} + b \overline{h} - c \overline{j}, \quad (17)$$

$$0 = d \overline{g} + e \overline{h} - f \overline{j}. \quad (18)$$
Similarly from the relation $A^{-1}A = I$ we have

\begin{align}
1 &= |a|^2 + |d|^2 - |g|^2, \quad (19) \\
1 &= |b|^2 + |e|^2 - |h|^2, \quad (20) \\
1 &= -|c|^2 - |f|^2 + |j|^2, \quad (21) \\
0 &= \overline{ab} + \overline{de} - \overline{gh}, \quad (22) \\
0 &= \overline{ac} + \overline{df} - \overline{gj}, \quad (23) \\
0 &= \overline{bc} + \overline{ef} - \overline{hj}. \quad (24)
\end{align}

### 3.2 The unitary groups of the second Hermitian form

Let $A$ be a matrix which preserves the second Hermitian form, that is a unitary matrix. In other words for all $v$ and $w$ in $\mathbb{C}^2$ we have

$$w^* A^* J_2 A v = \langle A v, A w \rangle_2 = \langle v, w \rangle_2 = w^* J_2 v.$$ 

As before, letting $v$ and $w$ run through a basis of of $\mathbb{C}^2$, we see that this means $A^* J_2 A = J_2$. In other words, $A^{-1} = J_2^{-1} A^* J_2$.

\[ A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}, \quad A^{-1} = J_2^{-1} A^* J_2 = \begin{bmatrix} \overline{j} & \overline{f} & \overline{c} \\ \overline{h} & \overline{e} & \overline{b} \\ \overline{g} & \overline{d} & \overline{a} \end{bmatrix}. \quad (25) \]

Again, we use this expression to find useful identities between the entries of $A$.

Using the expression of $A^{-1}$ in terms of the adjugate matrix we obtain

\begin{align}
\overline{a} \Delta &= ae - bd \quad (26) \\
\overline{b} \Delta &= cd - af \quad (27) \\
\overline{c} \Delta &= bf - ce \quad (28) \\
\overline{d} \Delta &= bg - ah \quad (29) \\
\overline{e} \Delta &= aj - cg \quad (30) \\
\overline{f} \Delta &= ch - bj \quad (31) \\
\overline{g} \Delta &= dh - eg \quad (32) \\
\overline{h} \Delta &= fg - dj \quad (33) \\
\overline{j} \Delta &= ej - fh \quad (34)
\end{align}

From the equations $AA^{-1} = I$ and (25) we have the following identities relating the entries of $A$:

\begin{align}
1 &= a \overline{j} + b \overline{h} + c \overline{g}, \quad (35) \\
1 &= d \overline{j} + |e|^2 + f \overline{d}, \quad (36) \\
0 &= a \overline{j} + b \overline{e} + c \overline{d}, \quad (37) \\
0 &= a \overline{x} + |h|^2 + e \overline{a}, \quad (38) \\
0 &= d \overline{j} + e \overline{h} + f \overline{g}, \quad (39) \\
0 &= g \overline{j} + |h|^2 + j \overline{g}. \quad (40)
\end{align}
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Similarly from the relation $A^{-1}A = I$ we have
\begin{align*}
1 & = j a + f d + c g, \quad (41) \\
1 & = h b + |e|^2 + f h, \quad (42) \\
0 & = j b + f e + c h, \quad (43) \\
0 & = j c + |f|^2 + c j, \quad (44) \\
0 & = h a + c d + b g, \quad (45) \\
0 & = g a + |d|^2 + e g. \quad (46)
\end{align*}

3.3 PU(2, 1) and its action on complex hyperbolic space

We now show unitary matrices act on complex hyperbolic space. Any matrix in U(2, 1) which is a (non-zero) complex scalar multiple of the identity maps each line in $\mathbb{C}^{2,1}$ to itself and so acts trivially on complex hyperbolic space. Since this matrix is unitary with respect to $\langle \cdot , \cdot \rangle$ then the scalar must have unit norm. Because of this, we define the projective unitary group $PU(2, 1) = U(2, 1)/U(1)$ where $U(1)$ is canonically identified with $\{ e^{i\theta}I | 0 \leq \theta < 2\pi \}$, where $I$ is the identity matrix in $U(2, 1)$. Sometimes it will be useful to consider $SU(2, 1)$, the group of matrices with determinant 1 which are unitary with respect to $\langle \cdot , \cdot \rangle$. The group $SU(2, 1)$ is a 3-fold covering of $PU(2, 1)$:

$$PU(2, 1) = SU(2, 1)/\{ I, \omega I, \omega^2 I \}$$

where $\omega = (-1 + i\sqrt{3})/2$ is a cube root of unity. This is completely analogous to the fact that $SL(2, \mathbb{C})$ is a double cover of $PSL(2, \mathbb{C})$. Cube roots of unity are used because $SU(2, 1)$ comprises $3 \times 3$ matrices.

We now use Hermitian linear algebra to show that $PU(2, 1)$ acts transitively on $H^2_C$ and doubly transitively on $\partial H^2_C$.

We will begin by working with the first Hermitian form. Let $z \in \mathbb{C}^{2,1}$ be any negative vector. That is $\langle z, z \rangle_1 < 0$. Then $\hat{z} = z/\sqrt{-\langle z, z \rangle_1}$ is a negative vector with $\langle \hat{z}, \hat{z} \rangle_1 = -1$. We can now construct a matrix $A$ in $U(2, 1)$ whose third column is $\hat{z}$. In order to do this, we take any basis for $\mathbb{C}^{2,1}$ containing $\hat{z}$. We can then use a version of the Gram-Schmidt process in signature $(2, 1)$ to produce vectors $e_1$ and $e_2$ so that $\langle e_1, e_1 \rangle_1 = \langle e_2, e_2 \rangle_1 = 1$ and $\langle e_1, e_2 \rangle_1 = \langle e_2, \hat{z} \rangle_1 = \langle e_1, \hat{z} \rangle_1 = 0$. The vectors $e_j$ now form the first two columns of $A$. By construction $A^T J_1 A = J_1$ and so $A \in U(2, 1)$. Moreover, the image of the column vector $(0, 0, 1)^t$, that is the canonical lift of the origin $o$, under $A$ is just $\hat{z}$. This process leads to the following result which shows that $PU(2, 1)$ acts transitively on $H^2_C$.

**Proposition 3.1** For any point $z$ in $H^2_C$ (using the ball model) there is an element of $PU(2, 1)$ sending the origin $o$ to $z$.

**Proof:** We work with the unit ball model. Let $z$ be the canonical lift of $z$ to $\mathbb{C}^{2,1}$. As above we can scale $z$ to form $\hat{z} = z/\sqrt{-\langle z, z \rangle_1}$ and find a matrix $A$ in $U(2, 1)$ sending the canonical lift of the origin, $o$, to $\hat{z}$. Projectivising, we can view $A$ as an element of $PU(2, 1)$ sending $o$ to $z$ as required. $\square$
If we know a vector $e_1$ Hermitian orthogonal to $z$, then instead of using the Gram-Schmidt process, we could find $e_2$ using the Hermitian cross product. That is, if

$$p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}, \quad q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

then define $n$ by

$$n = \begin{bmatrix} p_1 q_3 - p_3 q_1 \\ p_3 q_2 - p_2 q_3 \\ p_2 q_1 - p_1 q_2 \end{bmatrix}.$$  \hspace{1cm} (47)

Then $n$ is orthogonal to $p$ and $q$ with respect to the first Hermitian form and

$$\langle n, n \rangle_1 = \langle p, q \rangle_1 \langle q, p \rangle_1 - \langle p, p \rangle_1 \langle q, q \rangle_1.$$

**Corollary 3.2** The stabiliser of a point in $\mathbb{H}_C^2$ under $\text{PU}(2, 1)$ is $\text{P}(\text{U}(2) \times \text{U}(1))$ which is conjugate to $\text{U}(2)$. Moreover, the stabiliser of the origin $o$ in the ball model acts on $\mathbb{B}^2$ with the usual action of $\text{U}(2)$ on $\mathbb{C}^2$.

**Proof:** We work in the unit ball model. By the above proposition we can conjugate so that the point in question is the origin. Now any matrix in $\text{PU}(2, 1)$ fixing the origin is the projectivisation of a block diagonal matrix in $\text{U}(2) \times \text{U}(1)$ in $\text{U}(2, 1)$. In other words, it has the form

$$\begin{pmatrix} A & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

where $A \in \text{U}(2)$ and $e^{i\theta} \in \text{U}(1)$. Projectivising we may assume that $e^{i\theta} = 1$. Clearly all matrices of this form stabilise the origin. This gives the result. \hspace{1cm} \square

We now consider the action of $\text{PU}(2, 1)$ on the boundary. We choose to work with the second Hermitian form. We first show how to find a vector $n$ that is Hermitian orthogonal to $p$ and $q$. If

$$p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}, \quad q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

then

$$n = \begin{bmatrix} p_1 q_3 - p_3 q_1 \\ p_3 q_2 - p_2 q_3 \\ p_2 q_1 - p_1 q_2 \end{bmatrix}.$$  \hspace{1cm} (48)

A short computation shows that we again have

$$\langle n, n \rangle_2 = \langle p, q \rangle_2 \langle q, p \rangle_2 - \langle p, p \rangle_2 \langle q, q \rangle_2.$$

We now show that $\text{PU}(2, 1)$ acts doubly transitively on $\partial \mathbb{H}_C^2$. For this we use the Siegel domain and the group preserving the second Hermitian form.

**Proposition 3.3** For any pair of points $p$ and $q$ in $\partial \mathbb{H}_C^2$ (using the Siegel domain model) there is an element of $\text{PU}(2, 1)$ sending the origin $o$ to $p$ and $\infty$ to $q$. 

Proof: Choose any lifts \( p \) and \( q \) of \( p \) and \( q \) to \( \mathbb{C}^2 \). Consider \( \tilde{p} = p/\langle p, q \rangle_2 \). So \( \langle \tilde{p}, \tilde{p} \rangle_2 = \langle q, q \rangle_2 = 0 \) and \( \langle \tilde{p}, q \rangle_2 = 1 \). Let \( n \) be the Hermitian orthogonal to \( \tilde{p} \) and \( q \) given by \( (48) \). Then \( \langle n, n \rangle_2 = 1 \) and \( \langle n, \tilde{p} \rangle_2 = \langle n, q \rangle_2 = 0 \). Let \( A \) be the matrix whose columns are \( q, n, \tilde{p} \) respectively. Then \( A^* J_2 A = J_2 \), that is \( A \) is unitary with respect to the second Hermitian form. Moreover, projectivising to a matrix in \( \text{PU}(2, 1) \), we see \( A \) sends \( o \) to \( p \) and \( \infty \) to \( q \) as required.

Lemma 3.4 Let \( p \) and \( q \) be null vectors with \( \langle p, q \rangle = -1 \). Let \( n \) be normal to \( p \) and \( q \) with \( \langle n, n \rangle = 1 \). Then for any \( z \in \mathbb{C}^2 \),
\[
\langle z, p \rangle \langle q, z \rangle + \langle z, q \rangle \langle p, z \rangle = |\langle z, n \rangle|^2 - \langle z, z \rangle.
\]
Proof: Write \( z \) in terms of \( p, q \) and \( n \). Then
\[
z = -\langle z, q \rangle p - \langle z, p \rangle q + \langle z, n \rangle n.
\]
Then
\[
\langle z, z \rangle = -\langle z, p \rangle \langle q, z \rangle - \langle z, q \rangle \langle p, z \rangle + \langle z, n \rangle \langle n, z \rangle.
\]

3.4 Complex hyperbolic isometries

Since the Bergman metric is given in terms of the Hermitian form \( \langle \cdot, \cdot \rangle \) it is clear that if \( A \) is unitary with respect to \( \langle \cdot, \cdot \rangle \) then \( A \) acts isometrically on the projective model of complex hyperbolic space. Thus \( \text{PU}(2, 1) \) is a subgroup of the complex hyperbolic isometry group.

There are isometries of \( \mathbf{h}^2_\mathbb{C} \) not in \( \text{PU}(2, 1) \). For example, consider coordinate-wise complex conjugation \( z \mapsto \bar{z} \). Then
\[
\cosh^2 \left( \frac{\rho(z, w)}{2} \right) = \frac{\langle z, w \rangle}{\langle z, z \rangle} = \frac{\langle w, z \rangle}{\langle w, w \rangle} = \cosh^2 \left( \frac{\rho(z, w)}{2} \right).
\]
Therefore complex conjugation is also an isometry of complex hyperbolic space.

We now show that the holomorphic isometry group of \( \mathbf{H}^2_\mathbb{C} \) is \( \text{PU}(2, 1) \) and that the full isometry group is generated by \( \text{PU}(2, 1) \) and complex conjugation.

Theorem 3.5 Every isometry of \( \mathbf{H}^2_\mathbb{C} \) is either holomorphic or else anti-holomorphic. Moreover, each holomorphic isometry of \( \mathbf{H}^2_\mathbb{C} \) is given by a matrix in \( \text{PU}(2, 1) \) and each anti-holomorphic isometry is given by complex conjugation followed by a matrix in \( \text{PU}(2, 1) \).

Proof: We use the ball model \( \mathbb{B}^2 \) of \( \mathbf{H}^2_\mathbb{C} \) and the first Hermitian form. Let \( \Phi \) be any isometry of \( \mathbf{H}^2_\mathbb{C} \). By applying an element of \( \text{PU}(n, 1) \) and using Proposition 3.1, we may assume that \( \Phi \) fixes the origin. Also, using Corollary 3.2, we may assume that \( \Phi \) maps \((1/2, 0) \subset \mathbb{B}^2 \) to some point \((x, 0) \subset \mathbb{B}^2 \) with \( 0 \leq x < 1 \).
Let \((z_1, z_2)\) be any point in \(\mathbb{B}^2\) and let \((w_1, w_2) = \Phi(z_1, z_2)\). Then
\[
\frac{1}{1 - |z_1|^2 - |z_2|^2} = \cosh^2 \left( \frac{\rho((z_1, z_2), (0, 0))}{2} \right)
= \cosh^2 \left( \frac{\rho((w_1, w_2), (0, 0))}{2} \right)
= \frac{1}{1 - |w_1|^2 - |w_2|^2}.
\]
Thus \(|z_1|^2 + |z_2|^2 = |w_1|^2 + |w_2|^2\).

Using this identity applied to \(\Phi(1/2, 0) = (x, 0)\) we can see that \(x = 1/2\) and so \(\Phi\) fixes \((1/2, 0) \in \mathbb{B}^2\).

Now consider \(\Phi(r, 0) = (a + ib, c + id)\) for any \(0 < r < 1\). From the above remark we see that
\[r^2 = a^2 + b^2 + c^2 + d^2.\]
In particular, \(a \leq r < 1\) and so \((1 - r/2)^2 \leq (1 - a/2)^2\). Also
\[
\frac{(1 - r/2)^2}{(1 - r^2)(1 - 1/4)} = \cosh^2 \left( \frac{\rho((r, 0), (1/2, 0))}{2} \right)
= \cosh^2 \left( \frac{\rho((a + ib, c + id), (1/2, 0))}{2} \right)
= \frac{(1 - a/2)^2 + (b/2)^2}{(1 - a^2 - b^2 - c^2 - d^2)(1 - 1/4)}.
\]
Thus we have \((1 - r/2)^2 = (1 - a/2)^2 + b^2\). In other words
\[(1 - a/2)^2 \geq (1 - r/2)^2 = (1 - a/2)^2 + b^2 \geq (1 - a/2)^2.\]
Thus \(a = r\) and \(b = c = d = 0\). Hence \(\Phi\) fixes \((r, 0)\) for all \(0 \leq r < 1\).

Now consider \(\Phi(0, 1/2) = (a + ib, c + id)\). As before \(1/4 = a^2 + b^2 + c^2 + d^2\). Moreover, for all \(0 < r < 1\) we have:
\[
\frac{1}{(1 - 1/4)(1 - r^2)} = \cosh^2 \left( \frac{\rho((0, 1/2), (r, 0))}{2} \right)
= \cosh^2 \left( \frac{\rho((a + ib, c + id), (r, 0))}{2} \right)
= \frac{(1 - ar)^2 + (br)^2}{(1 - a^2 - b^2 - c^2 - d^2)(1 - r^2)}.
\]
In other words \((a^2 + b^2)r^2 - 2ar + 1 = 1\) for all \(0 < r < 1\). Therefore \(a = b = 0\) and \(\Phi(0, 1/2) = (0, c + id)\). We may apply an element of \(PU(2, 1)\) fixing \((r, 0)\) and sending \((0, c + id)\) to \((0, s)\) for \(0 < s < 1\). It is then clear that \(s = 1/2\) and, reasoning as above, we can show \(\Phi(0, r) = (0, r)\) for all \(0 \leq r < 1\).
Finally consider $\Phi(z_1, z_2) = (w_1, w_2)$ for any $(z_1, z_2) \in \mathbb{B}^2$. Then, arguing as above, $|z_1|^2 + |z_2|^2 = |w_1|^2 + |w_2|^2$ and, for all $r$ with $0 < r < 1$, we have

$$\frac{(1 - rz_1)(1 - r\overline{z}_1)}{(1 - |z_1|^2 - |z_2|^2)(1 - r^2)} = \cosh^2 \left( \frac{\rho((z_1, z_2), (r, 0))}{2} \right)$$

$$= \cosh^2 \left( \frac{\rho((w_1, w_2), (r, 0))}{2} \right)$$

$$= \frac{(1 - rw_1)(1 - r\overline{w}_1)}{(1 - |w_1|^2 - |w_2|^2)(1 - r^2)}.$$ 

Thus $|1 - rz_1|^2 = |1 - rw_1|^2$ and equating coefficients of $r$ we see $|z_1|^2 = |w_1|^2$ and $\Re(z_1) = \Re(w_1)$. In other words $z_1 = w_1$ or $z_1 = \overline{w}_1$. Also

$$\frac{(1 - rz_2)(1 - r\overline{z}_2)}{(1 - |z_1|^2 - |z_2|^2)(1 - r^2)} = \cosh^2 \left( \frac{\rho((z_1, z_2), (0, r))}{2} \right)$$

$$= \cosh^2 \left( \frac{\rho((w_1, w_2), (0, r))}{2} \right)$$

$$= \frac{(1 - rw_2)(1 - r\overline{w}_2)}{(1 - |w_1|^2 - |w_2|^2)(1 - r^2)}.$$ 

A similar argument gives that $z_2 = w_2$ or $z_2 = \overline{w}_2$.

It is easy to check that $(z_1, z_2) \mapsto (z_1, \overline{z}_2)$ and $(z_1, z_2) \mapsto (\overline{z}_1, z_2)$ are not isometries. Thus $\Phi$ is either the identity, or complex conjugation.

Therefore any isometry of $\mathbb{H}_c^2$ is either in $\text{PU}(2,1)$, which means it is holomorphic, or it is an element of $\text{PU}(2,1)$ followed by complex conjugation, which means it is anti-holomorphic. \hfill \Box

### 3.5 Classification of isometries

In this section we will classify holomorphic isometries. The familiar trichotomy from real hyperbolic geometry applies in the complex hyperbolic setting as well: A holomorphic complex hyperbolic isometry $A$ is said to be:

(i) **loxodromic** if it fixes exactly two points of $\partial \mathbb{H}_c^2$;

(ii) **parabolic** if it fixes exactly one point of $\partial \mathbb{H}_c^2$;

(iii) **elliptic** if it fixes at least one point of $\mathbb{H}_c^2$.

We will analyse each of these types in more detail. As we do so, this coarse classification will be refined. We will give three different approaches to classifying isometries. First, we will discuss an algebraic approach by considering eigenvectors and eigenvalues of matrices in $\text{SU}(2,1)$. Secondly, we will discuss a dynamical approach by looking at fixed points and subsets that are preserved. Finally, (in Section 5.6) we will discuss a geometrical approach by considering products of involutions in totally real Lagrangian planes.
First we explain the classification in terms of the corresponding matrices in SU(2,1). It is clear that a fixed point of an isometry $A$ lying in $H^2_C$ or its boundary corresponds to an eigenvector of the corresponding matrix lying in $V_-$ or $V_0$ respectively. The goal of the first part of this section is to prove the following theorem, which verifies that the trichotomy above exhausts all possibilities:

**Theorem 3.6** Let $A$ be a matrix in SU(2,1). Then one of the following possibilities occurs:

(i) $A$ has two null eigenvectors with eigenvalues $\lambda$ and $\overline{\lambda}^{-1}$ where $|\lambda| \neq 1$, in which case $A$ is loxodromic;

(ii) $A$ has a repeated eigenvalue of unit modulus whose eigenspace is spanned by a null vector, in which case $A$ is parabolic;

(iii) $A$ has a negative eigenvector, in which case $A$ is elliptic.

We will prove Theorem 3.6 by way of a series of lemmas. First we investigate some general properties of eigenvalues of matrices in SU(2,1).

**Lemma 3.7** Let $A \in$ SU(2,1) and let $\lambda$ be an eigenvalue of $A$. Then $\overline{\lambda}^{-1}$ is an eigenvalue of $A$.

**Proof:** We know that $A$ preserves the Hermitian form defined by $J$. Hence, $A^*JA = J$ and so $A = J^{-1}(A^*)^{-1}J$. Thus $A$ has the same set of eigenvalues as $(A^*)^{-1}$ (they are conjugate). Since the characteristic polynomial of $A^*$ is the complex conjugate of the characteristic polynomial of $A$, we see that if $\lambda$ is an eigenvalue of $A$ then $\overline{\lambda}$ is an eigenvalue of $A^*$. Therefore $\overline{\lambda}^{-1}$ is an eigenvalue of $(A^*)^{-1}$ and hence of $A$. \(\square\)

**Corollary 3.8** If $\lambda$ is an eigenvalue of $A \in$ SU(2,1) with $|\lambda| \neq 1$ then $\overline{\lambda}^{-1}$ is a distinct eigenvalue. In particular, either $A$ has all three eigenvalues of absolute value 1 or else $A$ has a pair of eigenvalues $\lambda$ and $\overline{\lambda}^{-1}$ with $|\lambda| \neq 1$ and the third eigenvalue $\overline{\lambda}^{-1}$ of absolute value 1.

Next we show that any eigenvalue not of unit modulus corresponds to a null eigenvector and that any eigenvectors that are not (Hermitian) orthogonal have eigenvalues $\lambda$ and $\mu = \overline{\lambda}^{-1}$.

**Lemma 3.9** Let $\lambda$, $\mu$ be eigenvalues of $A \in$ SU(2,1) and let $v$, $w$ be any eigenvectors with eigenvalues $\lambda$, $\mu$ respectively.

(i) Either $|\lambda| = 1$ or $\langle v, v \rangle = 0$.

(ii) Either $\lambda \mu = 1$ or $\langle v, w \rangle = 0$. 
Thus either $|\lambda| = 1$ or $\langle v, v \rangle = 0$.

(ii) $\langle v, w \rangle = \langle Av, Aw \rangle = \langle \lambda v, \mu w \rangle = \lambda \mu \langle v, w \rangle$.

Thus either $\lambda \mu = 1$ or $\langle v, w \rangle = 0$. \hfill \Box

The following lemma is an easy consequence of the signature of $\mathbb{C}^{2,1}$.

Lemma 3.10 If $v, w \in \mathbb{C}^{2,1} - \{0\}$ with $\langle v, v \rangle \leq 0$ and $\langle w, w \rangle \leq 0$ then either $w = \lambda v$ for some $\lambda \in \mathbb{C}$ or $\langle v, w \rangle \neq 0$.

Proof: Let $e_1, e_2, e_3$ be the standard basis vectors for $\mathbb{C}^{2,1}$ with the first Hermitian form. That is

\[ 1 = \langle e_1, e_1 \rangle_1 = \langle e_2, e_2 \rangle_1 = -\langle e_3, e_3 \rangle_1, \]
\[ 0 = \langle e_1, e_2 \rangle_1 = \langle e_2, e_3 \rangle_1 = \langle e_3, e_1 \rangle_1. \]

Write $v = v_1 e_1 + v_2 e_2 + v_3 e_3$, $w = w_1 e_1 + w_2 e_2 + w_3 e_3$.

Because $\langle v, v \rangle_1 \leq 0$ and $\langle w, w \rangle_1 \leq 0$ we have

\[ |v_1|^2 + |v_2|^2 \leq |v_3|^2, \quad |w_1|^2 + |w_2|^2 \leq |w_3|^2. \]

Therefore $v_3$ and $w_3$ are both non-zero. Suppose that $\langle v, w \rangle_1 = 0$. This is equivalent to

\[ v_1 \overline{w_1} + v_2 \overline{w_2} = v_3 \overline{w_3}. \]

Hence, for all $\lambda \in \mathbb{C}$ we have

\[ |v_3 - \lambda w_3|^2 = |v_3|^2 - \lambda w_3 \overline{v_3} - \overline{\lambda} v_3 w_3 + |\lambda|^2 |w_3|^2 \]
\[ \geq |v_1|^2 - \lambda w_1 \overline{v_1} - \overline{\lambda} v_1 w_1 + |\lambda|^2 |w_1|^2 \]
\[ + |v_2|^2 - \lambda w_2 \overline{v_2} - \overline{\lambda} v_2 w_2 + |\lambda|^2 |w_2|^2 \]
\[ = |v_1 - \lambda w_1|^2 + |v_2 - \lambda w_2|^2. \]

Choose $\lambda = -v_3/w_3$. The left hand side of this inequality is then zero. Since the right hand side is non-negative, it too must be zero. This means that $v_1 - w_1 v_3/w_3 = v_2 - w_2 v_3/w_3 = 0$.

In other words, $v_1/w_1 = v_2/w_2 = v_3/w_3 = \lambda$ and so $v = \lambda w$. \hfill \Box

Using these lemmas we can analyse the eigenvectors of any $A$ in $\text{SU}(2,1)$ with an eigenvalue not of unit modulus. This will prove Theorem 3.6 (i).

Lemma 3.11 Suppose that the eigenvalues of $A \in \text{SU}(2,1)$ are $re^{i\theta}$, $r^{-1}e^{i\theta}$, $e^{-2i\theta}$ where $r \neq 1$ and they have eigenvectors $u$, $v$ and $w$ respectively then

\[ \langle u, u \rangle = \langle v, v \rangle = \langle u, w \rangle = \langle v, w \rangle = 0, \quad \langle w, w \rangle > 0, \quad \langle u, v \rangle \neq 0. \]
Proof: We know that \( \langle u, u \rangle \langle v, v \rangle = 0 \) by Lemma 3.9 (i) as \( r \neq 1 \). We also know that \( \langle u, w \rangle \langle v, w \rangle = 0 \) by part Lemma 3.9 (ii) as \( re^{i\theta} \neq 1 \) and \( r^{-1}e^{3i\theta} \neq 1 \). Using the Lemma 3.10 we see that \( \langle u, v \rangle \neq 0 \). If \( \langle w, w \rangle \leq 0 \) then, by Lemma 3.10 we would have \( \langle u, w \rangle \neq 0 \) which is a contradiction. \( \square \)

From now on we may assume that all the eigenvalues of \( A \) have unit modulus. We begin with the case where they are all distinct. We show that such a matrix satisfies the condition of Theorem 3.6 (iii).

Lemma 3.12 Suppose that the eigenvalues of \( A \in SU(2,1) \) are distinct with absolute value 1, and suppose they have eigenvectors \( u, v \) and \( w \) respectively then

\[
\langle u, v \rangle = \langle v, w \rangle = \langle w, u \rangle = 0
\]

and two of \( \langle v, v \rangle, \langle v, v \rangle, \langle w, w \rangle \) are positive while the other is negative.

Proof: Since the eigenvalues are distinct we have \( \langle u, v \rangle = \langle v, w \rangle = \langle w, u \rangle = 0 \), from Lemma 3.9 (ii) Now using Lemma 3.10, this implies that at most one of \( \langle v, v \rangle, \langle v, v \rangle, \langle w, w \rangle \) is non-positive. Since \( \langle \cdot, \cdot \rangle \) is non-degenerate and indefinite none of \( \langle v, v \rangle, \langle v, v \rangle, \langle w, w \rangle \) are zero and at least one of them is negative. \( \square \)

We need to consider the case of repeated eigenvalues. We begin by supposing the eigenvalues are \( e^{i\theta}, e^{i\theta} \) and \( e^{-2i\theta} \) where \( e^{i\theta} \neq e^{-2i\theta} \).

Lemma 3.13 Suppose that \( A \in SU(2,1) \) has a repeated eigenvalue \( e^{i\theta} \). Suppose that the \( e^{i\theta} \)-eigenspace is spanned by \( v \), then \( \langle v, v \rangle = 0 \).

Proof: Since \( e^{i\theta} \) is a repeated eigenvalue, there exists a vector \( u \) that is not a multiple of \( v \) and which satisfies \( Au = e^{i\theta}u + \lambda v \). Since \( u \) is not in the \( e^{i\theta} \)-eigenspace we must have \( \lambda \neq 0 \). Then

\[
\langle u, v \rangle = \langle Au, Av \rangle = \langle u, v \rangle + \lambda e^{-i\theta} \langle v, v \rangle.
\]

This implies that \( \langle v, v \rangle = 0 \) as claimed. \( \square \)

Lemma 3.14 Suppose that \( A \in SU(2,1) \) has eigenvalues \( e^{i\theta}, e^{i\theta} \) and \( e^{-2i\theta} \). Let \( v \) be an eigenvector corresponding to \( e^{i\theta} \) and \( w \) be the eigenvector corresponding to \( e^{-2i\theta} \). Suppose that \( \langle w, w \rangle \leq 0 \). Then in fact \( \langle w, w \rangle < 0 \), \( \langle v, v \rangle = 0 \), \( \langle v, v \rangle > 0 \) and the \( e^{i\theta} \)-eigenspace is two dimensional.

Proof: As \( v \) and \( w \) correspond to distinct eigenvalues of unit modulus then, by part Lemma 3.9 (ii) \( \langle v, w \rangle = 0 \). If \( \langle v, v \rangle \leq 0 \) then by Lemma 3.10 \( \langle v, w \rangle \neq 0 \), a contradiction. Using Lemma 3.13, this means that the \( e^{i\theta} \)-eigenspace cannot be spanned by \( v \). Therefore there are orthogonal \( e^{i\theta} \)-eigenvectors \( v \) and \( u \) with \( \langle v, v \rangle > 0 \) and \( \langle u, u \rangle > 0 \). Since \( \langle \cdot, \cdot \rangle \) is indefinite and non-degenerate we have \( \langle w, w \rangle < 0 \). \( \square \)

Lemma 3.15 Suppose that \( A \in SU(2,1) \) has eigenvalues \( e^{i\theta}, e^{i\theta} \) and \( e^{-2i\theta} \). Let \( v \) be an eigenvector corresponding to \( e^{i\theta} \) and \( w \) be the eigenvector corresponding to \( e^{-2i\theta} \). Suppose that \( \langle w, w \rangle > 0 \). Then either
Lemma 3.10, we may suppose that 

\[ A v = v \]

Then \( v \) is linearly independent, we see that \( v \) is linearly independent. Suppose that there exist non-trivial, linearly independent vectors \( w \). Therefore, \( u, v \) and \( w \) are linearly independent and hence form a basis for \( \mathbb{C}^{2,1} \). Suppose that \( z \) is a negative vector in \( \mathbb{C}^{2,1} \) and write \( z = \alpha u + \beta v + \gamma w \). Then

\[ 0 > \langle z, z \rangle = |\alpha|^2 \langle u, u \rangle + |\beta|^2 \langle v, v \rangle + |\gamma|^2 \langle w, w \rangle. \]

As \( \langle w, w \rangle > 0 \), at least one of \( \langle u, u \rangle \) or \( \langle v, v \rangle \) is negative. As \( \mathbb{C}^{2,1} \) has signature \((2,1)\) one of them is negative and the other positive. This gives the result. \( \Box \)

Combining these lemmas we see that if \( A \) has two distinct eigenvalues, one of which is repeated, then \( A \) either satisfies the condition of Theorem 3.6 (ii) or Theorem 3.6 (iii).

Finally we need to consider the case where all three eigenvalues are the same, necessarily a cube root of unity. We will show that \( A \) satisfies the conditions of Theorem 3.6 (ii). So this will complete the proof of Theorem 3.6.

Lemma 3.16 Suppose that \( A \in SU(2,1) \) has exactly one eigenvalue. Then either \( A \) is a multiple of the identity or else the eigenspace is spanned by a null vector.

Proof: The eigenspace of \( A \) has dimension 1, 2 or 3. If it has dimension 3 then, necessarily, \( A \) is a multiple of the identity. If it has dimension 1 then, by Lemma 3.13, this eigenspace is spanned by a null vector.

Thus we only have to rule out the case where the eigenspace has dimension 2. Thus we suppose that there exist non-trivial, linearly independent vectors \( v, w \in \mathbb{C}^{2,1} \) so that \( Av = e^{i\theta}v \) and \( Aw = e^{i\theta}w \). Taking linear combinations of \( v \) and \( w \) if necessary and using Lemma 3.10, we may suppose that \( \langle v, v \rangle \neq 0 \), \( \langle w, w \rangle \neq 0 \) and \( \langle v, w \rangle = 0 \). Let \( n \) be the common (Hermitian) orthogonal of \( v \) and \( w \), as found in (47) or (48). As \( v \) and \( w \) are linearly independent we see that \( n \neq 0 \). Also, since neither \( v \) nor \( w \) is null, \( n \neq v, w \). Then \( v, w \) and \( n \) form a basis for \( \mathbb{C}^{2,1} \). Therefore

\[ An = \frac{\langle An, v \rangle}{\langle v, v \rangle} v + \frac{\langle An, w \rangle}{\langle w, w \rangle} w + \frac{\langle An, n \rangle}{\langle n, n \rangle} n. \]

Now we have

\[ \langle An, v \rangle = e^{i\theta} \langle An, Av \rangle = e^{i\theta} \langle n, v \rangle = 0 \]

and so \( \langle An, v \rangle = 0 \). Similarly, \( \langle An, w \rangle = 0 \). Thus \( An \) is a multiple of \( n \). Hence \( n \) is an eigenvector of \( A \). Since \( A \) only has one eigenvalue, namely \( e^{i\theta} \), then \( n \) is in the span of \( v \) and \( w \). This is a contradiction. \( \Box \)

We now show that we can use the trace of \( A \in SU(2,1) \) to decide the class of \( A \). From Corollary 3.8 we see that, if \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) are eigenvalues of \( A \), then \( \lambda_1^{-1}, \lambda_2^{-1} \) and \( \lambda_3^{-1} \) are
form some permutation of \( \lambda_1, \lambda_2 \) and \( \lambda_3 \). Let \( \chi_A(x) \) be the characteristic polynomial of \( A \). Suppose that
\[
\chi_A(x) = x^3 - a_2 x^2 + a_1 x - a_0.
\]
Then \( a_2 = \lambda_1 + \lambda_2 + \lambda_3 = \text{tr}(A) \) and \( a_0 = \lambda_1 \lambda_2 \lambda_3 = \det(A) = 1 \). The other coefficient is
\[
a_1 = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1 = \frac{1}{2} \lambda_1 \lambda_2 + \frac{1}{2} \lambda_2 \lambda_3 + \frac{1}{2} \lambda_3 \lambda_1 = \frac{1}{2} \text{tr}(A).
\]
We denote the trace of \( A \) by \( \text{tr}(A) = \tau \). Thus we have
\[
\chi_A(x) = x^3 - \tau x^2 + \tau x - 1.
\]

We want to find out when \( A \in \text{SU}(2,1) \) has repeated eigenvalues. In other words, we want to find conditions on \( \tau \) for which \( \chi_A(x) = 0 \) has repeated solutions. This is true if and only if \( \chi_A(x) \) and its derivative \( \chi_A'(x) \) have a common root. Two polynomials have a common root if and only if their resultant vanishes (see Kirwan [17]). Now
\[
\chi_A'(x) = 3x^2 - 2\tau x + \tau.
\]
Therefore a brief calculation shows that the resultant of \( \chi_A(x) \) and \( \chi_A'(x) \) is given by:
\[
R(\chi_A, \chi_A') = \begin{vmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & -\tau & \tau \\
3 & -2\tau & 0 & 0 \\
0 & 3 & -2\tau & 0
\end{vmatrix} = |\tau|^4 - 8\Re(\tau^3) + 18|\tau|^2 - 27.
\]

**Theorem 3.17** Let \( f(\tau) = |\tau|^4 - 8\Re(\tau^3) + 18|\tau|^2 - 27 \). Let \( A \in \text{SU}(2,1) \) then:

(i) \( A \) has an eigenvalue \( \lambda \) with \( |\lambda| \neq 1 \) if and only if \( f(\text{tr}(A)) > 0 \),

(ii) \( A \) has a repeated eigenvalue if and only if \( f(\text{tr}(A)) = 0 \),

(iii) \( A \) has distinct eigenvalues of unit modulus if and only if \( f(\text{tr}(A)) < 0 \).

**Proof:** Part (ii) follows from the reasoning given above. In part (i) we know that the eigenvalues of \( A \) have the form \( re^{i\theta}, r^{-1}e^{i\theta}, e^{-2i\theta} \) with \( r \neq 1 \) by the corollary. A brief calculation (exercise) shows that
\[
f(re^{i\theta} + r^{-1}e^{i\theta} + e^{-2i\theta}) = (r - r^{-1})^2(r + r^{-1} - 2\cos(3\theta))^2 > 0.
\]
In part (iii) write the eigenvalues as \( e^{i\theta}, e^{i\phi}, e^{i\psi} \) where \( \theta, \phi \) and \( \psi \) are distinct and sum to zero mod 2\pi. Then another brief calculation (exercise) shows that
\[
f(e^{i\theta} + e^{i\phi} + e^{i\psi}) = -4(\sin(\theta - \phi) + \sin(\phi - \psi) + \sin(\phi - \psi))^2 < 0.
\]
This proves the theorem. \( \square \)
The curve \( f(\tau) = 0 \) is a classical curve called a deltoid, see Chapter 8 of Lockwood [20] or page 26 of Kirwan [17] where it is written in terms of \( x = \Re(\tau) \) and \( y = \Im(\tau) \). The points outside correspond to case (i) in the theorem. This may be seen by considering \( A \) with eigenvalues \( r, r^{-1} \) and 1 which lie in the interval \((3, \infty)\) and those with eigenvalues \( e^{i\theta}, e^{i\theta} \) and 1 which lie in \((-1, 3)\). The rest follows by continuity.

**Lemma 3.18** Suppose that \( A \) is an elliptic element of \( SU(2,1) \) with real trace, that is \( \tr(A) \in [-1,3) \). Then the eigenvalues of \( A \) are 1, \( e^{i\theta} \) and \( e^{-i\theta} \) where \( 2\cos(\theta) = \tr(A) - 1 \).

**Proof:** If the eigenvalues of \( A \) are \( e^{i\theta}, e^{i\phi} \) and \( e^{-i\theta-i\phi} \) then

\[
0 = \Im(\tr(A)) = \sin(\theta) + \sin(\phi) - \sin(\theta + \phi) = 4\sin(\theta/2)\sin(\phi/2)\sin(\theta/2 + \phi/2)
\]

and so at least one of \( \theta, \phi \) or \( \theta + \phi \) is an integer multiple of \( 2\pi \). Hence \( A \) has eigenvalue +1. The rest of the lemma follows easily. \( \square \)

We now sum up the previous results about eigenvectors and eigenvalues and interpret them in terms of information about fixed points.

First, consider a loxodromic map \( A \). We know that \( A \) corresponds to a matrix with eigenvectors \( re^{i\theta}, r^{-1}e^{i\theta}, e^{-2i\theta} \), where \( r > 1 \), with eigenvectors \( p, q \in V_0 \) and \( n \in V_+ \) respectively. These correspond to an attractive fixed point \( p \) and a repulsive fixed point \( q \) on \( \partial H^2_\C \). The complex line \( L \) spanned by \( p \) and \( q \) is preserved by \( A \). This line has polar vector \( n \).

Next, consider a parabolic map \( A \). Such a map corresponds to a matrix with a repeated eigenvalue of unit modulus whose eigenspace is spanned by a null eigenvector \( p \). This vector corresponds to a neutral fixed point \( p \) on \( \partial H^2_\C \). There are two cases to consider, namely when \( A \) has a single eigenvalue of multiplicity 3 and when \( A \) has two distinct eigenvalues, one of which is repeated. In the first case we say that \( A \) is pure parabolic. Later we shall see that pure parabolic maps correspond to Heisenberg translations. A pure parabolic map has trace 3 or \((-3 \pm 3i\sqrt{3})/2\), that is it corresponds to one of the three corners of the deltoid. In the second case we say that \( A \) is screw parabolic. In this case the non-repeated eigenvalue has an eigenvector \( n \) in \( V_+ \). The complex line polar to \( n \) is preserved by \( A \), and \( A \) acts as a translation there. Moreover, \( A \) rotates \( H^2_\C \) around this complex line. Screw parabolic maps correspond to smooth points of the deltoid.

Finally, consider an elliptic map \( A \). There are now three cases. First, suppose that \( A \) has a repeated eigenvalue with a two dimensional eigenspace containing both positive and negative vectors. This eigenspace corresponds to a complex line \( L \) on which \( A \) acts as the identity. In particular, there are points of \( \partial H^2_\C \) fixed by \( A \) and so \( A \) is called boundary elliptic. As \( A \) fixes \( L \) and rotates \( H^2_\C \) around \( L \), it is complex reflection in the line \( L \). If \( A \) is not boundary elliptic then it has an eigenspace spanned by a negative vector \( w \). This corresponds to a fixed point \( w \in H^2_\C \). In this case \( A \) is called regular elliptic. There are two possibilities. Either \( A \) has a repeated eigenvalue with an eigenspace spanned by two positive vectors. In this case \( A \) is complex reflection in the point \( w \). Otherwise, \( A \) has three distinct eigenvalues. Complex reflections again correspond to smooth points of the deltoid while other elliptic maps correspond to points of the deltoid’s interior.
4 The boundary

4.1 Relation to the Heisenberg group

We recall that one model of real hyperbolic $n$-space $\mathbb{H}^n_\mathbb{R}$ is the upper half space in $\mathbb{R}^n$, that is $\{(x_1, \ldots, x_n) | x_1, \ldots, x_n \in \mathbb{R}, x_n > 0\}$. The boundary of this model is the one point compactification of $\mathbb{R}^{n-1}$ thought of as the subspace of $\mathbb{R}^n$ given by $x_n = 0$. This model of real hyperbolic $n$ space is foliated by horospheres $H_u$ for $u > 0$. The horosphere $H_u$ is a copy of $\mathbb{R}^{n-1}$ given by points with $x_n = u$. We want to form the analogous construction for complex hyperbolic space.

In this section we work in the Siegel domain model and we consider $\text{PU}(2, 1)$ preserving the second Hermitian form. First we study the boundary a little more carefully. A finite point $z$ is in the boundary of the Siegel domain if its standard lift to $\mathbb{C}^2$ is $z$ where

$$z = \begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix} \quad \text{where} \quad z_1 + \overline{z}_1 + |z_2|^2 = 0.$$

We write $\zeta = z_2/\sqrt{2} \in \mathbb{C}$ and this condition becomes $2\Re(z_1) = -2|\zeta|^2$. Hence we may write $z_1 = -|\zeta|^2 + iv$ for $v \in \mathbb{R}$. That is for $\zeta \in \mathbb{C}$ and $v \in \mathbb{R}$:

$$z = \begin{bmatrix} -|\zeta|^2 + iv \\ \sqrt{2}\zeta \\ 1 \end{bmatrix}$$

Therefore we may identify the boundary of the Siegel domain with the one point compactification of $\mathbb{C} \times \mathbb{R}$.

We now investigate the effect of isometries in $\text{PU}(2, 1)$ on these finite boundary points. We will show that the collection of these points has a group law giving it the structure of the Heisenberg group. Thus the boundary of the Siegel domain is the one point compactification of the Heisenberg group.

Lemma 4.1 Suppose that $A \in \text{PU}(2, 1)$ has the standard form (25). Then the following are equivalent:

(i) $A$ fixes $\infty$,

(ii) $A$ is upper triangular,

(iii) $g = 0$.

Proof: Using the notation of (25), we see that $A$ fixes $\infty$, if and only if $d = g = 0$. Moreover, $A$ fixes $\infty$ if and only if $A^{-1}$ also fixes $\infty$. Using the expression for $A^{-1}$ given in (25) we see that $A^{-1}$ fixes $\infty$ if and only if $h = g = 0$. Thus $A$ fixes $\infty$ if and only if it is upper triangular.

Clearly if $A$ is upper triangular then $g = 0$. Conversely, assume $g = 0$. Using (40) and (46) we see that this implies $|d|^2 = |h|^2 = 0$. This proves the result. $\square$
Consider the map $T$ from $\mathbb{C} \times \mathbb{R}$ to $\text{GL}(3, \mathbb{C})$ given by

$$T(\zeta, v) = \begin{bmatrix} 1 & -\sqrt{2} \zeta & -|\zeta|^2 + iv \\ 0 & 1 & \sqrt{2} \zeta \\ 0 & 0 & 1 \end{bmatrix}.$$ 

It is easy to check that this fixes infinity and sends the origin to the point $(\zeta, v)$. It is also easy to check that $T(\zeta, v)$ is in $\text{PU}(2, 1)$ since

$$(T(\zeta, v))^{-1} = J_2(T(\zeta, v))^* J_2 = \begin{bmatrix} 1 & \sqrt{2} \zeta & -|\zeta|^2 - iv \\ 0 & 1 & -\sqrt{2} \zeta \\ 0 & 0 & 1 \end{bmatrix} = T(-\zeta, -v).$$

In order to find the group law we multiply two such matrices

$$T(\zeta, v)T(\xi, t) = \begin{bmatrix} 1 & -\sqrt{2} \zeta & -|\zeta|^2 + iv \\ 0 & 1 & \sqrt{2} \zeta \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & \sqrt{2} \xi & -|\xi|^2 + it \\ 0 & 1 & -\sqrt{2} \xi \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\sqrt{2}(\zeta + \xi) & -|\zeta + \xi|^2 + iv + it + \sqrt{2} \zeta - \sqrt{2} \xi \\ 0 & 1 & \sqrt{2}(\zeta + \xi) \\ 0 & 0 & 1 \end{bmatrix} = T(\zeta + \xi, v + t + 2\Im(\xi \zeta)).$$

This means that $T$ is a group homomorphism to $\text{PU}(2, 1)$ from $\mathbb{C} \times \mathbb{R}$ with the group law

$$(\zeta, v) * (\xi, t) = (\zeta + \xi, v + t + 2\Im(\xi \zeta)).$$

This group law gives $\mathbb{C} \times \mathbb{R}$ the structure of the 3 dimensional Heisenberg group $\mathcal{N}$. We also remark that $\Im(\xi \zeta) = \omega(\zeta, \xi)$ where $\omega$ is the standard symplectic form on $\mathbb{C}$.

The Heisenberg group is not Abelian but is 2-step nilpotent. In order to see this observe that

$$(\zeta, v) * (\xi, t) * (-\zeta, -v) * (-\xi, -t) = (0, 4\Im(\xi \zeta)).$$

Therefore any point in $\mathcal{N}$ of the form $(0, t)$ is central and the commutator of any two elements lies in the centre.

Geometrically, we think of the $\mathbb{C}$ factor of $\mathcal{N}$ as being horizontal and the $\mathbb{R}$ factor as being vertical. We refer to $T(\zeta, v)$ as Heisenberg translation by $(\zeta, v)$. A Heisenberg translation by $(0, t)$ is called vertical translation by $t$. It is easy to see the Heisenberg translations are ordinary translations in the horizontal direction and shears in the vertical direction. The fact that $\mathcal{N}$ is nilpotent means that translating around a horizontal square gives a vertical translation, rather like going up a spiral staircase. We define vertical projection $\Pi : \mathcal{N} \to \mathbb{C}$ to be the map $\Pi(\zeta, v) = \zeta$.

### 4.2 Horospherical coordinates

Fix $u \in \mathbb{R}_+$ and consider all those points $z \in \mathbf{H}_k^3$ for which the standard lift $\mathbf{z}$ has $\langle \mathbf{z}, \mathbf{z} \rangle = -2u$. This is equivalent to saying

$$z = \begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix} \text{ where } z_1 + \overline{z}_1 + |z_2|^2 = -2u.$$
In other words, $2\Re(z_1) = -|z_2|^2 - 2u$. We again write $z_2 = \sqrt{2}\zeta$ which means that $z_1 = -|\zeta|^2 - u + iv$. Thus $z$ corresponds to a point $(\zeta, v, u) \in \mathbb{C} \times \mathbb{R} \times \mathbb{R}_+$ via

$$z = \begin{bmatrix} -|\zeta|^2 - u + iv \\ \sqrt{2}\zeta \\ 1 \end{bmatrix}.$$ 

Let $H_u$ denote the set of points in $H^2_\mathbb{C}$ with $(z, z)_2 = -2u$. This is called the horosphere of height $u$. It carries the structure of the Heisenberg group. Thus, for example (left) Heisenberg translation by $(\tau, t)$ is given by

$$T(\tau, t) : (\zeta, v, u) \mapsto (\zeta + \tau, v + t + 2\Im(\zeta\tau), u).$$

In this way we canonically identify a point $z$ in the Siegel domain with $(\zeta, v, u) \in \mathbb{N} \times \mathbb{R} \times \mathbb{R}_+$ and we call $(\zeta, v, u)$ the horospherical coordinates of $z$. Sometimes it is useful to identify the finite boundary points with the horosphere of height zero, that is $H_0 = \partial H^2_\mathbb{C} - \{\infty\}$. This means that $(\zeta, v) = (\zeta, v, 0) \in \partial H^2_\mathbb{C} - \{\infty\}$.

Likewise, we define the horoball $B_t$ of height $t$ to be the union of all horospheres of height $u > t$. This is an open (topological) ball of dimension 4. Thus $H^2_\mathbb{C}$ is itself the horoball of height 0.

With respect to horospherical coordinates the second Hermitian form is given by

$$(\zeta, v, u, (\zeta, t, s))_2 = -|\zeta - \xi|^2 - u - s + i(v - t + 2\Im(\zeta\xi)).$$

With respect to horospherical coordinates $(x + iy, v, u)$ the Bergman metric is given by

$$ds^2 = \frac{-4}{\langle z, z \rangle^2} \det \begin{pmatrix} \langle z, z \rangle & \langle dz, z \rangle \\ \langle z, dz \rangle & \langle dz, dz \rangle \end{pmatrix}$$

$$= \frac{1}{u^2} \left( 4u \, dx^2 + 4u \, dy^2 + du^2 + (dv + 2x \, dy - 2y \, dx)^2 \right)$$

$$= \left( dx \, dy \, du \, dv \right) \begin{pmatrix} 4(u + y^2)/u^2 & -4xy/u^2 & 0 & -2y/u^2 \\ -4xy/u^2 & 4(u + x^2)/u^2 & 0 & 2x/u^2 \\ 0 & 0 & 1/u^2 & 0 \\ -2y/u^2 & 2x/u^2 & 0 & 1/u^2 \end{pmatrix} \left( dx \, dy \, du \, dv \right).$$

Hence, as a Riemannian metric, the Bergman metric is given by the inner product on the (real) tangent space to the Siegel domain defined, with respect to the basis $(dx, dy, du, dv)$, by the matrix $g$, where

$$g = \begin{pmatrix} 4(u + y^2)/u^2 & -4xy/u^2 & 0 & -2y/u^2 \\ -4xy/u^2 & 4(u + x^2)/u^2 & 0 & 2x/u^2 \\ 0 & 0 & 1/u^2 & 0 \\ -2y/u^2 & 2x/u^2 & 0 & 1/u^2 \end{pmatrix}. \quad (49)$$

Therefore the volume form on the Siegel domain is given by

$$d\text{Vol} = \sqrt{\det(g)} \, dx \, dy \, du \, dv = \frac{4}{u^3} \, dx \, dy \, du \, dv. \quad (50)$$
Alternatively, we could have used 
\[-z_1 - \overline{z}_1 - |z_2|^2 = 2u\]
to derive this from Proposition 2.2 and the Jacobian relating horospherical coordinates and Siegel domain coordinates:

\[
dx \, dy \, du \, dv = 2 \, dx_1 \, dy_1 \, dx_2 \, dy_2.
\]

We now investigate the how the Cayley transform from Section 2.3 changes between horospherical and ball coordinates. Consider \((x_1 + iy_1, x_2 + iy_2)\) in the unit ball in \(\mathbb{C}^2\).

Taking the canonical lift to \(\mathbb{C}^2\) and then applying the Cayley transform gives

\[
\frac{1}{\sqrt{2}} \begin{bmatrix}
1 & 0 & 1 \\
0 & \sqrt{2} & 0 \\
1 & 0 & -1
\end{bmatrix} \begin{bmatrix}
x_1 + iy_1 \\
x_2 + iy_2 \\
1
\end{bmatrix} = \begin{bmatrix}
(x_1 + 1 + iy_1)/\sqrt{2} \\
x_2 + iy_2 \\
(x_1 - 1 + iy_1)/\sqrt{2}
\end{bmatrix}.
\]

Projectivising so that the third coordinate is 1 this becomes

\[
\frac{1}{\sqrt{2}} \begin{bmatrix}
(x_1 + 1 + iy_1)/(x_1 - 1 + iy_1) \\
\sqrt{2}(x_2 + iy_2)/(x_1 - 1 + iy_1) \\
1
\end{bmatrix} = \begin{bmatrix}
(x_1^2 + y_1^2 - 1 - 2iy_1)/((x_1 - 1)^2 + y_1^2) \\
\sqrt{2}(x_1x_2 + y_1y_2 - x_2 + ix_1y_2 - iy_1x_2 - iy_2)/((x_1 - 1)^2 + y_1^2) \\
1
\end{bmatrix}.
\]

Thus the horospherical coordinates become \((\zeta, vu)\) where

\[
\zeta = \frac{x_1x_2 + y_1y_2 - x_2 + ix_1y_2 - iy_1x_2 - iy_2}{(x_1 - 1)^2 + y_1^2},
\]

\[
v = \frac{-2y_1}{(x_1 - 1)^2 + y_1^2},
\]

\[
u = \frac{1 - x_1^2 - y_1^2 - x_2^2 - y_2^2}{(x_1 - 1)^2 + y_1^2}.
\]

A straightforward, though lengthy, computation shows that the Jacobian of this transformation is

\[
J = \frac{4}{((x_1 - 1)^2 + y_1^2)^3}.
\]

Hence horospherical coordinates and ball coordinates are related by

\[
dx \, dy \, du \, dv = \frac{4}{((x_1 - 1)^2 + y_1^2)^3} \, dx_1 \, dy_1 \, dx_2 \, dy_2.
\]

Using Proposition 2.1, we can see again how the volume form transforms between different sets of coordinates:

\[
dVol = \frac{16}{(1 - x_1^2 - y_1^2 - x_2^2 - y_2^2)^3} \, dx_1 \, dy_1 \, dx_2 \, dy_2 = \frac{4}{u^3} \, dx \, dy \, du \, dv.
\]

We may also define horospherical coordinates based at a point other than infinity. Later we will want to do this for horospherical coordinates based at the origin \(o = (0, 0) \in \mathcal{N}\).
Because horospherical coordinates are not defined intrinsically but require some normalisation we need to be careful about what the horosphere centred at $o$ of height $u$ means.

We define horospheres and horoballs based at $o$ as the image of those based at $\infty$ under the inversion $\iota$ given by the matrix

$$
\iota = \begin{bmatrix}
0 & 0 & -1 \\
0 & -1 & 0 \\
-1 & 0 & 0
\end{bmatrix}.
$$

We now investigate the effect of $\iota$ on horospherical coordinates

$$
\begin{bmatrix}
0 & 0 & -1 \\
0 & -1 & 0 \\
-1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
-|z|^2 - u + iv \\
\sqrt{2z} \\
1
\end{bmatrix}
= \begin{bmatrix}
-1 \\
-\sqrt{2z} \\
1
\end{bmatrix}
\approx \begin{bmatrix}
-\frac{|z|^2 - u - iv}{(|z|^2 + u)^2 + v^2} \\
-\frac{\sqrt{2z}}{|z|^2 + u - iv} \\
1
\end{bmatrix}.
\tag{51}
$$

Thus the map $\iota$ carries the point of $H^2_C$ with horospherical coordinates $(\zeta, v, u)$ to the point with coordinates

$$
\iota(\zeta, v, u) = \left( \frac{-\zeta}{|\zeta|^2 + u - iv}, \frac{-v}{(|\zeta|^2 + u)^2 + v^2}, \frac{u}{(|\zeta|^2 + u)^2 + v^2} \right). \tag{52}
$$

Using this we define horospherical coordinates about $o$ as the image under $\iota$ of horospherical coordinates about $\infty$.

Similarly elements of $PU(2,1)$ fixing $o$ may be obtained from those fixing $\infty$ by conjugating by $\iota$. Thus we may speak of Heisenberg translation by $(\tau, t)$ fixing $o$. This is just the conjugate by $\iota$ of the Heisenberg translation by $(\tau, t)$ fixing $\infty$. As a matrix in $PU(2,1)$ it is given by

$$
\begin{bmatrix}
1 & 0 & 0 \\
\sqrt{2\tau} & 1 & 0 \\
-|\tau|^2 + it & -\sqrt{2\tau} & 1
\end{bmatrix}.
$$

4.3 The Cygan metric

In this section we define a metric on the Heisenberg group, the Cygan metric. We extend the Cygan metric to an incomplete metric on $H^2_C$ which agrees with the Cygan metric on each horosphere. This metric should be thought of as the counterpart to the Euclidean metric on the upper half space model of real hyperbolic space.

The Heisenberg norm is given by

$$
|\langle \zeta, v \rangle| = \left| |\zeta|^2 - iv \right|^{1/2}.
$$

This gives rise to a metric, the Cygan metric, on $N$ by

$$
\rho_0((\zeta_1, v_1), (\zeta_2, v_2)) = \left| (\zeta_1, v_1)^{-1} * (\zeta_2, v_2) \right|.
$$
In other words,

\[ \rho_0((\zeta_1, v_1), (\zeta_2, v_2)) = \left| \frac{-|\zeta_1|^2 + iv_1}{\sqrt{2}\zeta_1} - \frac{-|\zeta_2|^2 + iv_2}{\sqrt{2}\zeta_2} \right|^{1/2}. \]

If we take the standard lift of points on \( \partial \mathbb{H}_C^2 - \{ \infty \} \) to \( \mathbb{C}^{2,1} \) we can write the Cygan metric in terms of the second Hermitian form:

\[ \rho_0((\zeta_1, v_1), (\zeta_2, v_2)) = \left| \begin{bmatrix} -|\zeta_1|^2 + iv_1 \\ \sqrt{2}\zeta_1 \\ 1 \end{bmatrix}, \begin{bmatrix} -|\zeta_2|^2 + iv_2 \\ \sqrt{2}\zeta_2 \\ 1 \end{bmatrix} \right|^{1/2}. \]

We remark that the Cygan metric is not a path metric. That is, there exist pairs of points such that the Cygan distance between them is strictly shorter than the Cygan length of any path joining them (see section 3.1 of [7] for more details of the connection between metrics and path metrics). In order to demonstrate this fact it suffices to give a pair of points \((\zeta_1, v_1)\) and \((\zeta_2, v_2)\) so that for all points \((\zeta_3, v_3)\) with \((\zeta_3, v_3) \neq (\zeta_1, v_1), (\zeta_2, v_2)\) the triangle inequality is strict. That is

\[ \rho_0((\zeta_1, v_1), (\zeta_2, v_2)) < \rho_0((\zeta_1, v_1), (\zeta_3, v_3)) + \rho_0((\zeta_3, v_3), (\zeta_2, v_2)). \]

By the triangle inequality, the Cygan length of any path joining \((\zeta_1, v_1)\) and \((\zeta_2, v_2)\) is at least as big as the right hand side of this inequality. It is easiest to demonstrate this when \((\zeta_1, v_1) = (0, 0)\) and \((\zeta_2, v_2) = (0, 1)\). Then we have

\[
\begin{align*}
\rho_0((0, 0), (\zeta_3, v_3)) &+ \rho_0((\zeta_3, v_3), (0, 1)) \\
&= \left| |\zeta_3|^2 + iv_3 \right|^{1/2} + \left| |\zeta_3|^2 + i(v_3 - 1) \right|^{1/2} \\
&\geq |v_3|^{1/2} + |v_3 - 1|^{1/2} \\
&\geq 1 \\
&= \rho_0((0, 0), (0, 1)).
\end{align*}
\]

In the first inequality we have have equality if and only if \(\zeta_3 = 0\) and in the second inequality we have equality if and only if \(v_3 = 0\) or \(v_3 = 1\). Thus we have strict inequality whenever \((\zeta_3, v_3) \neq (0, 0), (0, 1)\).

We can extend the Cygan metric to an incomplete metric on \( \overline{\mathbb{H}_C^2} - \{ \infty \} \) as follows

\[ \rho_0((\zeta_1, v_1, u_1), (\zeta_2, v_2, u_2)) = \left| |\zeta_1 - \zeta_2|^2 + |u_1 - u_2| - iv_1 + iv_2 - 2i\Im(\zeta_1\zeta_2) \right|^{1/2}. \] (53)

We remark that this agrees with \(|(z_1, z_2)|^{1/2}\) if and only if one (or both) of \(z_1\) or \(z_2\) is null, that is it corresponds to a point of \( \partial \mathbb{H}_C^2 \). We now show that the extended Cygan metric on \( \overline{\mathbb{H}_C^2} - \{ \infty \} \) is indeed a metric. By restricting to points on \( \partial \mathbb{H}_C^2 - \{ \infty \} \), this will also show that the original Cygan metric on \( \mathcal{N} \) is a metric.

**Proposition 4.2** The function \( \overline{\mathbb{H}_C^2} - \{ \infty \} \) given by equation (53) is a metric.

**Proof:** It is completely obvious that that both \( \rho_0((\zeta_1, v_1, u_1), (\zeta_2, v_2, u_2)) = 0 \) if and only if \((\zeta_1, v_1, u_1) = (\zeta_2, v_2, u_2)\) and that

\[
\rho_0((\zeta_1, v_1, u_1), (\zeta_2, v_2, u_2)) = \rho_0((\zeta_2, v_2, u_2), (\zeta_1, v_1, u_1))
\]
for all points in $\overline{H_C} - \{\infty\}$. Therefore, it suffices to show that
\[
\rho_0((\zeta_1, v_1, u_1), (\zeta_2, v_2, u_2)) \leq \rho_0((\zeta_1, v_1, u_1), (\zeta_3, v_3, u_3)) + \rho_0((\zeta_3, v_3, u_3), (\zeta_2, v_2, u_2)).
\]

Now we have
\[
|\zeta_1 - \zeta_2|^2 - 2i\Im(\zeta_1\overline{\zeta}_2)
= |\zeta_1|^2 - 2\zeta_1\overline{\zeta}_2 + |\zeta_2|^2
= |\zeta_1 - \zeta_3|^2 - 2(\zeta_1 - \zeta_3)(\overline{\zeta_2} - \overline{\zeta}_3) + |\zeta_2 - \zeta_3|^2 - 2i\Im(\zeta_1\overline{\zeta}_3) - 2i\Im(\zeta_3\overline{\zeta}_2).
\]

Therefore
\[
\rho_0^2((\zeta_1, v_1, u_1), (\zeta_2, v_2, u_2))
= \left| |\zeta_1 - \zeta_2|^2 + |u_1 - u_2| - iv_1 + iv_2 - 2i\Im(\zeta_1\overline{\zeta}_2) \right|
\leq \left| |\zeta_1 - \zeta_2|^2 + |u_1 - u_3| + |u_3 - u_2| - iv_1 + iv_3 - iv_2 - 2i\Im(\zeta_1\overline{\zeta}_3) \right|
\leq \left| |\zeta_1 - \zeta_3|^2 + |u_1 - u_3| - iv_1 + iv_3 - 2i\Im(\zeta_1\overline{\zeta}_3) \right|
+ 2|\zeta_1 - \zeta_3| |\zeta_3 - \zeta_2| + |\zeta_3 - \zeta_2|^2 + |u_3 - u_2| - iv_3 + iv_2 - 2i\Im(\zeta_3\overline{\zeta}_2)
\leq \left( \rho_0((\zeta_1, v_1, u_1), (\zeta_3, v_3, u_3)) + \rho_0((\zeta_3, v_3, u_3), (\zeta_2, v_2, u_2)) \right)^2.
\]

We now discuss Cygan spheres. The Cygan sphere of radius $r \in \mathbb{R}_+$ and centre $z_0 = (\zeta_0, v_0) = (\zeta_0, v_0, 0) \in \partial H_C^2$ is defined by
\[
S_r(z_0) = \{ z = (\zeta, v, u) : \rho_0(z, z_0) = r \}.
\]

In terms of coordinates, $S_r(z_0)$ is given by
\[
S_r(z_0) = \left\{ z = (\zeta, v, u) : \left| |\zeta - \zeta_0|^2 + u + iv - iv_0 - 2i\Im(\zeta\overline{\zeta}_0) \right| = r^2 \right\}.
\]

Suppose that $\zeta_0 = 0$. Cygan spheres centred at $z_0 = (0, v_0)$ are ovoids with the property that along the locus $\zeta = 0$ they have fourth order contact with their tangent plane. The diameter of their equator, that is the points $(\zeta, 0, 0)$ with $|\zeta| = r$, grows linearly with $r$. On the other hand, the diameter of their meridians, that is the points $(0, v, u)$ with $|u + iv - iv_0| = r^2$, grows quadratically with $r$. Thus, as $r$ tends to zero, Cygan spheres become very short and fat, like a pancake, and, as $r$ tends to infinity, Cygan spheres of radius $r$ become very long and thin, like a cigar.

When $\zeta_0 \neq 0$, Cygan spheres are sheared ovoids, the magnitude of the shear being proportional to $|\zeta_0|$. Otherwise they enjoy the same properties outlined above. In particular, Cygan spheres are always convex.

We conclude this section by considering the subgroup of $\text{PU}(2, 1)$ stabilising the point at infinity. Such maps will be called Heisenberg similarities. We have already seen the group of Heisenberg translations. This is a normal subgroup of the group of Heisenberg
similarities. Using this subgroup, it is sufficient to classify those elements of PU(2,1) fixing both ∞ and the origin o = (0,0). For example, we have Heisenberg rotations. These are given by (ζ, v) ↦ (e^{iθ}ζ, v) and are boundary elliptic. Also we have (real) dilations (ζ, v) ↦ (rζ, r^2 v) where r ∈ ℜ+. A product of a Heisenberg rotation and a real dilation is a complex dilation

\[ A : (ζ, v) \mapsto (re^{iθ}ζ, r^2 v) = (λζ, |λ|^2 v). \]

Here λ = re^{iθ} is the multiplier or complex dilation factor of A. Complex dilations are isomorphic to ℜ × U(1). The group of Heisenberg similarities is the semi-direct product of the complex dilations and the Heisenberg translations, isomorphic to (ℜ+ × U(1)) ⋉ N.

**Lemma 4.3** As a matrix in PU(2,1) (with respect to the second Hermitian form) the complex dilation A : (ζ, v, u) ↦ (λζ, |λ|^2 v, |λ|^2 u) is given by the loxodromic matrix, also denoted A:

\[ A = \begin{bmatrix}
λ & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & λ^{-1}
\end{bmatrix}. \]

**Proof:** We may write

\[ A = \begin{bmatrix}
λ & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & λ^{-1}
\end{bmatrix}. \]

Choose any if \( z = (ζ, v, u) \in H^2_ℂ \). Then taking the canonical lift \( z \) of \( z \) to \( ℂ^{2,1} \), we see that \( A(ζ, v, u) \) is

\[ \begin{bmatrix}
λ & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & λ^{-1}
\end{bmatrix} \begin{bmatrix}
-|ζ|^2 - u + iv \\
\sqrt{2}ζ \\
1
\end{bmatrix} = \begin{bmatrix}
λ(-|ζ|^2 - u + iv) \\
\sqrt{2}ζ \\
λ^{-1}
\end{bmatrix} = λ^{-1} \begin{bmatrix}
|λ|^2(-|ζ|^2 - u + iv) \\
\sqrt{2}λζ \\
1
\end{bmatrix}. \]

Thus the canonical lift \( A(z) \) of \( A(ζ, v, u) \) to \( ℂ^{2,1} \) is

\[ A(z) \begin{bmatrix}
|λ|^2(-|ζ|^2 - u + iv) \\
\sqrt{2}λζ \\
1
\end{bmatrix}. \]

The following lemma shows how complex dilations distort the Cygan metric and also how their Cygan translation lengths vary. These will be very useful to us when considering the action of complex dilations on \( ∂H^2_ℂ \).

**Lemma 4.4** Suppose that A ∈ PU(n,1) fixes o and ∞ and has complex multiplier λ. Writing \( M = |λ - 1| + |λ^{-1} - 1| \), we have

(i) \( ρ_0(A(z), A(w)) = |λ|ρ_0(z, w) \) for all \( z, w \in H^2_ℂ \),
(ii) \( \rho_0(A(z), z) \leq |\lambda|^{1/2} M^{1/2} \rho_0(z, o) \) for all \( z \in \partial H^2_\infty - \{\infty\} \).

**Proof:** Let \( z = (\zeta, v, u) \) and \( w = (\xi, t, s) \). The canonical lifts \( A(z) \) of \( A(\zeta, v, u) \) and \( A(\omega) \) of \( A(\xi, t, s) \) to \( \mathbb{C}^{2,1} \) are

\[
A(z) = \begin{bmatrix}
\frac{|\lambda|^2 (|\zeta|^2 - u + iv)}{\sqrt{2} \lambda \zeta} \\
\frac{1}{1}
\end{bmatrix}
\quad \text{and} \quad
A(\omega) = \begin{bmatrix}
\frac{|\lambda|^2 (|\xi|^2 - s + it)}{\sqrt{2} \lambda \xi} \\
\frac{1}{1}
\end{bmatrix}.
\]

From this we see that

\[
\rho_0(A(z), A(\omega)) = \left| |\lambda|^2 \left(|\zeta - \xi|^2 + |u - s| - iv - 2i \Im(\zeta \overline{\xi}) \right) \right|^{1/2}
\]

\[
= |\lambda| \left|\frac{|\zeta - \xi|^2 + |u - s| - iv - 2i \Im(\zeta \overline{\xi})}{\rho_0(z, w)} \right|^{1/2}
\]

This proves (i). For part (ii) we argue similarly with \( z = (\zeta, v, 0) \):

\[
\rho_0(A(z), z) = \left| |\lambda|^2 |\zeta|^2 - 2(\lambda \zeta \overline{\zeta}) - |\zeta|^2 - i|\lambda|^2 v + iv \right|^{1/2}
\]

\[
= |\lambda(\lambda - 1) (|\zeta|^2 - iv) - (\lambda - 1)(|\zeta|^2 + iv) |^{1/2}
\]

\[
\leq (|\lambda| + 1)^{1/2} |\lambda - 1|^{1/2} |\zeta|^2 - iv |^{1/2}
\]

\[
= |\lambda|^{1/2} M^{1/2} \rho_0(z, 0).
\]

This completes the proof of (ii). \( \square \)

The next lemma shows how a map in \( \text{PU}(2, 1) \) not fixing \( \infty \) distorts the Cygan metric on the boundary.

**Lemma 4.5** Let \( B \) be any element of \( \text{PU}(2, 1) \) that does not fix \( \infty \). Then there exists a positive real number \( r_B \) depending only on \( B \) so that for all \( z, w \in \partial H^2_\infty - \{\infty, B^{-1}(\infty)\} \) we have

(i) \[
\rho_0(B(z), B(w)) = \frac{r_B^2 \rho_0(z, w)}{\rho_0(z, B^{-1}(\infty)) \rho_0(w, B^{-1}(\infty))},
\]

(ii) \[
\rho_0(B(z), B(\infty)) = \frac{r_B^2}{\rho_0(z, B^{-1}(\infty))}.
\]

**Proof:** As above let \( z = (\zeta, v, 0) \) and \( w = (\xi, t, 0) \) have canonical lifts

\[
z = \begin{bmatrix}
-|\zeta|^2 + iv \\
\sqrt{2} \zeta \\
1
\end{bmatrix}
\quad \text{and} \quad
w = \begin{bmatrix}
-|\xi|^2 + it \\
\sqrt{2} \xi \\
1
\end{bmatrix}.
\]
Moreover, the canonical lifts $B$ of $B^{-1}(\infty)$ and $B(\infty)$ are

$$\begin{bmatrix} j/g \\ h/g \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a/d \\ d/g \\ 1 \end{bmatrix}. $$

Hence

$$\rho_0(z, B^{-1}(\infty)) = |-\zeta|^2 + iv + \sqrt{2}\zeta h/g + j/g|^{1/2},$$

$$\rho_0(w, B^{-1}(\infty)) = |-\zeta|^2 + it + \sqrt{2}\xi h/g + j/g|^{1/2}.$$

Moreover, the canonical lifts $B(z)$ and $B(w)$ of $Bz$ and $Bw$ are

$$B(z) = \frac{(a(|\zeta|^2 + iv) + b\sqrt{2}\zeta + c)/(g(-|\zeta|^2 + iv) + h\sqrt{2}\zeta + j)}{d(|\zeta|^2 + iv) + e\sqrt{2}\zeta + f)/(g(-|\zeta|^2 + iv) + h\sqrt{2}\zeta + j)},$$

$$B(w) = \frac{(a(|\zeta|^2 + it) + b\sqrt{2}\xi + c)/(g(-|\zeta|^2 + it) + h\sqrt{2}\xi + j)}{d(|\zeta|^2 + it) + e\sqrt{2}\xi + f)/(g(-|\zeta|^2 + it) + h\sqrt{2}\xi + j)}.$$

Hence

$$\rho_0(B(z), B(w)) = \frac{|-\zeta|^2 + iv + 2\zeta \xi - |\zeta|^2 - it|^{1/2}}{|g(-|\zeta|^2 + iv) + h\sqrt{2}\zeta + j|^{1/2}|g(-|\zeta|^2 + it) + h\sqrt{2}\xi + j|^{1/2}} = \frac{r_B^2 \rho_0(z, w)}{\rho_0(z, B^{-1}(\infty)) \rho_0(w, B^{-1}(\infty))}.$$

This proves (i). Similarly

$$\rho_0(B(z), B(\infty)) = \frac{1}{|g(-|\zeta|^2 + iv) + h\sqrt{2}\zeta + j|^{1/2}|g|^{1/2}} = \frac{r_B^2}{\rho_0(z, B^{-1}(\infty))}.$$

This proves (ii). \qed

An important consequence of this proposition is that $B$ sends the Cygan sphere of radius $r_B$ with centre $B^{-1}(\infty)$ to the Cygan sphere of radius $r_B$ with centre $B(\infty)$. Motivated by the analogous Euclidean spheres in real hyperbolic space, we define the isometric sphere of $B$ to be the Cygan sphere of radius $r_B$ and centre $B^{-1}(\infty)$.
Lemma 4.6 Let $B$ be a loxodromic map with multiplier $\lambda \in \mathbb{C}$, attractive fixed point $p$ and repulsive fixed point $q$ and isometric sphere of radius $r_B$. Suppose that $p, q \neq \infty$, and let $M = |\lambda - 1| + |\lambda^{-1} - 1|$. Then

(i) $\rho_0(p, B^{-1}(\infty)) = |\lambda|^{1/2}r_B$ and $\rho_0(p, B(\infty)) = |\lambda|^{-1/2}r_B$,

(ii) $\rho_0(q, B^{-1}(\infty)) = |\lambda|^{-1/2}r_B$ and $\rho_0(q, B(\infty)) = |\lambda|^{1/2}r_B$,

(iii) $\rho_0(p, q) \leq M^{1/2}r_B$.

Proof: Let $C$ be any element of $\text{PU}(2,1)$ with $C(o) = p$ and $C(\infty) = q$ as found in Proposition 3.3. Let $r_C$ be the radius of its isometric sphere. Then $A = C^{-1}BC$ is a complex dilation with multiplier $\lambda$. Using Lemma 4.5 (ii) with $z = B(z) = q$ we have

$$r_B^2 = \rho_0(q, B(\infty))\rho_0(q, B^{-1}(\infty)).$$

Also, substituting for $B = CAC^{-1}$, $q = C(\infty)$ and using Lemma 4.5 (ii) again, but this time with $C$, we have

$$\rho_0(q, B(\infty)) = \rho_0(C(\infty), CAC^{-1}(\infty)) = \frac{r_C^2}{\rho_0(AC^{-1}(\infty), C^{-1}(\infty))} = \frac{|\lambda|\rho_0(C^{-1}(\infty), A^{-1}C^{-1}(\infty))}{|\lambda|^{-1}\rho_0(C(\infty), CAC^{-1}(\infty))} = |\lambda|^{-1}\rho_0(q, B^{-1}(\infty)).$$

Part (ii) follows immediately. Part (i) follows by applying part (ii) to $B^{-1}$.

For part (iii) we begin with

$$r_B^2 = \rho_0(q, B(\infty))\rho_0(q, B^{-1}(\infty)) = \rho_0(C(\infty), CAC^{-1}(\infty))\rho_0(C(\infty), CA^{-1}C^{-1}(\infty)) = \frac{r_C^4}{\rho_0(C^{-1}(\infty), AC^{-1}(\infty))\rho_0(C^{-1}(\infty), A^{-1}C^{-1}(\infty))}.$$ 

Now using Lemma 4.4 (ii) we have

$$\rho_0(C^{-1}(\infty), A^{-1}C^{-1}(\infty)) \leq |\lambda|^{1/2}M^{1/2}\rho_0(o, C^{-1}(\infty))$$

and

$$\rho_0(C^{-1}(\infty), AC^{-1}(\infty)) \leq |\lambda|^{-1/2}M^{1/2}\rho_0(o, C^{-1}(\infty)).$$

Therefore

$$r_B^2 \geq \frac{r_C^4}{M\rho_0(o, C^{-1}(\infty))^2} = \frac{\rho_0(C(o), C(\infty))^2}{M}.$$
where we have used Lemma 4.5 (ii) again. Substituting \( p = C(o) \) and \( q = C(\infty) \) gives the result.

We now discuss parabolic maps. As a matrix in \( \text{PU}(2,1) \) (left) Heisenberg translation by \((\tau, t)\) is given by

\[
A = \begin{bmatrix}
1 & -\sqrt{2}\tau & -|\tau|^2 + it \\
0 & 1 & \sqrt{2}\tau \\
0 & 0 & 1
\end{bmatrix}.
\]

(54)

At a point \( z = (\zeta, v) \in \mathcal{N} \) the Cygan translation length of \( A \) is given by

\[
t_A(z) = \rho_0(A(z), z) = \left| |\tau|^2 + it + 4i\Im(\tau\overline{\zeta}) \right|^{1/2}.
\]

We now estimate how this translation length changes with \( z \).

**Lemma 4.7** Let \( A \) be Heisenberg translation by \((\tau, t)\) and let \( t_A(z) \), \( t_A(w) \) denote the Cygan translation length of \( A \) at the points \( z = (\zeta, v) \) and \( w = (\xi, s) \) in \( \mathcal{N} \). Then

\[
t_A(w)^2 \leq t_A(z)^2 + 4|\tau| |\xi - \zeta|.
\]

**Proof:** We have

\[
t_A(w)^2 = \left| |\tau|^2 + it + 4i\Im(\tau\overline{\xi}) \right|
\]

\[
= \left| |\tau|^2 + it + 4i\Im(\tau\overline{\xi}) + 4i\Im(\tau(\overline{\xi} - \zeta)) \right|
\]

\[
\leq \left| |\tau|^2 + it + 4i\Im(\tau\overline{\xi}) \right| + 4|\tau| |\xi - \zeta|.
\]

\[
\square
\]

Let \( A \) be the screw parabolic map with fixed point \( \infty \), multiplier \( e^{i\theta} \) and axis the complex line \( L_A = \{(\zeta, v, u) : \zeta = \zeta_0 \in \mathbb{C} \} \). Suppose that \( A \) acts as Heisenberg translation by \((0, t)\) on \( L_A \). In horospherical coordinates \( A \) is given by

\[
A : (\zeta, v, u) \mapsto (\zeta e^{i\theta} + \zeta_0(1 - e^{i\theta}), v + t + 2\Im((\zeta - \zeta_0)\overline{\zeta_0}(1 - e^{i\theta})), u).
\]

(55)

As a matrix in \( \text{PU}(2,1) \) the map \( A \) is given by

\[
A = \begin{bmatrix}
1 & \sqrt{2}\zeta_0(1 - e^{i\theta}) & -2|\zeta_0|^2(1 - e^{i\theta}) + it \\
0 & e^{i\theta} & \sqrt{2}\zeta_0(1 - e^{i\theta}) \\
0 & 0 & 1
\end{bmatrix}.
\]

At a point \( z = (\zeta, v) \in \mathcal{N} \) the Cygan translation length of \( A \) is

\[
t_A(z) = \rho_0(A(z), z) = \left| 2|\zeta - \zeta_0|^2(e^{i\theta} - 1) + it \right|^{1/2}.
\]

**Lemma 4.8** Let \( A \) be the screw parabolic map given by (55). Suppose that \( t\sin(\theta) \geq 0 \). Let \( t_A(z) \), \( t_A(w) \) denote the Cygan translation length of \( A \) at the points \( z = (\zeta, v) \) and \( w = (\xi, s) \) in \( \mathcal{N} \). Then

\[
t_A(w) \leq t_A(z) + \left| 2|\xi - \zeta|^2(e^{i\theta} - 1) \right|^{1/2}.
\]
Proof: Since $t \sin(\theta) \geq 0$ it is easy to see that
\[ 2|\zeta - \zeta_0|^2(e^{i\theta} - 1) \leq t_A(z)^2. \]

Using this fact and the triangle inequality we have
\[
t_A(w)^2 = 2|\xi - \zeta_0|^2(e^{i\theta} - 1) + it \leq 2|\xi - \zeta_0|^2(e^{i\theta} - 1) + t \leq 2|\xi - \zeta_0|^2(e^{i\theta} - 1) + 2(\zeta - \zeta_0)^2 - |\xi - \zeta_0|^2) + (e^{i\theta} - 1) \leq \frac{t_A(z)^2}{2}\left(2|\xi - \zeta_0|^2(e^{i\theta} - 1) + 2\right) \leq t_A(z)^2 + 2t_A(z)|\xi - \zeta_0|^2(e^{i\theta} - 1) + 2|\xi - \zeta_0|^2(e^{i\theta} - 1) = \left(t_A(z) + 2|\xi - \zeta_0|^2(e^{i\theta} - 1)\right)^2. \]

\[ \boxdot \]

5 Subspaces of complex hyperbolic space

5.1 Geodesics

Consider a pair distinct null vectors $p, q \in V_0$. Without loss of generality normalise so that $\langle p, q \rangle = -1$. These vectors correspond to a pair of points $p$ and $q$ in $\partial H^2_C$. We want to describe the geodesic $\gamma$ with endpoints $p$ and $q$.

Proposition 5.1 Let $p, q \in V_0$ be null vectors with $\langle p, q \rangle = -1$. For all real $t$ let $\gamma(t)$ be the point in $H^2_C$ corresponding to the vector $e^{t/2}p + e^{-t/2}q$ in $\mathbb{C}^{2,1}$. Then $\gamma = \{ \gamma(t) \mid t \in \mathbb{R} \}$ is the geodesic in $H^2_C$ with endpoints $p$ and $q$ parametrised by arc length $t$.

Proof: First observe that $\gamma(t)$ is in $H^2_C$. This is because
\[
\langle e^{t/2}p + e^{-t/2}q, e^{t/2}p + e^{-t/2}q \rangle = e^{t}\langle p, p \rangle + \langle p, q \rangle + \langle q, p \rangle + e^{-t}\langle q, q \rangle = -2.
\]
It suffices to show that $ρ(γ(t), γ(s)) = |t − s|$ for all real $s$ and $t$.

\[
\cosh^2\left(\frac{ρ(γ(t), γ(s))}{2}\right) = \frac{(e^{s/2}p + e^{-s/2}q)(e^{s/2}p + e^{-s/2}q)}{(e^{t/2}p + e^{-t/2}q)(e^{t/2}p + e^{-t/2}q)} = \left(-\frac{e(t−s)/2 − e^{−(t+s)/2}}{2}\right)^2.
\]

This proves the result. \qed

Any pair of points $z, w \in H_2^e$ lie on a unique geodesic. We now use the description of geodesics given above to find an expression for this geodesic.

**Proposition 5.2** Let $γ(t)$ be a geodesic in $H_2^e$ parametrised by arc length $t$. Suppose that $γ(r)$ and $γ(s)$ correspond to the points $z$ and $w$ in $V_-$ where $(z, z) = (w, w) = −2$. Then $γ(t)$ is given by the vector

\[
\frac{\sinh((t−s)/2)}{\sinh((r−s)/2)}z + \frac{\sinh((r−t)/2)}{\sinh((r−s)/2)}w.
\]

**Proof:** Suppose that the endpoints of $γ$ correspond to the null vectors $p$ and $q$ with $\langle p, q \rangle = −1$. Then

\[
z = e^{r/2}p + e^{-r/2}q, \quad w = e^{s/2}p + e^{-s/2}q.
\]

Then we see that

\[
2\sinh((r−s)/2)p = e^{-s/2}z − e^{-r/2}w, \quad 2\sinh((r−s)/2)q = −e^{s/2}z + e^{r/2}w.
\]

The point $γ(t)$ then corresponds to

\[
e^{t/2}p + e^{-t/2}q = \frac{\sinh((t−s)/2)}{\sinh((r−s)/2)}z + \frac{\sinh((r−t)/2)}{\sinh((r−s)/2)}w.
\]

\qed

We now find the height of a point on a geodesic neither of whose endpoints is $∞$.

**Proposition 5.3** Let $p$ and $q$ be points of $∂H_2^e$ neither of which is $∞$. Let $u_{pq}$ be the maximal height of a point on the geodesic with endpoints $p$ and $q$. Then $u_{pq} \leq ρ_0(p, q)^2/2$.

**Proof:** Suppose that the points $p$ and $q$ correspond to the vectors $p$ and $q$ with $\langle p, q \rangle = −1$. Let $n$ denote their cross product, as defined in (48). As given above an arbitrary point $γ(t)$ of the geodesic with endpoints $p$ and $q$ corresponds to the vector $e^{t/2}p + e^{-t/2}q$. Let $e_1$ be the column vector $(1, 0, 0)^t$ corresponding to $∞$. Thus using Lemma 3.4 we have

\[
0 = ⟨e_1, e_1⟩_2 = ⟨n, e_1⟩_2⟨e_1, n⟩_2 − ⟨p, e_1⟩_2⟨e_1, q⟩_2 − ⟨q, e_1⟩_2⟨e_1, p⟩_2.
\]
Now \( p_3 = \langle p, e_1 \rangle_2 \) and \( q_3 = \langle q, e_1 \rangle_2 \) and so

\[
p_3 \overline{p_3} + q_3 \overline{q_3} = |\langle e_1, n \rangle_2|^2 \geq 0.
\]

In order to calculate Cygan distances using the Hermitian form we must re-normalise the vectors, in this case \( p, q \) and \( e^{t/2}p + e^{-t/2}q \), so that their third coordinate is 1. This means that the Cygan distance from \( p \) to \( q \) is given by

\[
\rho_0(p, q) = \left| \left\langle \begin{bmatrix} p_1/p_3 \\ p_2/p_3 \\ 1 \end{bmatrix}, \begin{bmatrix} q_1/q_3 \\ q_2/q_3 \\ 1 \end{bmatrix} \right\rangle_2 \right|^{1/2} = \frac{1}{|p_3|^{1/2} |q_3|^{1/2}}.
\]

Similarly, the height \( u_t \) of the point \( \gamma(t) \) is given by

\[
u_t = \frac{1}{|e^{t/2}p_3 + e^{-t/2}q_3|^2} = \frac{1}{e^t |p_3|^2 + p_3 \overline{p_3} + q_3 \overline{q_3} + e^{-t} |q_3|^2}.
\]

Using elementary calculus we see that the right hand side is maximised when \( e^t = |q_3|/|p_3| \).

Thus

\[
u_{pq} = \frac{1}{2 |p_3||q_3| + p_3 \overline{p_3} + q_3 \overline{q_3}} \\
\leq \frac{1}{2 |p_3||q_3|} \\
= \frac{\rho_0(p, q)^2}{2}. \]

\[\Box\]

### 5.2 Complex lines

Consider a complex line \( L \) in \( \mathbb{C}^2 \) that intersects the unit ball (which we think of as \( \mathbb{H}^2_{\mathbb{C}} \)). Let \( z \) be any point in \( L \cap \mathbb{H}^2_{\mathbb{C}} \). We can apply an element of \( \text{PU}(2, 1) \) to \( L \) so that it becomes the last coordinate axis \( \{ (0, z_2) | z_2 \in \mathbb{C} \} \). The intersection of this with the unit ball is the disc \( |z_2| < 1 \). We claim that the restriction of the Bergman metric to this disc is the Poincaré metric of constant curvature \(-1\).

In order to see this, let \( z = (0, z_2) \) and \( w = (0, w_2) \) have lifts to \( \mathbb{C}^{2,1} \) given by

\[
z = \begin{bmatrix} 0 \\ z_2 \\ 1 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} 0 \\ w_2 \\ 1 \end{bmatrix}
\]

The distance between these points is given by

\[
\cosh^2 \left( \frac{\rho(z, w)}{2} \right) = \frac{\langle z, w \rangle_1 \langle w, z \rangle_1}{\langle z, z \rangle_1 \langle w, w \rangle_1} \\
= \frac{|z_2 w_2 - 1|^2}{(|z_2|^2 - 1)(|w_2|^2 - 1)}.
\]

This is just the Poincaré metric (see page 132 of [2]).
In other words any complex line $L$ is an embedded copy of $\mathbb{H}^2_C$. This subgroup of $\text{PU}(2,1)$ preserving this disc is the projectivisation of the block diagonal matrices

$$
\begin{bmatrix}
e^{i\theta} & 0 \\
0 & A 
\end{bmatrix}
$$

where $e^{i\theta} \in U(1)$ acts on $L^\perp$ rotating $\mathbb{H}^2_C$ around $L$ and $A \in U(1,1)$ is an isometry of the Poincaré metric on $L$ acting by M"obius transformations. The group of such transformations is then $\text{P}(U(1) \times U(1,1)) < \text{PU}(2,1)$. Clearly this group is isomorphic to $U(1,1)$.

Any other complex line intersecting $\mathbb{H}^2_C$ is the image of $V$ under an element $B$ of $\text{PU}(2,1)$.

The stabiliser of this complex line is the conjugate of $\text{P}(U(1) \times U(1,1))$ by $B$.

Taking the second Hermitian form and $z$ lying in the subset given by

$\{z = (z_1,0) | z \in \mathbb{C}\}$

we see that $z$ lies in $\mathbb{H}^2_C$ if and only if $\langle z, z \rangle_2 = 2\Re(z_1) < 0$. This is a half-plane, in fact the left half plane. In order to get the more familiar upper half plane we write $z = (iz_1,0)$. This point corresponds to a negative vector if and only if $\Im(z_1) > 0$. Lifting the points $z$ and $w$ to $z = \begin{bmatrix} iz_1 \\
0 \\
1 \end{bmatrix}$ and $w = \begin{bmatrix} iw_1 \\
0 \\
1 \end{bmatrix}$

we find that the distance function is given by

$$
\cosh^2\left(\frac{\rho(z,w)}{2}\right) = \frac{\langle z,w \rangle_2 \langle w,z \rangle_2}{\langle z,z \rangle_2 \langle w,w \rangle_2} = \frac{|z_1 - \overline{w_1}|^2}{2\Im(z_1)2\Im(w_1)}.
$$

This is just the Poincaré metric on the upper half plane (see Theorem 7.2.1(iv) of [2]).

### 5.3 Totally real Lagrangian planes

Now consider a totally real Lagrangian plane $R$. This may be characterised by $\langle v, w \rangle \in \mathbb{R}$ for all $v, w \in R$.

Any totally real Lagrangian plane $R$ is the image under an element of $\text{PU}(2,1)$ of the subspace comprising those points of $\mathbb{H}^2_C$ with real coordinates, that is an embedded copy of real hyperbolic space $\mathbb{H}^2_R = \{(x_1, x_2) | x_1, x_2 \in \mathbb{R}\}$. This subspace intersects the unit ball in the subset consisting of those points with $x_1^2 + x_2^2 < 1$. We claim that the Bergman metric restricted to this disc is just the Klein-Beltrami metric on the unit ball in $\mathbb{R}^2$ with constant curvature $-1/4$. To see this, write $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{H}^2_R$. Lift $x$ and $y$ to column vectors $x$ and $y$ in $\mathbb{C}^{2,1}$ as

$$
x = \begin{bmatrix} x_1 \\
x_2 \\
1 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} y_1 \\
y_2 \\
1 \end{bmatrix}.
$$

The we have

$$
\cosh^2\left(\frac{\rho(x,y)}{2}\right) = \frac{\langle x,y \rangle_1 \langle y,x \rangle_1}{\langle x,x \rangle_1 \langle y,y \rangle_1} = \frac{(x_1y_1 + x_2y_2 - 1)^2}{(x_1^2 + x_2^2 - 1)(y_1^2 + y_2^2 - 1)}.
$$
This is the Klein-Beltrami metric on the unit ball in $\mathbb{R}^2$ with constant curvature $-1/4$. It is more usual to define the Klein-Beltrami metric with curvature $-1$. In order to do this, replace $\rho(x,y)/2$ in this formula with $\rho(x,y)$ (see Chapter 3 of [24]).

Thus we obtain an embedded copy of $H^2_\mathbb{R}$. The isometries preserving this space lie in the projectivisation of the natural inclusion $O(2,1) < U(2,1)$. It is also preserved by complex conjugation. Any other totally real Lagrangian plane is the image of this one under $B$ in $PU(2,1)$ and is stabilised by the conjugate by $B$ of the projectivisation of $O(2,1)$.

### 5.4 Totally geodesic subspaces

In this section we show that complex lines and totally real Lagrangian planes are totally geodesic. Together with geodesics, these are the only totally geodesic proper subspaces of $H^2_\mathbb{C}$. We will not show the latter fact.

**Proposition 5.4** All complex lines $L$ in $H^2_\mathbb{C}$ are totally geodesic.

**Proof:** Let $L$ be a complex line. We need to show that for all choices of $z$ and $w$ in $L$ the geodesic segment joining $z$ to $w$ lies in $L$. We may represent $z$ and $w$ by negative vectors with $\langle z,z \rangle = \langle w,w \rangle = -2$. From Proposition 5.2 we see that a general point on this geodesic segment is given by

$$\frac{\sinh((t-s)/2)}{\sinh((r-s)/2)} z + \frac{\sinh((t-r)/2)}{\sinh((s-r)/2)} w.$$ 

As this is a linear combination of $z$ and $w$ it corresponds to a point of $L$. \qed

**Proposition 5.5** All totally real Lagrangian planes $R$ in $H^2_\mathbb{C}$ are totally geodesic.

**Proof:** Let $R$ be a totally real Lagrangian plane. We must show that for all choices of $z$ and $w$ in $R$ the geodesic segment joining $z$ to $w$ lies in $R$. Totally real Lagrangian planes are characterised by $\langle v,w \rangle \in \mathbb{R}$ for all choices of $v$ and $w$ in $R$. Therefore we must show that, every point on the geodesic segment joining $z$ and $w$ corresponds to a vector whose Hermitian product with every point in $R$ is real.

As before we may lift $z$ and $w$ in $R$ to vectors with $\langle z,z \rangle = \langle w,w \rangle = -2$ and $\langle z,w \rangle \in \mathbb{R}$. Then a general point on the geodesic segment joining $z$ and $w$ is given by

$$\frac{\sinh((t-s)/2)}{\sinh((r-s)/2)} z + \frac{\sinh((t-r)/2)}{\sinh((s-r)/2)} w.$$ 

Let $v$ be any vector corresponding to a point of $R$. Then $\langle v,z \rangle$ and $\langle v,w \rangle$ are both real. Thus

$$\langle v, \frac{\sinh((t-s)/2)}{\sinh((r-s)/2)} z + \frac{\sinh((t-r)/2)}{\sinh((s-r)/2)} w \rangle = \frac{\sinh((t-s)/2)}{\sinh((r-s)/2)} \langle v,z \rangle + \frac{\sinh((t-r)/2)}{\sinh((s-r)/2)} \langle v,w \rangle$$

is real for all $t$. Hence the geodesic segment from $z$ to $w$ is in $R$. \qed
Alternatively, we could have used the following lemma together with the fact that a complex line is the fixed point set of a boundary elliptic isometry and that any totally real Lagrangian plane is fixed by an anti-holomorphic isometry conjugate to complex conjugation.

**Lemma 5.6** Any subset $S$ of $H^2_\mathbb{C}$ that is precisely the fixed point set of an isometry is totally geodesic.

**Proof:** Consider two points $z_0, z_1 \in S$. By negative curvature there is a unique geodesic $\alpha : [0, 1] \to H^2_\mathbb{C}$ joining $z_0 = \alpha(0)$ to $z_1 = \alpha(1)$. The distance of $z_t = \alpha(t)$ from $z_0$ is a monotone increasing function of $t$.

We need to show that the geodesic $\alpha$ lies in $S$. Suppose it does not. Then there is a point $z_t = \alpha(t)$ for some $t \in (0, 1)$ on this geodesic not lying in $S$. By assumption, there exists an $A \in PU(2, 1)$ so that $A(z) = z$ if and only if $z \in S$. In particular $A(z_0) = z_0$, $A(z_1) = z_1$ but $A(z_t) \neq z_t$. Therefore $A(\alpha)$ is a geodesic joining $z_0$ to $z_1$. Now $z_t$ is the unique point of $\alpha$ a distance $\rho(z_0, z_t)$ from $z_0$ and $A(z_t)$ is the unique point of $A(\alpha)$ a distance $\rho(z_0, z_t)$ from $z_0$. As these two points are different we see that $A(\alpha) \neq \alpha$. \qed

The following theorem follows using the general theory of symmetric spaces. Its proof is beyond the scope of these notes A sketch proof is given in Section 3.1.11 of Goldman [11].

**Theorem 5.7** All totally geodesic subspaces of $H^2_\mathbb{C}$ are either complex linear or totally real.

**Corollary 5.8** Every totally geodesic subspace of $H^2_\mathbb{C}$ has real dimension which is either 1 or 2. In particular, there are no totally geodesic real hypersurfaces.

This corollary means that there are no polyhedra in the standard meaning of the term. We therefore have to generalise the notion of polyhedra by defining suitable classes of real hypersurface for their boundaries.

We conclude this section by showing how totally real and complex linear subspaces fit together. The real line $\{(0, x)|x \in \mathbb{R}\}$ is a geodesic. It is contained in a unique complex line, namely $\{(0, z)|z \in \mathbb{C}\}$. It is also contained in a one parameter family of totally real subspaces, namely for each $\theta \in [0, \pi)$ there is a totally real subspace $\{(re^{i\theta}, x)|r, x \in \mathbb{R}\}$.

5.5 **Boundaries of totally geodesic subspaces**

We now describe the intersection of complex lines and totally real planes with the boundary of the Siegel domain.

First consider a complex line $L$ passing through the point at infinity. By applying a suitable Heisenberg translation, we may suppose that $L$ also passes through the origin $o = (0, 0) \in \mathcal{N}$. In other words, $L$ is the complex line spanned by

\[
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}.
\]
This complex line \( L \) consists of points
\[
\begin{bmatrix}
-u + iv \\
0 \\
1
\end{bmatrix}.
\]
These points have horospherical coordinates \((0, v, u)\). Hence, \( L \) intersects the finite part of the boundary in the vertical axis \( \{(0, v)|v \in \mathbb{R}\} \) of \( \mathcal{N} \). By applying a Heisenberg translation, it is easy to see that any other complex line passing through infinity intersects the finite part of the boundary in the vertical line \( \{(\zeta_0, v)|v \in \mathbb{R}\} \) for some fixed \( \zeta_0 \in \mathbb{C} \). This is called an infinite chain or infinite \( \mathbb{C} \)-circle.

Now consider a complex line not passing through the point at infinity. The simplest example of such a line which intersects complex hyperbolic space is the line \( L \) spanned by
\[
\begin{bmatrix}
-1 \\
0 \\
1
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}.
\]
This complex line \( L \) consists of points
\[
\begin{bmatrix}
-1 \\
\sqrt{2}\zeta \\
1
\end{bmatrix} = \begin{bmatrix}
-|\zeta|^2 - (1 - |\zeta|^2) \\
\sqrt{2}\zeta \\
1
\end{bmatrix}.
\]
These points have horospherical coordinates \((\zeta, 0, 1 - |\zeta|^2)\). Hence, \( L \) intersects the boundary of the Siegel domain in such points with \(|\zeta| = 1\). In other words, \( \partial L \) is the circle \( \{(e^{i\theta}, 0)|\theta \in [0, 2\pi]\} \). By applying a Heisenberg dilation we see that the circle \( \{(r_0 e^{i\theta}, 0)|\theta \in [0, 2\pi]\} \) for any fixed \( r_0 \in \mathbb{R}_+ \) is also the boundary of a complex line. Now applying Heisenberg translation by \((x_0 + iy_0, v_0)\), we see that the most general complex line not passing through infinity intersects the boundary of the Siegel domain in the following ellipse whose vertical projection is a circle
\[
\left\{(r_0 e^{i\theta} + x_0 + iy_0, v_0 + 2r_0 y_0 \cos(\theta) - 2r_0 x_0 \sin(\theta)|\theta \in [0, 2\pi]\right\}
\]
for fixed \( r_0 \in \mathbb{R}_+ \) and \((x_0 + iy_0, v_0) \in \mathcal{N}\). Observe that the eccentricity of the ellipse increases with \(|x_0 + iy_0|\). This is called a finite chain or finite \( \mathbb{C} \)-circle.

We do the same for totally real subspaces. First consider the totally real subspace \( R \) passing through \( o \) and \( \infty \) which is fixed by complex conjugation. Hence \( R \) consists of vectors in \( \mathbb{C}^{2,1} \) with real entries. Finite points in the Siegel domain with real entries have the form
\[
\begin{bmatrix}
-x^2 - u \\
\sqrt{2}x \\
1
\end{bmatrix}
\]
where \( x \in \mathbb{R} \). These points have horospherical coordinates \((x, 0, u)\). Hence \( L \) intersects the boundary at \( \infty \) and in the points
\[
\begin{bmatrix}
-x^2 \\
\sqrt{2}x \\
1
\end{bmatrix}
\]
where $x \in \mathbb{R}$. In other words $\partial R$ is the $x$ axis of the Heisenberg group, that is the subset of the Heisenberg group given by $\{(x,0)|x \in \mathbb{R}\}$. By applying Heisenberg rotations we see that for any fixed $\theta_0 \in [0,2\pi]$ the line $\{(xe^{i\theta_0},0)|x \in \mathbb{R}\}$ is also the finite part of the boundary of a totally real plane containing $o$ and $\infty$. In particular, the $y$ axis of the Heisenberg group is such a line. By applying Heisenberg translation by $(x_0 + iy_0,v_0)$ we find the general form for the boundary of a totally real plane passing through $\infty$. It is

$$\left\{(xe^{i\theta_0} + x_0 + iy_0, v_0 + 2xy_0 \cos(\theta_0) - 2x_0 \sin(\theta_0) | x \in \mathbb{R}\right\}$$

for fixed $\theta_0 \in [0,2\pi)$ and $(x_0 + iy_0, v_0) \in \mathcal{N}$. Observe that the gradient of the line is proportional to its distance from $o$. Such boundaries of totally real Lagrangian planes containing $\infty$ are called infinite $\mathbb{R}$-circles.

We now do the same for the boundaries of totally real subspaces not passing through $\infty$. This is more complicated. We begin with the boundary of the totally real subspace $R$ fixed by the following involution

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} z_3 \\ \bar{z}_2 \\ \bar{z}_1 \end{bmatrix}.$$  

Points in the boundary of the Siegel domain fixed by this involution have the form

$$\begin{bmatrix} -e^{2i\theta} \\ i\sqrt{2\cos(2\theta)}e^{i\theta} \\ 1 \end{bmatrix}$$

where $\theta \in [-\pi/4,\pi/4) \cup (3\pi/4,5\pi/4]$.

The values of $\theta$ are chosen to make $\cos(2\theta)$ non-negative. In other words the subset of the Heisenberg group given by

$$\left\{(i\sqrt{\cos(2\theta)})e^{i\theta}, -\sin(2\theta)) | \theta \in [-\pi/4,\pi/4) \cup (3\pi/4,5\pi/4]\right\}.$$  

This is an example of a finite $\mathbb{R}$-circle, and is called the standard finite $\mathbb{R}$-circle.

This $\mathbb{R}$-circle $R$ is a non-planar space curve and we now discuss it slightly more carefully. In order to simplify notation, define

$$p(\theta) = (i\sqrt{\cos(2\theta)}e^{i\theta}, -\sin(2\theta)).$$

Observe that $R$ is connected in spite of the fact that the values of the parameter $\theta$ are contained in two disjoint intervals. To see this, observe that $p(-\pi/4) = p(3\pi/4) = (0,1)$ and $p(\pi/4) = p(5\pi/4) = (0,-1)$. Alternatively, we can see that $R$ is homeomorphic to a circle using the following re-parametrisation of $R$:

$$\begin{bmatrix} -(1 + i \cos \phi)/(1 - i \cos \phi) \\ \sqrt{2}i \sin \phi/(1 - i \cos \phi) \\ 1 \end{bmatrix} = \begin{bmatrix} -(\sin^2 \phi - 2i \cos \phi)/(1 + \cos^2 \phi) \\ \sqrt{2}i \sin \phi(1 + i \cos \phi)/(1 + \cos^2 \phi) \\ 1 \end{bmatrix}$$

where $\phi \in [0,2\pi)$.

The vertical projection of $R$ is a plane curve given by the parametric equation

$$i\sqrt{\cos(2\theta)}e^{i\theta}$$

where $\theta \in [-\pi/4,\pi/4) \cup (3\pi/4,5\pi/4]$. 


In Cartesian coordinates \( x = -\sin(\theta)\sqrt{\cos(2\theta)} \) and \( y = \cos(\theta)\sqrt{\cos(2\theta)} \) we have
\[
x^2 + y^2 = \cos(2\theta), \quad x^2 - y^2 = -\cos^2(2\theta).
\]
Thus the equation of this curve is
\[
(x^2 + y^2)^2 + x^2 - y^2 = 0.
\]
This is a lemniscate of Bernoulli, see Chapter 12 of Lockwood [20].

We rewrite \( v = -\sin(2\theta) \) as
\[
v = -\sin(2\theta) = \frac{-\sin(\theta)}{\cos(\theta)}(\cos^2(\theta) - \sin^2(\theta) + 1) = \frac{x}{y}(x^2 + y^2 + 1).
\]
Observe that as \( x \) and \( y \) tend to 0 they do so along the lines \( x = y \) and \( x = -y \). Therefore
when \( x = y = 0 \) we have \( v = \pm 1 \). Thus the \( \mathbb{R} \)-circle is a space curve given points
\((x + iy, v) \in \mathcal{N}\) satisfying the equations
\[
(x^2 + y^2)^2 + x^2 - y^2 = 0, \quad v = \frac{x}{y}(x^2 + y^2 + 1) \quad \text{for} \quad y^2 > x^2 > 0
\]
together with the points \((0, \pm 1)\).

From our parametrisation we have
\[
v^2 = 1 - \cos^2(2\theta), \quad 2x^2 = \cos(2\theta)(1 - \cos(2\theta)), \quad 2y^2 = \cos(2\theta)(1 + \cos(2\theta)).
\]
Therefore the projection of this \( \mathbb{R} \)-circle onto the \((x, v)\) plane is given by all real solutions to the equation
\[
(v^2 - 2x^2)^2 = v^2 - 4x^2.
\]
Similarly the projection onto the \((y, v)\) plane is given by all real solutions other than \((0, 0)\) to the equation
\[
(v^2 + 2y^2)^2 = v^2 + 4y^2.
\]
In order to obtain all finite \( \mathbb{R} \)-circles we need to apply Heisenberg rotations, dilations and translations to this one.

5.6 Classification of isometries revisited

Inversion in a totally real Lagrangian plane is an anti-holomorphic involution. A product of two of these inversions is holomorphic and so is necessarily in \( \text{PU}(2, 1) \). The theorem below shows, first, that all elements of \( \text{PU}(2, 1) \) may be written as the product of inversions in two \( \mathbb{R} \)-circles and, secondly, they may be classified as loxodromic, parabolic, boundary elliptic or regular elliptic by the intersection and linking properties of these \( \mathbb{R} \)-circles.

**Theorem 5.9** Any \( A \in \text{PU}(2, 1) \) may be decomposed as the product of a pair of inversions in totally real Lagrangian planes. Moreover, if these totally real Lagrangian planes have boundary \( \mathbb{R} \)-circles \( R_1 \) and \( R_2 \) then:

(i) if \( R_1 \) and \( R_2 \) are disjoint and unlinked then \( A \) is loxodromic,
(ii) if \( R_1 \) and \( R_2 \) are disjoint and linked then \( A \) is regular elliptic,
(iii) if \( R_1 \) and \( R_2 \) intersect in exactly one point then \( A \) is parabolic,
(iv) if \( R_1 \) and \( R_2 \) intersect in two points then \( A \) is boundary elliptic.

Every complex hyperbolic isometry is either holomorphic, and so is given by a matrix in \( \text{PU}(2,1) \), or anti-holomorphic, so is an inversion in a totally real Lagrangian subspace followed by an element of \( \text{PU}(2,1) \). Thus we have the following corollary.

**Corollary 5.10** Any complex hyperbolic isometry may be written as a product of at most three inversions in totally real Lagrangian planes.

We prove Theorem 5.9 by conjugating \( A \) to a normalised form and then showing such an \( A \) may be written as a product of inversions in totally real Lagrangian planes. We do this case by case. It will follow from our reasoning that the \( \mathbb{R} \)-circles have the required properties. As the cases exhaust all possibilities (except for \( R_1 = R_2 \) which gives \( A = I \)) this proves the result. For simplicity we work on the boundary \( \partial \mathbb{H}_\mathbb{C}^2 = \mathcal{N} \cup \{ \infty \} \).

**Lemma 5.11** Suppose that \( A : (\zeta, v) \mapsto (\lambda \zeta, |\lambda|^2 v) \) with \( |\lambda| \neq 1 \) is a loxodromic map fixing \( o \) and \( \infty \). Then \( A \) may be written as \( A = \iota_2 \iota_1 \) where \( \iota_j \) is inversion in the \( \mathbb{R} \)-circle \( R_j \). Here \( R_1 \) is the standard imaginary \( \mathbb{R} \)-circle and \( R_2 \) is the image of the standard imaginary \( \mathbb{R} \)-circle under \( B : (\zeta, v) \mapsto (\lambda^{1/2} \zeta, |\lambda| v) \). These two \( \mathbb{R} \)-circles are disjoint and unlinked.

**Proof:** Inversion in the standard imaginary \( \mathbb{R} \)-circle \( R_1 \) and the loxodromic map \( B \) are given by
\[
\iota_1 \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} z_3 \\ z_2 \\ z_1 \end{bmatrix}, \quad B \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\zeta} \zeta_{1/2} z_1 \\ \zeta_{1/2} z_2 \\ \frac{1}{\zeta} \zeta_{1/2} z_3 \end{bmatrix}.
\]

Inversion in \( R_2 = B(R_1) \) is given by
\[
\iota_2 \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = B \iota_1 B^{-1} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = B \iota_1 \begin{bmatrix} \frac{1}{\zeta} \zeta_{-1/2} z_1 \\ \zeta_{-1/2} z_2 \\ \frac{1}{\zeta} \zeta_{-1/2} z_3 \end{bmatrix} = B \begin{bmatrix} \frac{1}{\zeta} \zeta_{1/2} z_3 \\ \zeta_{1/2} z_2 \\ \frac{1}{\zeta} \zeta_{1/2} z_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\zeta} \zeta_{1/2} z_3 \\ \zeta_{1/2} z_2 \\ \frac{1}{\zeta} \zeta_{1/2} z_1 \end{bmatrix}.
\]

Therefore
\[
\iota_2 \iota_1 \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \iota_2 \begin{bmatrix} \frac{1}{\zeta} \zeta_{1/2} z_3 \\ \zeta_{1/2} z_2 \\ \frac{1}{\zeta} \zeta_{1/2} z_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\zeta} \zeta_{1/2} z_3 \\ \zeta_{1/2} z_2 \\ \frac{1}{\zeta} \zeta_{1/2} z_1 \end{bmatrix} = A \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix},
\]
using Lemma 4.3. Thus \( A = \iota_2 \iota_1 \) as required. Observe \( R_1 \) is on the unit Cygan sphere centred at \( o \) and \( R_2 \) is on the Cygan sphere of radius \( |\lambda|^{1/2} \neq 1 \) centred at \( o \). These two spheres are disjoint and nested. Hence the two \( \mathbb{R} \)-circles are disjoint and unlinked. □

**Lemma 5.12** Suppose that \( A : (\zeta, v) \mapsto (\zeta + \tau, v + t - 2 \Im(\tau z)) \) is Heisenberg translation by \( (\tau, t) \in \mathcal{N} - \{ o \} \) fixing \( \infty \). Then \( A \) may be written as \( \iota_2 \iota_1 \) where \( \iota_j \) is inversion in the \( \mathbb{R} \)-circle \( R_j \). Here \( R_1 \) is the infinite \( \mathbb{R} \)-circle given by \( R_1 = \{ (\zeta, v) = (k \tau, 0) : k \in \mathbb{R} \} \) and \( R_2 \) is the image of \( R_1 \) under the map \( B : (\zeta, v) \mapsto (\zeta + \tau/2, v + t/2 - \Im(\tau \zeta)) \). These two \( \mathbb{R} \)-circles only intersect in the point \( \infty \).
Proof: Without loss of generality, assume that $\tau \in \mathbb{R}$. Inversion in $R_1$ is given by $\iota_1(\zeta, v) = (-\bar{\zeta}, -v)$. Then $R_2 = B(R_1)$ is given by

$$R_2 = \{(ki + 1/2)\tau, -k|\tau|^2 + t/2 : k \in \mathbb{R}\}.$$ 

If $\tau \neq 0$ then the $\zeta$ coordinates of $R_1$ and $R_2$ are distinct and if $\tau = 0$ the $v$ coordinates are distinct. Hence they only intersect at $\infty$. Inversion in $R_2$ is given by $\iota_2(\zeta, v) = B\iota_1 B^{-1}(\zeta, v)$

$$= B(\zeta - \tau/2, v - t/2 + \tau \Im(\zeta))$$
$$= B(-\bar{\zeta} + \tau/2, -v + t/2 - \tau \Im(\zeta))$$
$$= (-\bar{\zeta} + \tau, -v + t - 2\tau \Im(\zeta)).$$

Thus $\iota_2 \iota_1(\zeta, v) = (\zeta + \tau, v + t - 2\tau \Im(\zeta)) = A(\zeta, v)$. Hence $A = \iota_2 \iota_1$ as claimed.

Lemma 5.13 Suppose that $A : (\zeta, v) \mapsto (e^{i\theta} \zeta, v + t)$ where $\theta \in (0, 2\pi)$ is screw parabolic, for $t \neq 0$, and boundary elliptic, for $t = 0$. Then $A$ may be written as $\iota_2 \iota_1$ where $\iota_j$ is inversion in the $\mathbb{R}$-circle $R_j$. Here $R_1$ is the $x$-axis in the Heisenberg group, that is $R_1 = \{(z, v) = (x, 0) : x \in \mathbb{R}\}$, and $R_2$ is the image of $R_1$ under the map $B : (\zeta, v) \mapsto (e^{i\theta/2} \zeta, v + t/2)$. If $t \neq 0$ then these two $\mathbb{R}$-circles only intersect in the point $\infty$. If $t = 0$ then these two $\mathbb{R}$-circles intersect at $0$ and $\infty$.

Proof: This is very similar to the previous lemmas. We have seen earlier that inversion in $R_1$ is complex conjugation. In horospherical coordinates this inversion is given by $\iota_1(\zeta, v, u) = (-\bar{\zeta}, -v, u)$. Hence

$$\iota_2(\zeta, v) = B\iota_1 B^{-1}(\zeta, v)$$
$$= B\iota_1(e^{-i\phi/2} \zeta, v - t/2)$$
$$= B(e^{i\phi/2} \bar{\zeta}, -v + t/2)$$
$$= (e^{i\theta} \bar{\zeta}, -v + t).$$

As before it is easy to see that $A = \iota_2 \iota_1$ and to find the intersections of the two $\mathbb{R}$-circles.

It remains to consider the case of regular elliptic maps. We begin by using the ball model. We know that a regular elliptic map may be written in the form

$$A = \begin{bmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{i\phi} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $\theta, \phi \in (0, 2\pi)$. The map $A$ can be decomposed as $A = \iota_2 \iota_1$ where

$$\iota_1 : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} \bar{z}_1 \\ e^{-i\phi} \bar{z}_2 \\ \bar{z}_3 \end{bmatrix}, \quad \iota_2 : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} e^{i\theta} \bar{z}_1 \\ \bar{z}_2 \\ \bar{z}_3 \end{bmatrix}.$$
are inversions in the totally real planes whose boundaries are the $R$-circles
\[
R_1 = \left\{ (z_1, z_2) = (\cos(\psi), e^{-i\phi/2} \sin(\psi)) : \psi \in [0, 2\pi) \right\},
\]
\[
R_2 = \left\{ (z_1, z_2) = (e^{i\theta/2} \cos(\psi), \sin(\psi)) : \psi \in [0, 2\pi) \right\}.
\]

Since $\theta/2$ and $\phi/2$ lie in the interval $(0, \pi)$, it is clear that these two $R$-circles are disjoint.

In order to be able to continue our analysis of the boundary using the Siegel domain model we conjugate $A$, $\iota_1$ and $\iota_2$ be the Cayley transform (2), but still use the same names. These maps become
\[
\iota_1 : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} \bar{z}_1 \\ e^{-i\phi/2} \bar{z}_2 \\ \bar{z}_3 \end{bmatrix}, \quad \iota_2 : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} \cos(\theta/2) e^{i\theta/2} \bar{z}_1 + i \sin(\theta/2) e^{i\theta/2} \bar{z}_3 \\ \bar{z}_2 \\ i \sin(\theta/2) e^{i\theta/2} \bar{z}_1 + \cos(\theta/2) e^{i\theta/2} \bar{z}_3 \end{bmatrix}
\]
and
\[
A = \begin{bmatrix} \cos(\theta/2) e^{i\theta/2} & 0 & i \sin(\theta/2) e^{i\theta/2} \\ 0 & e^{i\phi} & 0 \\ i \sin(\theta/2) e^{i\theta/2} & 0 & \cos(\theta/2) e^{i\theta/2} \end{bmatrix}. \tag{56}
\]

The $R$-circles $R_j$ fixed by the involutions $\iota_j$ become
\[
R_1 = \{ (x + iy, v) : \sin(\phi/2) x + \cos(\phi/2) y = v = 0 \} \tag{57}
\]
and
\[
R_2 = \left\{ (x + iy, v) : \sin(\theta/2) \left( (x^2 + y^2)^2 + y^2 - x^2 \right) - 2 \cos^2(\theta/2) x y = 0 \right\} \tag{58}
\]
It is clear that $R_2$ intersects the plane defined by $v = 0$ in the points where $y = 0$ and hence $x = \pm 1$. The $R$-circle $R_1$ is a line through the origin and gradient $-\tan(\phi/2) \neq 0$ in this plane. Each of the halfplanes determined by this line contains one of the points $(\pm 1, 0)$. Thus the two $R$-circles may easily be seen to be disjoint and linked. Thus we have proved:

**Lemma 5.14** Suppose that $A$ be a regular elliptic map of the form (56). Then $A$ may be written as $A = \iota_2 \iota_1$ where $\iota_j$ is inversion in the $R$-circles $R_j$ given in (57) and (58). These two $R$-circles are disjoint and linked.

### 6 Distance formulae

#### 6.1 Cross ratios

Cross-ratios were generalised to complex hyperbolic space by Korányi and Reimann [19]. Following their notation, we suppose that $z_1, z_2, w_1, w_2$ are four distinct points of $\mathbb{H}_C^2$, and we definite their complex cross-ratio to be
\[
[z_1, z_2, w_1, w_2] = \frac{\langle w_1, z_1 \rangle \langle w_2, z_2 \rangle}{\langle w_2, z_1 \rangle \langle w_1, z_2 \rangle}
\]
We will only use the absolute value $|\langle z_1, z_2, w_1, w_2 \rangle|$ which we call the cross-ratio. Observe that if two of the entries are the same then the cross ratio is still defined and equals one of 0, 1 or $\infty$. If $z_1$, $z_2$, $w_1$ and $w_2$ all lie on $\partial H^2_{\omega}$ then we can express the cross ratio in terms of the Cygan metric as follows:

$$\left|\langle z_1, z_2, w_1, w_2 \rangle\right| = \frac{\rho_0(w_1, z_1)^2 \rho_0(w_2, z_2)^2}{\rho_0(w_2, z_1)^2 \rho_0(w_1, z_2)^2}$$

provided none of the four points is $\infty$. If $w_1 = \infty$ then

$$\left|\langle z_1, z_2, \infty, w_2 \rangle\right| = \frac{\rho_0(w_2, z_2)^2}{\rho_0(w_2, z_1)^2}.$$

### 6.2 Distance between a point and a geodesic

Let $p$ and $q$ be points of $\partial H^2_{\omega}$ and let $z$ be any point of $H^2_{\omega}$. We choose lifts $p$, $q$ and $z$ in $\mathbb{C}^{2,1}$ of $p$, $q$ and $z$.

Following Goldman, we define

$$\eta(p, q, z) = [z, q, p, z] = \frac{\langle p, z \rangle \langle z, q \rangle}{\langle z, z \rangle \langle p, q \rangle}.$$

Then we have

**Proposition 6.1** Let $p$ and $q$ be points of $\partial H^2_{\omega}$ and let $z$ be any point of $H^2_{\omega}$. Let $\gamma$ be the geodesic with endpoints $p$ and $q$. The distance $\rho(\gamma, z)$ from $z$ to $\gamma$ is given by

$$\cosh^2 \left( \frac{\rho(\gamma, z)}{2} \right) = \left| \eta(p, q, z) \right| + \Re(\eta(p, q, z)).$$

**Proof:** Without loss of generality, normalise so that $\langle p, q \rangle = -1$. This means that a general point on the lift of $\gamma$ to $\mathbb{C}^{2,1}$ is given by $e^{t/2}p + e^{-t/2}q$. The corresponding point in $H^2_{\omega}$ will be denoted by $\gamma(t)$. Therefore

$$\cosh^2 \left( \frac{\rho(\gamma(t), z)}{2} \right) = \frac{|\langle e^{t/2}p + e^{-t/2}q, z \rangle|^2}{\langle e^{t/2}p + e^{-t/2}q, e^{t/2}p + e^{-t/2}q \rangle \langle z, z \rangle} = e^{t |\langle p, z \rangle|^2 + 2 \Re(\langle p, z \rangle \langle q, z \rangle) + e^{-t} |\langle q, z \rangle|^2}.$$

Using elementary calculus we see that, as $t$ varies over $\mathbb{R}$, the minimum of the right hand side is attained when

$$e^t = \left| \frac{\langle q, z \rangle}{\langle p, z \rangle} \right|.$$

Substituting this in the above expression we see that

$$\cosh^2 \left( \frac{\rho(\gamma, z)}{2} \right) = \frac{|\langle p, z \rangle \langle z, q \rangle| + \Re(\langle p, z \rangle \langle z, q \rangle)}{|\langle p, q \rangle \langle z, z \rangle|}.$$

As the denominator is real and positive this proves the result. \hfill $\Box$

We observe that if $z$ is on $\gamma$, then $z = e^{s/2}p + e^{-s/2}q$ and so $\eta(p, q, z) = 1/2$. 

6.3 Distance between pairs of geodesics

For \( j = 1, 2 \) let \( p_j \) and \( q_j \) be points of \( \partial \mathcal{H}_C^1 \) with lifts \( p_j \) and \( q_j \) in \( \mathbb{C}^{2,1} \). Normalise these lifts so that \( \langle p_j, q_j \rangle = -1 \). We want to investigate the distance between the geodesics \( \gamma_1 \) and \( \gamma_2 \) where \( \gamma_j \) has endpoints \( p_j \) and \( q_j \). These geodesics are given by

\[
\gamma_1 = \left\{ e^{t/2}p_1 + e^{-t/2}q_1 : t \in \mathbb{R} \right\}, \quad \gamma_2 = \left\{ e^{s/2}p_2 + e^{-s/2}q_2 : t \in \mathbb{R} \right\}.
\]

We want to find the distance between \( \gamma_1 \) and \( \gamma_2 \).

First we show that it is possible to find pairs of geodesics so that, as a function of the Hermitian products of \( p_j \) and \( q_j \), the distance between them cannot be expressed using radicals. This method and example is due to Hanna Sandler [25]. Suppose that \( \gamma_1(t) \) is the point on \( \gamma_1 \) with lift \( e^{t/2}p_1 + e^{-t/2}q_1 \). Using Proposition 6.1 we see that

\[
\cosh^2 \left( \frac{\rho(\gamma_1(t), \gamma_2)}{2} \right) = \left| \eta(p_2, q_2, \gamma_1(t)) \right| + \Re \left( \eta(p_2, q_2, \gamma_1(t)) \right).
\]

We now express this as a function of \( t \) and the inner products of the \( p_j \) and \( q_j \).

\[
\eta(p_2, q_2, \gamma_1(t)) = \frac{\langle p_2, e^{t/2}p_1 + e^{-t/2}q_1 \rangle \langle e^{t/2}p_1 + e^{-t/2}q_1, q_2 \rangle}{\langle e^{t/2}p_1 + e^{-t/2}q_1, e^{t/2}p_1 + e^{-t/2}q_1 \rangle} \langle p_2, q_2 \rangle
\]

\[
= \frac{1}{2} \left( \langle p_2, p_1 \rangle \langle p_1, q_2 \rangle e^t + \langle p_2, q_1 \rangle \langle p_1, q_2 \rangle \right)
\]

\[
+ \frac{1}{2} \left( \langle p_2, p_1 \rangle \langle q_1, q_2 \rangle + \langle p_2, q_1 \rangle \langle q_1, q_2 \rangle e^{-t} \right)
\]

\[
= \frac{1}{2} \left( ae^t + b + ce^{-t} \right)
\]

where

\[
a = \langle p_2, p_1 \rangle \langle p_1, q_2 \rangle, \\
b = \langle p_2, q_1 \rangle \langle p_1, q_2 \rangle + \langle p_2, p_1 \rangle \langle q_1, q_2 \rangle, \\
c = \langle p_2, q_1 \rangle \langle q_1, q_2 \rangle.
\]

Suppose that \( x(t) \) and \( y(t) \) are the real and imaginary parts of \( \eta(p_2, q_2, \gamma_1(t)) \). Then we need to find the minimum of the function

\[
g(t) = \sqrt{x(t)^2 + y(t)^2} + x(t).
\]

Differentiating and setting \( g'(t) = 0 \), we need to solve

\[
0 = \frac{x(t)x'(t) + y(t)y'(t)}{\sqrt{x(t)^2 + y(t)^2}} + x'(t).
\]

This is equivalent to

\[
x(t)x'(t) + y(t)y'(t) = -x'(t)\sqrt{x(t)^2 + y(t)^2}.
\]

Squaring both sides and simplifying we see that

\[
0 = y(t) \left( y(t)x'(t)^2 - y(t)y'(t)^2 - 2x(t)y(t)x'(t)y'(t) \right).
\]
If \( y(t) = 0 \) then \( g'(t) = 2x'(t) \). Thus a minimum occurs when \( y(t) = x'(t) = 0 \) and we have \( y(t)x'(t)^2 - y(t)y'(t)^2 - 2x(t)y(t)x'(t)y'(t) = 0 \) as well. Thus it suffices to solve

\[
0 = y(t)x'(t)^2 - y(t)y'(t)^2 - 2x(t)y(t)x'(t)y'(t) = 3 \left( \eta(p_2, q_2, \gamma_1(t)) \eta'(p_2, q_2, \gamma_1(t))^2 \right).
\]

Since \( \eta(p_2, q_2, \gamma_1(t)) = (ae^t + b + ce^{-t})/2 \) we have \( \eta'(p_2, q_2, \gamma_1(t)) = (ae^t - ce^{-t})/2 \). Thus,

\[
8\eta(p_2, q_2, \gamma_1(t)) \eta'(p_2, q_2, \gamma_1(t))^2 = (ae^t + b + ce^{-t}) \left( \pi^2 e^{-2t} - 2\pi c + c^2 e^{-2t} \right) = a|a|^2 e^{3t} + \pi^2 be^{2t} + (\pi^2 c - 2|a|^2 c)e^t - 2\pi b c + (\pi^2 c - 2|a|^2 c)e^t + b\pi^2 c e^{-2t} + c|c|^2 e^{-3t}.
\]

Multiplying by \( e^{3t} \) and taking the imaginary part, we see that finding the shortest distance between \( \gamma_1 \) and \( \gamma_2 \) is equivalent to solving a sixth order polynomial in \( e^t \). The coefficients of this polynomial may be expressed in terms of the Hermitian products of \( p_2 \) and \( q_2 \). The following example shows that there exist geodesics for which this sixth order polynomial is not solvable by radicals in terms of its coefficients.

Let

\[
p_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad q_1 = \begin{bmatrix} -(1 + 4i)/4 \\ -(1 + i)/2 \end{bmatrix}, \quad p_2 = \begin{bmatrix} -1/2 \\ 1/\sqrt{2} \end{bmatrix}, \quad q_2 = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}.
\]

These have been normalised so that \( \langle p_1, q_1 \rangle = \langle p_2, q_2 \rangle = -1 \). A short calculation shows that

\[
a = \frac{1}{4}, \quad b = \frac{5 - 2i\sqrt{2}}{8}, \quad c = \frac{25 - 4i\sqrt{2}}{64}.
\]

From this, we see that the minimum distance occurs when \( t \) satisfies

\[
0 = 2e^{3t} \left( \eta(p_2, q_2, \gamma_1(t)) \eta'(p_2, q_2, \gamma_1(t))^2 \right) = 3 \left( a|a|^2 e^{6t} + \pi^2 be^{5t} + (\pi^2 c - 2|a|^2 c)e^{4t} - 2\pi b c e^{3t} \right) + 3 \left( (\pi^2 c - 2|a|^2 c)e^{2t} + b\pi^2 c e^{t} + c|c|^2 \right) = \frac{-\sqrt{2}}{16} \left( (4e^t)^5 + 3(4e^t)^4 - 30(4e^t)^3 - 50(4e^t)^2 + 93(4e^t) - 657 \right).
\]

Thus writing \( x = 4e^t \) we need to find the roots of

\[
g(x) = x^5 + 3x^4 - 30x^3 - 50x^2 + 93x - 657.
\]

Evaluating \( g \) at \( x = -6, -5, -4, 5, 6 \) we see that \( g(x) \) has three real roots \( x_1, x_2, x_3 \) satisfying \(-6 < x_1 < -5 < x_2 < -4 \) and \( 5 < x_3 < 6 \). We claim that \( g(x) \) has no more real roots. In order to see this, consider

\[
g'(x) = 5x^4 + 12x^3 - 90x^2 - 100x + 93.
\]
Evaluating $g'$ at $x = -6, -5, 0, 1, 4$ we see that $g'(x)$ has roots $x_1, x_5, x_6, x_7$ with $-6 < x_4 < -5 < x_5 < 0 < x_6 < 1 < x_7 < 4$. Moreover, $x_6$ must be a local maximum of $g(x)$. However, when $0 < x < 1$ we have $g(x) < 1 + 3 + 93 - 657 < -560$. Therefore $g(x_6) < 0$ and so $g(x)$ has a local maximum on which it takes a negative value. Hence, $g$ cannot have the maximum number of real zeros and so has a pair of conjugate complex roots. Finally, we claim that $g(x)$ is irreducible over the integers and hence over the rationals. First we transform $g(x)$ to

$$h(x) = \frac{g(2x - 5)}{32} = x^5 - 11x^4 + 40x^3 - 50x^2 - 2x + 4.$$ Evaluating $h(x)$ at $-4, -2, -1, 2, 4$ we see that $h(x)$ has no linear factors. Now suppose that

$$h(x) = (x^2 + ax + b)(x^3 + cx^2 + dx + e)$$

for integers $a, b, c, d, e$. Then $b$ must be one of ±1, ±2, ±4. Expanding and equating coefficients and simplifying in each case we arrive at a contradiction.

(i) Suppose $b = 1$ and so $e = 4$. Evaluating the coefficients of $x$ and $x^2$ we have $-2 = ae + bd = 4a + d$ and $-50 = e + ad + bc = 4 + a(2 - 4a) + c$. Moreover, evaluating the coefficient of $x^3$ and substituting for $b, c$ and $d$ gives

$$40 = d + ac + b = -2 - 4a + a(-50 - 4 + 2a(1 + 2a)) + 1.$$ The right hand side of this is odd but the left is even, a contradiction.

(ii) Suppose $b = 2$ and $e = 2$. Then $-2 = 2a + 2d$ and $-11 = a + c$. Hence $d = -1 - a$ and $c = -11 - a$. Also, $-50 = e + ad + bc = 2 - a - a^2 - 22 - 2a$ and $40 = d + ac + b = -1 - a - 11a - a^2 + 2$. Combining these gives $a^2 = 30 - 3a = -39 - 12a$ which means $23 = 3a$, a contradiction.

(iii) Suppose that $b = 4$ and $e = 1$. Then $-2 = 4a + 4d$. Substituting for $a$ gives $-50 = e + ad + bc = 1 - 2d - 4d^2 + 4c$. This is a contradiction.

Similar arguments work when $b = -1, -2$ and $-4$.

Putting this together, we see that $g(x)$ is irreducible over $\mathbb{Z}$, and hence over $\mathbb{Q}$, and has exactly three real roots. Thus, using the same argument as Lemma 14.7 of [26] we see that $g(x)$ is not soluble by radicals. This argument says that the Galois group of $g(x)$ must contain a 5-cycle and an involution (complex conjugation). Hence this Galois group is $S_5$.

We conclude this section by giving a lower bound on the distance between geodesics. As at the start of this section, for $j = 1, 2, 6$, let $p_j, q_j$ be points on the boundary of $H^2_c$ with lifts $p_j$ and $q_j$ to $\mathbb{C}^{2,1}$ satisfying $(p_j, q_j) = -1$. Let $n_j$ be the unit polar vector to the complex line spanned by $p_j$ and $q_j$. The following lemma follows from Lemma 3.4:

**Lemma 6.2** With the definitions above we have

$$0 = \langle p_1, p_1 \rangle = \langle p_1, n_2 \rangle \langle n_2, p_1 \rangle - \langle p_1, q_2 \rangle \langle q_2, p_1 \rangle - \langle p_1, p_2 \rangle \langle q_2, p_1 \rangle,$$

$$-1 = \langle p_1, q_1 \rangle = \langle p_1, n_2 \rangle \langle n_2, q_1 \rangle - \langle p_1, q_2 \rangle \langle q_2, q_1 \rangle - \langle p_1, p_2 \rangle \langle q_2, q_1 \rangle,$$

$$0 = \langle q_1, q_1 \rangle = \langle q_1, n_2 \rangle \langle n_2, q_1 \rangle - \langle q_1, q_2 \rangle \langle q_2, q_1 \rangle - \langle q_1, q_2 \rangle \langle q_2, q_1 \rangle,$$

$$0 = \langle p_2, p_2 \rangle = \langle p_2, n_1 \rangle \langle n_1, p_2 \rangle - \langle p_2, q_1 \rangle \langle q_1, p_2 \rangle - \langle p_2, p_1 \rangle \langle q_1, q_2 \rangle,$$

$$-1 = \langle p_2, q_2 \rangle = \langle p_2, n_1 \rangle \langle n_1, q_2 \rangle - \langle p_2, q_1 \rangle \langle q_1, q_2 \rangle - \langle p_2, p_1 \rangle \langle q_1, q_2 \rangle,$$

$$0 = \langle q_2, q_2 \rangle = \langle q_2, n_1 \rangle \langle n_1, q_2 \rangle - \langle q_2, q_1 \rangle \langle q_1, q_2 \rangle - \langle q_2, p_1 \rangle \langle q_1, q_2 \rangle.$$
Lemma 6.3 For \( j = 1, 2 \), let \( \gamma_j \) be a geodesic with endpoints \( p_j \) and \( q_j \) in \( \partial \mathbb{H}^2 \). Let \( p_j \) and \( q_j \) be lifts of \( p_j \) and \( q_j \) to \( \mathbb{C}^{2,1} \) and let \( n_j \in \mathbb{C}^{2,1} \) be the normal vector to \( p_j \) and \( q_j \). Then

\[
\cosh^2 \left( \frac{p(\gamma_1, \gamma_2)}{2} \right) \geq \frac{1}{2} \left( \left| \langle p_1, p_2 \rangle \langle q_2, q_1 \rangle \right| + \left| \langle p_1, q_2 \rangle \langle p_2, q_1 \rangle \right| + \left| \langle p_1, n_2 \rangle \langle n_2, q_1 \rangle \right| + 1 \right).
\]

Proof: We normalise so that \( \langle p_j, q_j \rangle = -1 \) and \( \langle n_j, n_j \rangle = 1 \). Then

\[
\cosh^2 \left( \frac{p(\gamma_1(t), \gamma_2(s))}{2} \right) = \frac{\left| e^{t+s} \langle p_1, p_2 \rangle + e^{-t-s} \langle q_1, q_2 \rangle \right|^2 + \left| e^{t-s} \langle p_1, q_2 \rangle + e^{-t+s} \langle q_1, p_2 \rangle \right|^2}{4}.
\]

Equality in the penultimate line occurs if and only if all the following are true

\[
e^{t+s} = \frac{\langle q_1, q_2 \rangle}{\langle p_1, p_2 \rangle}, \quad e^{-t-s} = \frac{\langle q_1, p_2 \rangle}{\langle q_1, q_2 \rangle}, \quad e^t = \frac{\langle q_1, n_2 \rangle}{\langle p_1, q_2 \rangle}, \quad e^s = \frac{\langle n_1, q_2 \rangle}{\langle n_1, p_2 \rangle}.
\]

Equality in the last line happens if and only if \( \langle n_1, p_2 \rangle \langle q_2, n_1 \rangle \) is real and negative. \( \square \)

Corollary 6.4 The geodesics \( \gamma_1 \) and \( \gamma_2 \) intersect if and only if

\[
[p_2, q_1, p_1, q_2] \quad \text{and} \quad [q_2, q_1, p_1, p_2]
\]

are both real, non-negative and their sum is at most 1.

Proof: Again, we normalise so that \( \langle p_j, q_j \rangle = -1 \) and \( \langle n_j, n_j \rangle = 1 \). Using this, our hypotheses become that

\[
\langle p_1, p_2 \rangle \langle q_2, q_1 \rangle \quad \text{and} \quad \langle p_1, q_2 \rangle \langle p_2, q_1 \rangle
\]

are both real, non-negative and their sum is at most 1.
We see that
\[
-\langle p_1, n_2 \rangle \langle n_2, q_1 \rangle = -\langle n_1, p_2 \rangle \langle q_2, n_1 \rangle = 1 - \langle p_1, q_2 \rangle \langle p_2, q_1 \rangle - \langle p_1, p_2 \rangle \langle q_2, q_1 \rangle > 0.
\]
Define
\[
e^{t_0} = \frac{-\langle q_1, n_2 \rangle}{\langle p_1, n_2 \rangle} > 0 \quad \text{and} \quad e^{s_0} = \frac{-\langle n_1, q_2 \rangle}{\langle n_1, p_2 \rangle} > 0.
\]
Thus
\[
e^{t_0 + s_0} = \frac{\langle q_1, n_2 \rangle}{\langle p_1, n_2 \rangle} \frac{\langle n_1, q_2 \rangle}{\langle p_1, p_2 \rangle} \frac{\langle n_1, n_2 \rangle}{\langle p_1, n_2 \rangle} = \frac{\langle q_1, n_2 \rangle}{\langle n_1, n_2 \rangle} \frac{\langle n_1, q_2 \rangle}{\langle p_1, p_2 \rangle} \frac{\langle p_1, n_2 \rangle}{\langle n_1, p_2 \rangle}
\]
where we have multiplied top and bottom by \(\langle n_1, q_2 \rangle \langle p_2, n_1 \rangle = \langle p_1, n_2 \rangle \langle n_2, q_1 \rangle\). Using the identities from Lemma 6.2 to eliminate \(n_1\) and \(n_2\) we obtain:
\[
e^{t_0 + s_0} = \frac{\langle q_1, q_2 \rangle \langle p_2, q_1 \rangle + \langle q_1, p_2 \rangle \langle q_2, p_1 \rangle}{\langle p_1, p_2 \rangle \langle p_2, q_1 \rangle + \langle q_1, p_2 \rangle \langle p_2, p_1 \rangle}
\]
where we have used our hypothesis that
\[
\frac{\langle q_2, q_1 \rangle}{\langle p_2, p_1 \rangle} = \frac{\langle p_1, p_2 \rangle \langle q_2, q_1 \rangle}{|\langle p_1, p_2 \rangle|^2}
\]
is real.

A similar argument shows that
\[
e^{t_0 - s_0} = \frac{\langle q_1, p_2 \rangle}{\langle p_1, q_2 \rangle}
\]

Therefore we have equality at each stage in Lemma 6.3 and so we have
\[
cosh^2 \left( \frac{\rho(\gamma_1(t_0), \gamma_2(s_0))}{2} \right) = \langle p_1, q_2 \rangle \langle p_2, q_1 \rangle + \langle p_1, p_2 \rangle \langle q_2, q_1 \rangle - \langle p_1, n_2 \rangle \langle n_2, q_1 \rangle.
\]
\[
= 1.
\]

Hence the two points are the same.
Conversely, if
\[-\langle p_1, n_2 \rangle \langle n_2, q_1 \rangle, \quad \langle p_1, q_2 \rangle \langle p_2, q_1 \rangle, \quad \langle p_1, p_2 \rangle \langle q_2, q_1 \rangle \]
do not all lie in the interval \([0, 1]\), applying the triangle inequality to
\[-1 = \langle p_1, n_2 \rangle \langle n_2, q_1 \rangle - \langle p_1, q_2 \rangle \langle p_2, q_1 \rangle - \langle p_1, p_2 \rangle \langle q_2, q_1 \rangle \]
we have the strict inequality
\[\left| \langle p_1, p_2 \rangle \langle q_1, q_2 \rangle \right| + \left| \langle p_1, q_2 \rangle \langle q_1, p_2 \rangle \right| + \left| \langle p_1, n_2 \rangle \langle q_1, n_2 \rangle \right| > 1.\]
Hence the lower bound in Lemma 6.3 becomes
\[\cosh^2 \left( \frac{\rho(\gamma_1(t), \gamma_2(s))}{2} \right) > 1\]
and the two geodesics are disjoint. \(\square\)

**Proposition 6.5** For \(j = 1, 2\), let \(\gamma_j\) be a geodesic with endpoints \(p_j\) and \(q_j\) in \(\partial H_2^\circ\). Then
\[\cosh^2 \left( \frac{\rho(\gamma_1(t), \gamma_2(s))}{2} \right) \geq \min \left\{ \left| [p_2, q_1, p_1, q_2] \right|, \left| [q_2, q_1, p_1, p_2] \right| \right\}.\]

**Proof:** Again, using the triangle inequality on
\[-1 = \langle p_1, n_2 \rangle \langle n_2, q_1 \rangle - \langle p_1, q_2 \rangle \langle p_2, q_1 \rangle - \langle p_1, p_2 \rangle \langle q_2, q_1 \rangle \]
we see that
\[\frac{1}{2} \left( \left| \langle p_1, p_2 \rangle \langle q_1, q_2 \rangle \right| + \left| \langle p_1, q_2 \rangle \langle q_1, p_2 \rangle \right| + \left| \langle p_1, n_2 \rangle \langle q_1, n_2 \rangle \right| + 1 \right)\]
is greater or equal to both
\[\left| \langle p_1, p_2 \rangle \langle q_1, q_2 \rangle \right|, \quad \text{and} \quad \left| \langle p_1, q_2 \rangle \langle q_1, p_2 \rangle \right|.\]
Using \(\langle p_1, q_1 \rangle = \langle p_2, q_2 \rangle = -1\) we see that these are
\[\frac{\left| \langle p_1, q_2 \rangle \langle q_2, q_1 \rangle \right|}{\langle q_2, p_2 \rangle \langle p_1, q_1 \rangle} = \left| [p_2, q_1, p_1, q_2] \right|\]
and
\[\frac{\left| \langle p_1, q_2 \rangle \langle p_2, q_1 \rangle \right|}{\langle p_2, q_2 \rangle \langle p_1, q_1 \rangle} = \left| [q_2, q_1, p_1, p_2] \right|.\]
This completes the proof. \(\square\)
6.4 Distance to complex lines

We begin by finding the distance from a point to a complex line. This should be compared with Proposition 6.1.

**Proposition 6.6** Let $p$ and $q$ be points of $\partial \mathbb{H}^2_C$ and let $p$ and $q$ be lifts of $p$ and $q$. Let $L$ be the complex line spanned by $p$ and $q$. Then for any point $z$ in $\mathbb{H}^2_C$ we have

$$\cosh^2 \left( \frac{\rho(z, L)}{2} \right) = 2\Re \langle p, q, z \rangle.$$

**Proof:** Without loss of generality suppose that $\langle p, q \rangle = -1$. Let $n$ be the unit polar vector to $L$. Choose a lift $z$ of $z$ with $\langle z, z \rangle = -1$. Let $w = \lambda p + \mu q$ be a point on $L$ with $\langle w, w \rangle = -\lambda \overline{\mu} - \mu \overline{\lambda} = -1$. Then

$$\cosh^2 \left( \frac{\rho(z, w)}{2} \right) = \langle z, w \rangle \langle w, z \rangle = \langle z, p \rangle |\lambda| + \langle z, q \rangle |\mu|^2 + \langle z, p \rangle \langle q, z \rangle |\lambda| + \langle z, q \rangle \langle p, z \rangle |\mu|^2 + \langle z, p \rangle \langle q, z \rangle + \langle z, q \rangle \langle p, z \rangle \geq \langle z, p \rangle \langle q, z \rangle + \langle z, q \rangle \langle p, z \rangle.$$

We obtain equality in the last line with the point

$$w = \frac{\langle z, q \rangle}{\sqrt{\langle z, p \rangle \langle q, z \rangle + \langle z, q \rangle \langle p, z \rangle}} p + \frac{\langle z, p \rangle}{\sqrt{\langle z, p \rangle \langle q, z \rangle + \langle z, q \rangle \langle p, z \rangle}} q$$

where we have used Lemma 3.4 to check that the denominator is well defined. (This denominator is chosen to ensure $\langle w, w \rangle = -1$.) Using $\langle z, z \rangle = \langle p, q \rangle = -1$, this gives the result. \qed

**Corollary 6.7** Let $L$ be a complex line with polar vector $n$. Let $z$ be any point of $\mathbb{H}^2_C$ with lift $z$. Then

$$\cosh^2 \left( \frac{\rho(z, L)}{2} \right) = 1 - \frac{\langle z, n \rangle \langle n, z \rangle}{\langle z, z \rangle \langle n, n \rangle} \geq 1.$$

**Proof:** This follows from Proposition 6.6 using Lemma 3.4 and $\langle z, z \rangle = -1$. \qed

**Proposition 6.8** Let $L_1$ and $L_2$ be complex lines with polar vectors $n_1$ and $n_2$. Let

$$N(L_1, L_2) = \frac{\langle n_1, n_2 \rangle \langle n_2, n_1 \rangle}{\langle n_1, n_1 \rangle \langle n_2, n_2 \rangle}.$$
(i) If $N(L_1, L_2) > 1$ then $L_1$ and $L_2$ are ultraparallel and
\[
\cosh^2 \left( \frac{\rho(L_1, L_2)}{2} \right) = N(L_1, L_2).
\]

(ii) If $N(L_1, L_2) = 1$ then $L_1$ and $L_2$ are asymptotic or coincide.

(iii) If $N(L_1, L_2) < 1$ then $L_1$ and $L_2$ intersect.

**Proof:** First suppose that $N(L_1, L_2) > 1$. Let $L_2$ be spanned by $p_2$ and $q_2$. Suppose that $\langle p_2, q_2 \rangle = -1$. Also suppose, without loss of generality, that $\langle n_1, n_1 \rangle = \langle n_2, n_2 \rangle = 1$. Then a general point $w$ on $L_2$ has lift $w = \lambda p_2 + \mu q_2$ in $\mathbb{C}^2$. Without loss of generality, we suppose that $\langle w, w \rangle = -\lambda \mu - \mu \lambda = -1$. From Corollary 6.7, we know that
\[
\cosh^2 \left( \frac{\rho(L_1, w)}{2} \right) = 1 - \frac{\langle w, n_1 \rangle \langle n_1, w \rangle}{\langle w, w \rangle \langle n_1, n_1 \rangle} = 1 + \frac{|\langle n_1, p_2 \rangle \lambda + \langle n_1, q_2 \rangle \mu|^2}{2} \\
= 1 + \frac{|\langle n_1, p_2 \rangle \lambda - \langle n_1, q_2 \rangle \mu|^2}{2} \\
+ \langle n_1, p_2 \rangle \langle q_2, n_1 \rangle + \langle n_1, q_2 \rangle \langle p_2, n_1 \rangle \\
\geq 1 + \langle n_1, p_2 \rangle \langle q_2, n_1 \rangle + \langle n_1, q_2 \rangle \langle p_2, n_1 \rangle
\]
with equality when $\lambda/\mu = \langle n_1, q_2 \rangle / \langle n_1, p_2 \rangle$. Thus we choose $w$ to be the point
\[
w = \nu \langle n_1, q_2 \rangle p_2 + \nu \langle n_1, p_2 \rangle q_2
\]
where $\nu$ is chosen so that
\[
1 = -\langle w, w \rangle = |\nu|^2 \left( \langle n_1, p_2 \rangle \langle q_2, n_1 \rangle + \langle n_1, q_2 \rangle \langle p_2, n_1 \rangle \right).
\]

Now writing
\[
n_1 = -\langle n_1, q_2 \rangle p_2 - \langle n_1, p_2 \rangle q_2 + \langle n_1, n_2 \rangle n_2
\]
we have
\[
1 = \langle n_1, n_1 \rangle = -\langle n_1, p_2 \rangle \langle q_2, n_1 \rangle - \langle n_1, q_2 \rangle \langle p_2, n_1 \rangle + \langle n_1, n_2 \rangle \langle n_2, n_1 \rangle.
\]
Thus we see that
\[
|\nu|^2 = \frac{1}{N(L_1, L_2) - 1}
\]
and
\[
\cosh^2 \left( \frac{\rho(L_1, w)}{2} \right) = 1 + \langle n_1, p_2 \rangle \langle q_2, n_1 \rangle + \langle n_1, p_2 \rangle \langle q_2, n_1 \rangle \\
= \langle n_1, n_2 \rangle \langle n_2, n_1 \rangle.
\]

Using $\langle n_1, n_1 \rangle = \langle n_2, n_2 \rangle = 1$ this gives part (i).

If $L_1 = L_2$ then $n_2 = \lambda n_1$ and so $N(L_1, L_2) = 1$. If $L_1$ and $L_2$ are asymptotic, then we can write $L_j$ as the span of null vectors $p_j$ and $q_j$. As usual, suppose that $\langle p_j, q_j \rangle = -1$. Therefore
\[
n_1 = -\langle n_1, q_2 \rangle p - \langle n_1, p \rangle q_2 + \langle n_1, n_2 \rangle n_2 = -\langle n_1, q_2 \rangle p + \langle n_1, n_2 \rangle n_2
\]
since \( \langle n_1, p \rangle = 0 \). Hence
\[
1 = \langle n_1, n_1 \rangle = \langle n_1, n_2 \rangle \langle n_2, n_1 \rangle.
\]
Therefore \( N(L_1, L_2) = 1 \).

Finally, suppose that \( L_1 \) and \( L_2 \) intersect in a point \( w \). Then, lift \( w \) to the vector \( w = \lambda p_1 + \mu q_1 \). Since \( w \) is also on \( L_2 \) we have
\[
0 = \langle w, n_2 \rangle = \lambda \langle p_1, n_2 \rangle + \mu \langle q_1, n_2 \rangle.
\]
Therefore, using \( w \in V_\perp \) we see
\[
0 > \langle w, w \rangle = -\lambda \overline{\lambda} - \mu \overline{\mu} = \frac{|\lambda|^2}{|\langle q_1, n_2 \rangle|^2} \left( \langle p_1, n_2 \rangle \langle q_1, n_1 \rangle + \langle q_1, n_2 \rangle \langle n_2, p_1 \rangle \right) = \frac{|\lambda|^2}{|\langle q_1, n_2 \rangle|^2} \left( \langle n_1, n_2 \rangle \langle n_2, n_1 \rangle - 1 \right).
\]
Therefore \( N(L_1, L_2) = \langle n_1, n_2 \rangle \langle n_2, n_1 \rangle < 1 \). □

Suppose that two complex lines \( L_1 \) and \( L_2 \) intersect in the origin. This means their polar vectors \( n_j \) have the form
\[
n_1 = \begin{bmatrix} n_{11} \\ n_{12} \\ 0 \end{bmatrix}, \quad n_2 = \begin{bmatrix} n_{21} \\ n_{22} \\ 0 \end{bmatrix}.
\]
Therefore, in \( \mathbb{C}^2 \), the two lines have normal vectors
\[
\begin{pmatrix} n_{11} \\ n_{12} \end{pmatrix}, \quad \begin{pmatrix} n_{21} \\ n_{22} \end{pmatrix}.
\]
Using the Cauchy-Schwarz inequality, we see that the angle between the two lines is \( \theta \) where
\[
\cos^2(\theta) = \frac{|n_{21}n_{11} + n_{22}n_{12}|^2}{|n_{11}|^2 + |n_{12}|^2)(|n_{21}|^2 + |n_{22}|^2)}} = \frac{\langle n_1, n_2 \rangle \langle n_2, n_1 \rangle}{\langle n_1, n_1 \rangle \langle n_2, n_2 \rangle} = N(L_1, L_2).
\]
Motivated by this, we define the angle \( \theta \in [0, \pi/2] \) between any pair of intersecting complex lines by
\[
\cos^2(\theta) = N(L_1, L_2).
\]
This is clearly invariant under complex hyperbolic isometries.
7 Notes

2.1 The terms first and second Hermitian form were defined by Epstein [6]. The third Hermitian form was defined (but not named) by Chen and Greenberg [3] and has been used extensively by others.

2.2 The connection between Hermitian forms and models of complex hyperbolic space given is completely standard. It is an example of the more general connection between quadratic and Hermitian forms for symmetric spaces, see Chen and Greenberg [3] or Chapter 19 of Mostow [22]. The formula (1) holds for other rank 1 symmetric spaces, see page 135 of [22]. The formula for the Bergman distance using the cross ratio is contained in Giraud [10].

2.3 For other Cayley transforms see Section 4.1.1 of [11] or page 574 of [15], for example.

3.1 Similar formulae to those given in this section were given in [14].

3.2 The formulae in this section are analogous to those in the previous section.

3.3 The Hermitian cross product is defined in Section 2.2.7 of Goldman [11]. The fact that the isometry group of \( \mathbb{H}^2 \) acts transitively and acts doubly transitively on the boundary is a special case of similar results for other symmetric spaces.

3.4 Theorem 3.5 is a generalisation of Theorem 7.4.1 of [2].

3.5 The classification of complex hyperbolic isometries goes back to Giraud’s paper of 1921 [10], see also page 52 of Chen and Greenberg [3]. The use of trace to classify isometries may be found in Theorem 6.2.4 of Goldman [11].

4.1 The Heisenberg group is widely studied by analysts, see [18] or [19]. Its relationship to the boundary of complex hyperbolic space generalises to all rank 1 symmetric spaces of compact type. See for example Section 4.2 of [11] for more about this.

4.2 Horospherical coordinates were introduced by Goldman and Parker [12].

4.3 The Cygan metric was constructed for the Heisenberg group by Cygan [4], Lemma 2. See also Cygan [5] and Korányi [18], page 227. This metric was extended to the Siegel domain in [23]. The distortion result, Lemma 4.5, is due to Kamiya, Proposition 2.4 of [16] and is related to Theorem 5.22 of [1].

5.1 The expression of points on a geodesic in terms of their endpoints may be found in Theorem 3.3.3 of [11]. The treatment we give here follows Sandler, Section 3 of [25] (compare this with page 242 of [11] for example).

5.2 See Section 3.1.4 of [11]. For the Poincaré metric on the hyperbolic plane see Section 7.2 of [2].

5.3 See Section 2.2.1 of [11]. For the Klein-Beltrami metric on the hyperbolic plane see Section 3.3 of [24].

5.4 See Sections 3.1.4 and 3.1.9 of [11].
5.5 There is an extensive treatment of the material in this section in Sections 4.3 and 4.4 of [11].

5.6 The classification of elements of $\text{PU}(2,1)$ by means of products of pairs of real reflections is due to Falbel and Zocca [8]. It may be thought of as a generalisation of the classical idea that holomorphic (orientation preserving) isometries of the hyperbolic plane may be written as products of pairs of reflections in geodesics. Similarly, it generalises Fenchel’s idea that all orientation preserving isometries of real hyperbolic 3-space may be decomposed as a product of half-turns (rotation through $\pi$) about a pair of geodesics [9].

6.1 Cross ratios for complex hyperbolic space we introduced by Korányi and Reimann [19].

6.2 The $\eta$ invariant in Goldman [11], Sandler [25]. It is also related to the $A$ invariant of Kamiya, by $A(p, q; z) = 1/|\eta(p, q, z)|$ of Kamiya [15].

6.3 The first part of this section is mostly taken from Sandler’s paper [25]. This includes the example, which is the same as Sandler’s but using our conventions. The lower bound for the distance in the last part of this section is taken from Markham and Parker [21].

6.4 See page 100 of [11] or else [25].

References


REFERENCES


