3H NUMERICAL ANALYSIS 1997-98. PROBLEM SHEET 2.

2.1 Show that

$$B_n\left(e^{\alpha x};x\right) = \left(xe^{\alpha/n}+1-x\right)^n,$$

(a) Verify, in this particular case, the following result (Voronovskaya's theorem)

$$\lim_{n\to\infty} n[B_n(f;x)-f(x)] = \frac{1}{2}x(1-x)f''(x).$$

(b) Verify also, for $f(x)=e^{\alpha x}$, that

$$\lim_{n\to\infty} B_n'(f;x) = f'(x).$$

[This property of providing simultaneous approximation of a function and its derivative is not shared by many approximation sequences.]

2.2 Let f be the function on $[0,1]$ whose graph is the polygonal curve joining the points $(0,0)$, $(1/4,1/2)$, $(1/2,0)$, $(3/4,1/2)$ and $(1,0)$. Find the Bernstein polynomials $B_2(f;x)$ and $B_4(f;x)$, and sketch the graphs of these two polynomials. Will the sequence $\{B_n(f;x)\}$ converge to $f(x)$ on $[0,1]$ as $n\to\infty$?

2.3 For least-squares approximation by broken lines we found a set of linear equations which must be satisfied by the coefficients α_i. Use the bound $\max_i|\alpha_i| \le 3\max_i|\beta_i|$ to deduce that those linear equations have one and only one solution for a given function f.

2.4 (a) Let $I_1 f$ be the piecewise linear interpolant for f on a partition $a=x_0<x_1<\cdots<x_n=b$ of the interval $[a,b]$, and let p^* be the best (in the sense of minimising the maximum absolute error) piecewise linear approximation to f on that partition. Prove that

$$\|f-p^*\| \le \|f-I_1 f\| \le 2\|f-p^*\|.$$

(b) Let $0\le a<b$ and let p_1 be the straight line which interpolates $f(x)=\sqrt{x}$ at a and b. Show that

$$\max_{a\le x\le b}|f(x)-p_1(x)| = \frac{1}{4}\frac{(\sqrt{b}-\sqrt{a})^2}{\sqrt{b}+\sqrt{a}}.$$

(c) Use parts (a) and (b) to prove that the error in approximating $f(x)=\sqrt{|x|}$ on $[-1,1]$, by piecewise linear approximations with $n+1$ uniformly spaced knots, goes to zero no faster than $1/\sqrt{n}$.

2.5 A piecewise cubic s defined on $[a,b]$, with knots $a=x_0<x_1<\cdots<x_n=b$, may be written as

$$s(x) = \sum_{i=0}^{n}[\lambda_i(x)s(x_i)+\bar{\lambda}_i(x)s'(x_i)].$$

Write down explicit expressions for the cardinal functions $\lambda_i(x)$ and $\bar{\lambda}_i(x)$.

2.6 Show that

$$S(x) = \begin{cases} 28+25x+9x^2+x^3, & -3\le x\le -1 \\ 26+19x+3x^2-x^3, & -1\le x\le 0 \\ 26+19x+3x^2-2x^3, & 0\le x\le 3 \\ -163+208x-60x^2+5x^3, & 3\le x\le 4 \end{cases}$$

is a natural cubic spline function, and express it in truncated power form.

2.7 Find the natural cubic spline interpolant to x^3 at the knots $0,1,2,3$. What is the maximum error on $[0,3]$? What would happen if the "not a knot" conditions were used? What is the corresponding complete cubic spline?

2.8 For the partition $a = x_0 < x_1 < \cdots < x_n = b$ of the interval $[a, b]$, a cubic spline S is to be constructed to interpolate a function f at the knots. Prove that the conditions $S''(x_0) = S''(x_1)$ and $S''(x_n) = S''(x_{n-1})$ ensure the existence and uniqueness of S. What form does $S(x)$ take in the intervals $[x_0, x_1]$ and $[x_{n-1}, x_n]$?

2.9 Let $a = x_0 < x_1 < \cdots < x_n = b$ be a partition of the interval $[a, b]$. Let S be a cubic spline which interpolates a function f at the knots $x_0, x_1, \ldots, x_n$ and let $S''(x_i) = M_i$ for $i = 0, 1, \ldots, n$. Then, in the usual notation,

$$\mu_i M_{i-1} + 2M_i + \lambda_i M_{i+1} = d_i, \qquad i = 1, 2, \ldots, n-1. \qquad (1)$$

If the function f is periodic, i.e., $f(x + b - a) = f(x)$, we may wish to impose the periodicity conditions $S'(b) = S'(a)$ and $M_n = M_0$ on the approximating spline. Show that such a spline function S exists and is unique.
For $f(x) = \sin x$, with $n = 4$ and $x_i = i\pi/2$, find an explicit expression for $S(x)$ on $[0, \pi/2]$.

2.10 A spline function S on the partition $a = x_0 < x_1 < \cdots < x_n = b$ is to be constructed so that $S(x)$ is quadratic in each sub-interval, interpolates a function f at the knots, and has a continuous first derivative on $[a, b]$. Find equations which must be satisfied by the derivatives $S'(x_k) = m_k$. How many end conditions are required to specify S uniquely?

2.11 Prove that if f is a function in $C^2[0, 1]$ such that $f(0) = 0$, $f(\frac{1}{2}) = 1$ and $f(1) = 1$, then the inequality $\int_0^1 [f''(x)]^2\, dx \geq 12$ holds.

2.12 Let S be the cubic spline function

$$S(x) = x^3 - 4(x-1)_+^3 + 6(x-2)_+^3 - 4(x-3)_+^3 + (x-4)_+^3.$$

Show that S is identically zero if $x \geq 4$ but that severe cancellation occurs if $S(100)$ is evaluated from the definition of S.

2.13 Write down an expression for the quadratic B-spline with knots $0, 1, 2, 3$ and draw its graph.

2.14 A spline function S of degree $2k + 1$ on the partition $a = x_0 < x_1 < \cdots < x_n = b$ is called a natural spline if it satisfies the conditions

$$S^{(j)}(x_0) = 0 = S^{(j)}(x_n), \qquad \text{for } 1 \leq j \leq 2k.$$

Prove that, among the C^{k+1} functions y which agree with a function f at the knots, the natural cubic spline uniquely minimises the integral $\int_a^b [S^{(k+1)}(x)]^2\, dx$.
Let f be a function in $C^3[-2, 2]$ which has the values $f(-2) = f(-1) = f(1) = f(2) = 0$ and $f(0) = 1$. Show that the inequality $\|f'''\|_\infty \geq 4.5$ is satisfied. If it is known also that $f'(-2) = f'(2) = 0$, prove that the lower bound on $\|f'''\|_\infty$ may be increased to $(231 + 9\sqrt{33})/44 \approx 6.425$.

2.15 Let B_p^k be the B-spline of degree k with knots $x_j = j$, for $j = p, p+1, \ldots, p+k+1$. Use the B-spline recurrence relation to determine the value of the B-spline at each knot, for $k = 1, 2, 3, 4, 5$.

2.16 Problem 2.10 shows that we can't apply symmetric end conditions for a quadratic spline if we use the knots as interpolation nodes. An alternative approach is to take the interpolation nodes to be the two end points of the whole interval and the mid-points of the sub-intervals between successive knots. For simplicity, we assume constant knot-spacing h and use a B-spline basis.
Let $x_i = a + ih$, for $i = 0, \ldots, n$, with $h = (b - a)/n$, and let $t_0 = a$, $t_{n+1} = b$ and $t_j = x_{j-1} + h/2$ for $j = 1, \ldots, n$. For the B-splines based on the knots $\{x_i\}$, show that $B_i^2(x_{i+1}) = \frac{1}{2} = B_i^2(x_{i+2})$, $B_i^2(t_{i+3}) = \frac{1}{8} = B_i^2(t_{i+1})$ and $B_i^2(t_{i+2}) = \frac{3}{4}$. Let

$$S(x) = \sum_{i=-2}^{n-1} \alpha_i B_i^2(x).$$

Find the tridiagonal set of linear equations satisfied by the coefficients α_i when $S(t_i) = f(t_i)$, for $i = 0, \ldots, n+1$, where f is a given function.