Michaelmas 2012, NT III/IV, Problem Sheet 6.

1. Find the fundamental units of $\mathbb{Q}(\sqrt{d})$ for $d = 7$, 30 and 53.

2. Use a unit in $\mathbb{Z}[\sqrt{30}]$ to prove that the difference between 241/44 and $\sqrt{30}$ is less than 5×10^{-5}.

3. Let $n \in \mathbb{Z}$, $n > 2$ and put $d = n^2 - 2$. Show that $n^2 - 1 + n\sqrt{d}$ is a unit of $\mathbb{Z}[\sqrt{d}]$. Is it necessarily the fundamental unit? (Give a proof or a counterexample.)

   [Hint: A unit must be $\pm$ a power of the fundamental unit. Also, it may be helpful to use inequalities.]

4. Give formulae for all the solutions $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ (if any) to

$$9x^2 - 7y^2 = \pm 1\,.$$

5. Give formulae for all the solutions $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ (if any) to
   (i) $x^2 - 6y^2 = 1$; (ii) $x^2 - 6y^2 = -1$; (iii) $x^2 - 6y^2 = 5$;
   (iv) $x^2 - 6y^2 = -5$ and (v) $3x^2 - 2y^2 = 1$.
   [You may assume that $\mathbb{Z}[\sqrt{6}]$ is a PID.]

6. Give formulae for all the solutions $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ (if any) to
   (i) $x^2 - 13y^2 = -1$; (ii) $x^2 - 12y^2 = 13$;
   (iii) $x^2 - 375y^2 = \pm 11$ and (iv) $x^2 - 375y^2 = -51$.

7. Determine whether $299 + 10\sqrt{894}$ is the fundamental unit of $\mathbb{Q}(\sqrt{894})$.
   [Hint: A unit must be $\pm$ a power of the fundamental unit.]

8. Show that $\theta := (1 + \sqrt[3]{2})/\sqrt[3]{3}$ is a unit of $R = \mathcal{O}_{\mathbb{Q}(\sqrt[3]{2},\, \sqrt[3]{3})}$.
   [Remember that you need to show, amongst other things, that $\theta \in R$.]

9. Let d be a positive non-square integer and let u be the fundamental unit of $\mathbb{Z}[\sqrt{d}]$.
   (i) Show that if $n \in \mathbb{Z}^{>0}$ then the fundamental unit of $\mathbb{Z}[n\sqrt{d}]$ is u^m for some $m \in \mathbb{Z}^{>0}$.
   (ii) By considering the powers of u mod n, or otherwise, show that $m < n^2$.

PTO

Some challenging problems for long winter evenings

1. The two numbers
$$\sqrt{11+2\sqrt{29}}+\sqrt{16-2\sqrt{29}+2\sqrt{55-10\sqrt{29}}}$$
and
$$\sqrt{5}+\sqrt{22+2\sqrt{5}}$$
are equal up to the first 20 decimal places to
$$7.38117594089565797098\ldots$$
Are they equal?

2. Show that if
$$\frac{a^2+b^2}{ab+1}$$
(for a, b in $\mathbb{N}$) is integral, then it is a square.

3. Let $p=2m-1$ be an odd prime. Show that the polynomial
$$(1-x)^m+1+x^m$$
is twice a square in $F_p[x]$.

4. Show that for $k,n\in\mathbb{Z}_{>0}$, with k odd, $1^k+2^k+\cdots+n^k$ is divisible by $1+2+\cdots+n$.

*5. There exist no positive integers a, b such that both a^2+b^2 and ab are square.