Michaelmas 2012, NT III/IV, Problem Sheet 4.

1. With $K = \mathbb{Q}(\theta)$ and a, b and $c \in \mathbb{Q}$, calculate $Tr_K(a + b\theta + c\theta^2)$ and $N_K(a + b\theta + c\theta^2)$ where (i) $\theta = \sqrt[3]{7}$ and (ii) $\theta^3 + \theta^2 + 2 = 0$.
2. With $K = \mathbb{Q}(\theta)$ and a and $b \in \mathbb{Q}$, calculate $Tr_K(\theta^3)$ and $N_K(a+b\theta)$ where (i) $\theta^4 + 2\theta + 2 = 0$ and (ii) $\theta^4 = -1$.
3. (Some easy identities.) Show that (i) $X^3 + Y^3 = (X + Y)(X^2 - XY + Y^2)$; (ii) $X^3+Y^3+Z^3-3XYZ = (X+Y+Z)(X^2+Y^2+Z^2-XY-YZ-ZX)$ $= (X + Y + Z)(X + \omega Y + \overline{\omega} Z)(X + \overline{\omega} Y + \omega Z)$, where $\omega = \exp(2\pi i/3)$; [Note also that $(X^2 + Y^2 + Z^2 - XY - YZ - ZX) = ((X - Y)^2 + (Y - Z)^2 + (Z - X)^2)/2$.]

   (What do you get in (ii) if you take $X = a$, $Y = b\sqrt[3]{N}$ and $Z = c(\sqrt[3]{N})^2$?)

   (iii) $(X^n - Y^n) = \prod_{r=0}^{n-1}(X - \zeta^r Y)$, where $\zeta = \exp(2\pi i/n)$.
4. Let $\theta = \sqrt[3]{7}$. Put $R = \mathbb{Z}[\theta]$ and $K = \mathbb{Q}(\theta)$. Show that
   (i) $(1+\theta, 2)_R(1-\theta+\theta^2, 2)_R = (2)_R$ and
   (ii) $(1+\theta, 2)_R^3 = (1+\theta)_R$.
   (iii) Define, for $\alpha \in R$, $\psi(\alpha) = |N_K(\alpha)|$. Show that ψ is multiplicative and satisfies
   $$\psi(\alpha) = 1 \;\Rightarrow\; \alpha \in R^* .$$
   [Q3(ii) may be useful.]
   (iv) Show that $(1+\theta, 2)_R$ is not a principal ideal.
5. Let $L = \mathbb{Q}(i, \sqrt{2})$. Show that
   (i) $[L : \mathbb{Q}] = 4$ and
   (ii) $L = \mathbb{Q}(\theta)$ where $\theta = \frac{1+i}{\sqrt{2}}$.
   (iii) What is the minimum polynomial of θ over (a) $\mathbb{Q}$ and (b) $\mathbb{Q}(\sqrt{2})$?
6. Show that
   (i) $\mathbb{Q}(\sqrt{2}, \sqrt[3]{3}) = \mathbb{Q}(\sqrt{2} + \sqrt[3]{3})$ and
   (ii) $[\mathbb{Q}(\sqrt{2}, \sqrt[3]{3}) : \mathbb{Q}] = 6$.
   (iii) Find the minimum polynomial of $\sqrt{2} + \sqrt[3]{3}$ over $\mathbb{Q}$.
7. Let α be an algebraic number. Show that $n\alpha$ is an algebraic *integer* for some $n \in \mathbb{Z}^{>0}$.
8. Let S be a subring of a field K and suppose that there are α and $\beta \in S \setminus \{0\}$ such that
   (i) $\alpha/\beta \notin S$, yet
   (ii) α/β is a root of a monic polynomial in $S[X]$.
   Show that S is not a UFD.
9. Factorize 24 and $5 + 3\sqrt{-7}$ as products of irreducible elements
   (i) in $\mathbb{Z}\left[\frac{1+\sqrt{-7}}{2}\right]$ and
   (ii) in $\mathbb{Z}[\sqrt{-7}]$.
10. Show that (i) $\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$, (ii) $\mathbb{Z}\left[\frac{1+\sqrt{-11}}{2}\right]$ and (iii) $\mathbb{Z}[\sqrt{7}]$ are Euclidean domains.
11. Show that $\mathbb{Z}\left[\frac{1+\sqrt{D}}{2}\right]$ is not a UFD for $D \in \mathbb{Z}$ with $D < -7$ and $D \equiv 1 \mod 8$.
12. Show that an algebraic integer α is a unit if and only if its norm $\mathrm{N}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha)$ is equal to ± 1.