

Chapter 1

'Begin at the beginning,' the King said, gravely, 'and go on till you come to the end; then stop.'

Lewis Carroll, *Alice in Wonderland*, ch. 12

It is hard if I cannot start some game on these lone heaths.

William Hazlitt, *On Going a Journey*

Whose Game?

Who's game for an easy pencil-and-paper (or chalk-and-blackboard) game?

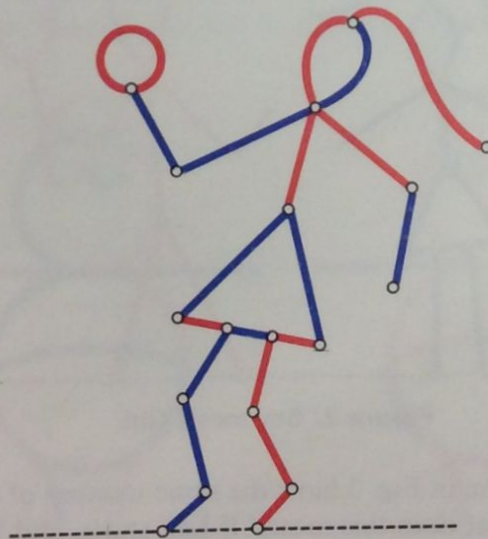


Figure 1. A Blue-Red Hackenbush Picture.

BLUE-RED HACKENBUSH

(or Red-Blue Hackenbush) is played with a picture such as that of Fig. 1. We shall call the two players **Left** and **Right**. Left moves by deleting any blue edge, together with any edges that are no longer connected to the ground (which is the dotted line in the figure), and Right moves by deleting a Red edge in a similar way. (Play it on a blackboard if you can, because it's easier to rub the edges out.) Quite soon, one of the players will find he can't move because there are no edges of his color in what remains of the picture, and whoever is first trapped in this way is the loser. You must make sure that doesn't happen to you!

Well, what can you do about it? Perhaps it would be a good idea to sit back and watch a game first, to make quite sure you understand the rules of the game before playing it with the professionals, so let's watch the effect of a few simple moves. Left might move first and rub out the girl's left foot. This would leave the rest of her left leg dangling rather lamely, but no other edges would actually disappear because every edge of the girl is still connected to the ground through her right leg. But Right at his next move could remove the girl completely, if he so wished, by rubbing out her right foot. Or Left could instead have used his first move to remove the girl's upper arm, when the rest of her arm and the apple would also disappear. So now you really understand the rules, and want to start winning. We think Fig. 1 might be a bit hard for you just yet, so let's look at Fig. 2, in which the blue and red edges are separated into parts that can't interact. Plainly the girl belongs to Left, in some sense, and the boy to Right, and the two players will alternately delete edges of their two people. Since the girl has more edges, Left can survive longer than Right, and can therefore win no matter who starts. In fact, since the girl has 14 edges to the boy's 11, Left ends with at least $14 - 11 = 3$ spare moves, if he chops from the top downwards, and Right can hold him down to this in a similar way.

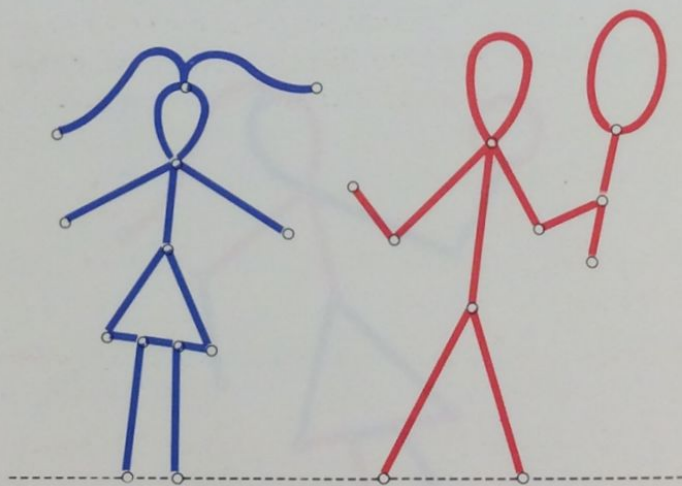


Figure 2. Boy meets Girl.

Tweedledum and Tweedledee in Fig. 3 have the same number of edges each, so that Left is $19 - 19 = 0$ moves ahead. What does this mean? If Left starts, and both players play sensibly from the top downwards, the moves will alternate Left, Right, Left, Right, until each player has made 19 moves, and it will be Left's turn to move when no edge remains. So if Left starts, Left will lose, and similarly if Right starts, Right will lose. So in this **zero position**, whoever starts loses.

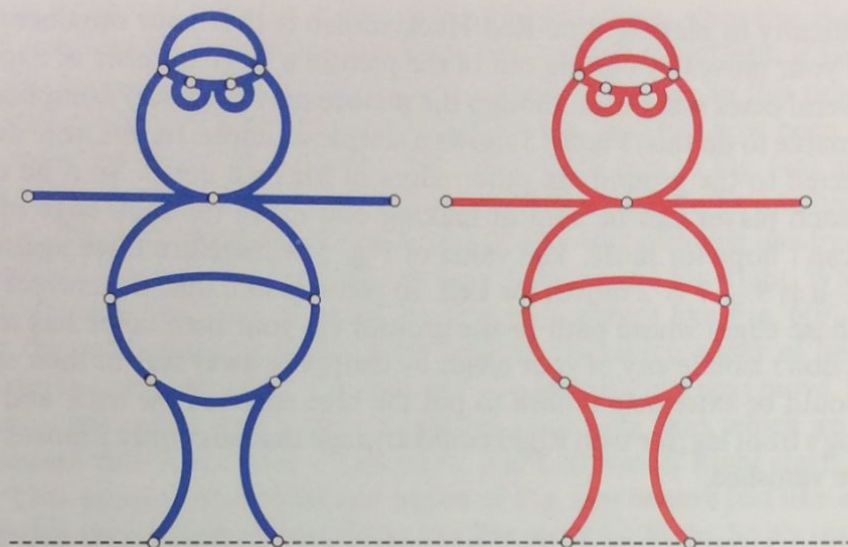


Figure 3. Tweedledum and Tweedledee, about to have a Battle.

THE TWEEDLEDUM AND TWEEDLEDEE ARGUMENT

In Fig. 4, we have swapped a few edges about so that Tweedledum and Tweedledee both have some edges of each color. But since we turn the new Dum into the new Dee exactly by interchanging blue with red, neither player seems to have any advantage. Is Fig. 4 still a zero position in the same sense that whoever starts loses? Yes, for the player second to move can copy any of his opponent's moves by simply chopping the corresponding edge from the other twin. If he does this throughout the game, he is sure to win, because he can never be without an available move. We shall often find games for which an argument like this gives a good strategy for one of the two players—we shall call it the **Tweedledum and Tweedledee Argument** (or **Strategy**) from now on.

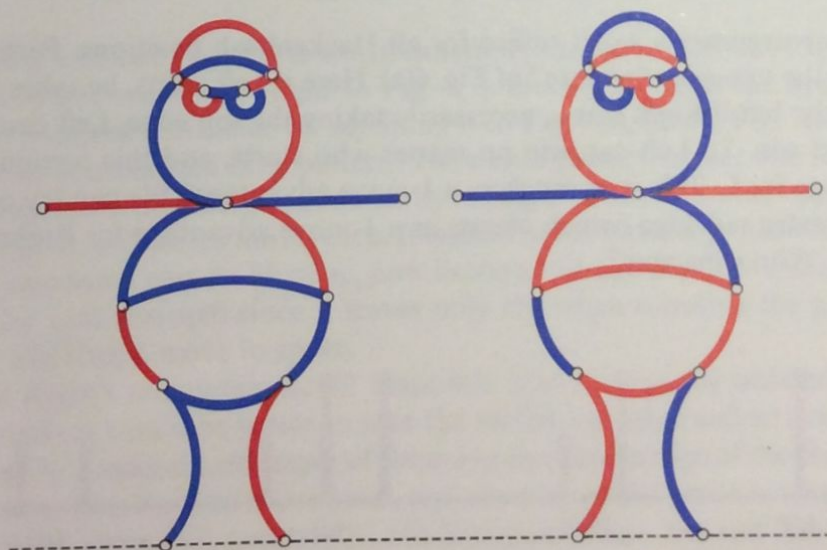


Figure 4. After their first Battle: Ready for the Next?

The main difficulty in playing Blue-Red Hackenbush is that your opponent might contrive to steal some of your moves by cutting out of the picture a large number of edges of your color. But there are several cases when even though the picture may look very complicated, you can be sure he will be unable to do this. Figure 5 shows a simple example. In this little dog, each player's edges are connected to the ground via other edges of his own color. So if he chops these in a suitable order, each player can be sure of making one move for each edge of his own color, and plainly he can't hope for more. The value of Fig. 5 is therefore once again determined by counting edges—it is $9 - 7 = 2$ moves for Left. In pictures like this, the correct chopping order is to take first those edges whose path to the ground via your own color has most edges—this makes sure you don't isolate any of your edges by chopping away any of their supporters. Thus in Fig. 5 Left would be extremely foolish to put the blue edges of the neck and head at risk by removing the dog's front leg; for then Right could arrange that after only 2 moves the 5 blue edges here would have vanished.

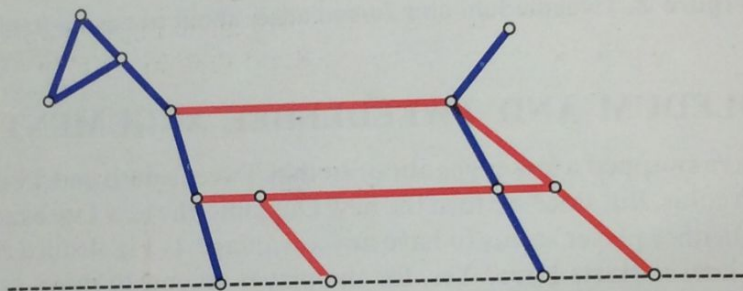


Figure 5. A Dog with Leftward Leanings.

HOW CAN YOU HAVE HALF A MOVE?

But these easy arguments won't suffice for all Hackenbush positions. Perhaps the simplest case of failure is the two-edge "picture" of Fig. 6(a). Here if Left starts, he takes the bottom edge and wins instantly, but if Right starts, necessarily taking the top edge, Left can still remove the bottom edge and win. So Left can win no matter who starts, and this certainly sounds like a positive advantage for Left. Is it as much as a 1-move advantage? We can try counterbalancing it by putting an extra red edge (which counts as a 1-move advantage for Right) on the ground, getting Fig. 6(b). Who wins now?

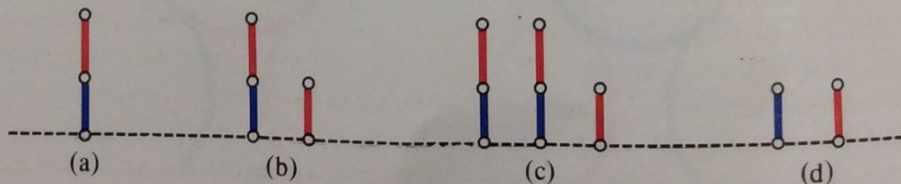


Figure 6. What do we mean by Half a Move?

If Right starts, he should take the higher of his two red edges, since this is clearly in danger. Then when Left removes his only blue edge, Right can still move and win. If Left starts, his only possible move still leaves Right a free edge, and so Right still wins. So this time, it is Right that wins, whoever starts, and Left's positive advantage of Fig. 6(a) has now been overwhelmed by adding the free move for Right. We can say that Left's advantage in Fig. 6(a), although positive, was strictly less than an advantage of one free move. Will it perhaps be one-half of a move?

We test this in Fig. 6(c), made up of *two* copies of Fig. 6(a) with just *one* free move for Right added, since if we are correct $\frac{1}{2} + \frac{1}{2}$ for Left will exactly balance 1 for Right. Who wins Fig. 6(c)? Left has essentially only one kind of move, leading to a picture like Fig. 6(b), which we know Right wins. On the other hand, if Right starts sensibly by taking either of his two threatened edges, Left will move to a picture like Fig. 6(d) and win after Right's next move. If Right has used up his free move at the outset, Left's reply would take us to Fig. 6(a), which we know he wins.

We've just shown that Right wins if Left starts and Left wins if Right starts, so that Fig. 6(c) is a zero game. This seems to show that *two* copies of Fig. 6(a) behave just like *one* free move for Left, in that together they exactly counterbalance a free move for Right. So it's really quite sensible to regard Fig. 6(a) as being a half-move's advantage for Left.

Putting Right's red edge partly under Left's control made Fig. 6(a) worse for him than Fig. 6(d). So perhaps Fig. 7(a) should be worth less to Right than Fig. 7(b) in which Right's edge is threatened by only one of Left's?

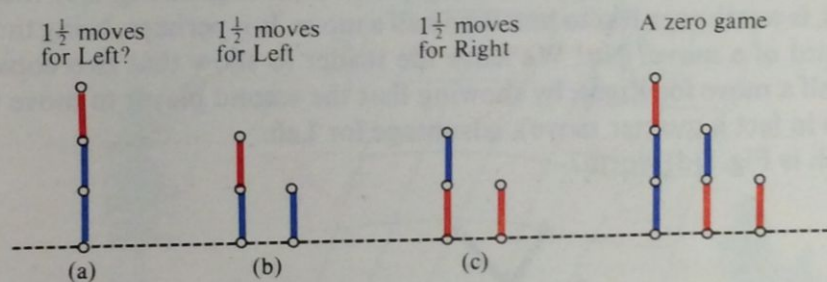


Figure 7. Is Right's Edge even more under Left's Control?

We are asking whether Fig. 7(a) is worth exactly $1\frac{1}{2}$ moves to Left like Fig. 7(b). We can test this by adding $1\frac{1}{2}$ free moves for Right to Fig. 7(a). Since Fig. 7(c) is the opposite of Fig. 7(b), we produce the required allowance by adjoining it to Fig. 7(a), giving Fig. 7(d).

Who wins this complicated little pattern? Here each player has just one risky edge partly in control of his opponent, and if a player starts by taking his risky edge, his opponent can remove the other, leaving two unfettered moves each. If instead he takes the edge just below his opponent's risky edge, the opponent can do likewise, now leaving just one free move each. The only other starting move for Left is stupid since it leaves only red edges touching the ground and indeed Right can now win with a move to spare.

What about Right's remaining move? Since this is to remove the isolated red edge, it *must* be stupid, for surely it would be better to take the middle red edge and so demolish a blue edge at the same time? And indeed Left's reply of chopping the middle edge of the chain of three proves perfectly adequate. So *every* first move loses, and once again the game is what we called a zero game. This seems to show that contrary to our first guess, Figs. 7(a) and 7(b) confer exactly the same advantage upon Left, namely one and a half free moves.

... AND QUARTER MOVES?

In Fig. 8(a), Right's topmost edge is partly under Left's control, but also partly under Right's as well, so it should perhaps be worth more to him than his middle one? Since we found that the middle edge was worth half a move to Right, the pair of red edges collectively would then be worth at least a whole move to him, counteracting Left's single edge. So maybe Right has the advantage here?

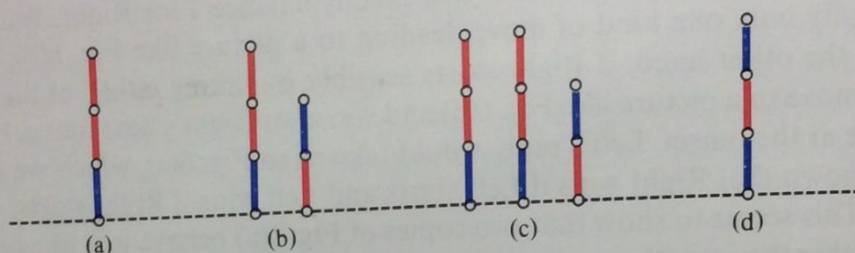


Figure 8. Are Right's Edges worth more than Left's?

This naive opinion is dispelled as soon as play starts, for Left's only move wins the game as soon as he makes it, showing that Fig. 8(a) gives a positive advantage to Left. But when we adjoin half a move for Right as in Fig. 8(b). Right can win playing first, by removing the topmost edge, or playing second, by removing the highest red edge then remaining. So Fig. 8(a), though a positive advantage for Left, is worth even less to him than half a move. Is it perhaps, being three edges high, worth just one-third of a move? No! We leave the reader to show that two copies of Fig. 8(a) exactly balance half a move for Right, by showing that the second player to move wins Fig. 8(c), so that Fig. 8(a) is in fact a quarter move's advantage for Left.

And how much is Fig. 8(d) worth?

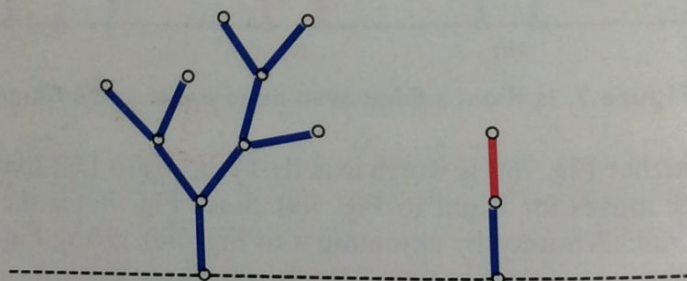


Figure 9. A Hackenbush Position worth $9\frac{1}{2}$.

Figure 9 shows a Hackenbush position of value $9\frac{1}{2}$, since the tree has value 9, and the rest value $\frac{1}{2}$. What are the moves here? Right has a unique red edge, and so a unique move, to a position of value $9 + 1 = 10$, but Left can move either at the top of the tree, leaving $8\frac{1}{2}$, or by removing the $\frac{1}{2}$ completely, which is a better move, since it leaves value 9. Since Left's best move is to value 9, and Right's to 10, we express this by writing the equation

$$\{9|10\} = 9\frac{1}{2} \quad (\text{"9 slash 10 equals } 9\frac{1}{2}\text{"}).$$

In a similar way, we have the more general equation

$$\{n|n+1\} = n + \frac{1}{2},$$

of which the simplest case is

$$\{0|1\} = \frac{1}{2},$$

with which we began. We also have the simpler equation

$$\{n|\} = n+1$$

for each $n = 0, 1, 2, \dots$, for if Left has just $n+1$ free moves, he can move so as to leave just n free moves, while Right cannot move at all. The very simplest equation of this type is

$$\{\mid\} = 0$$

which expresses the fact that if neither player has a legal move the game has zero value.

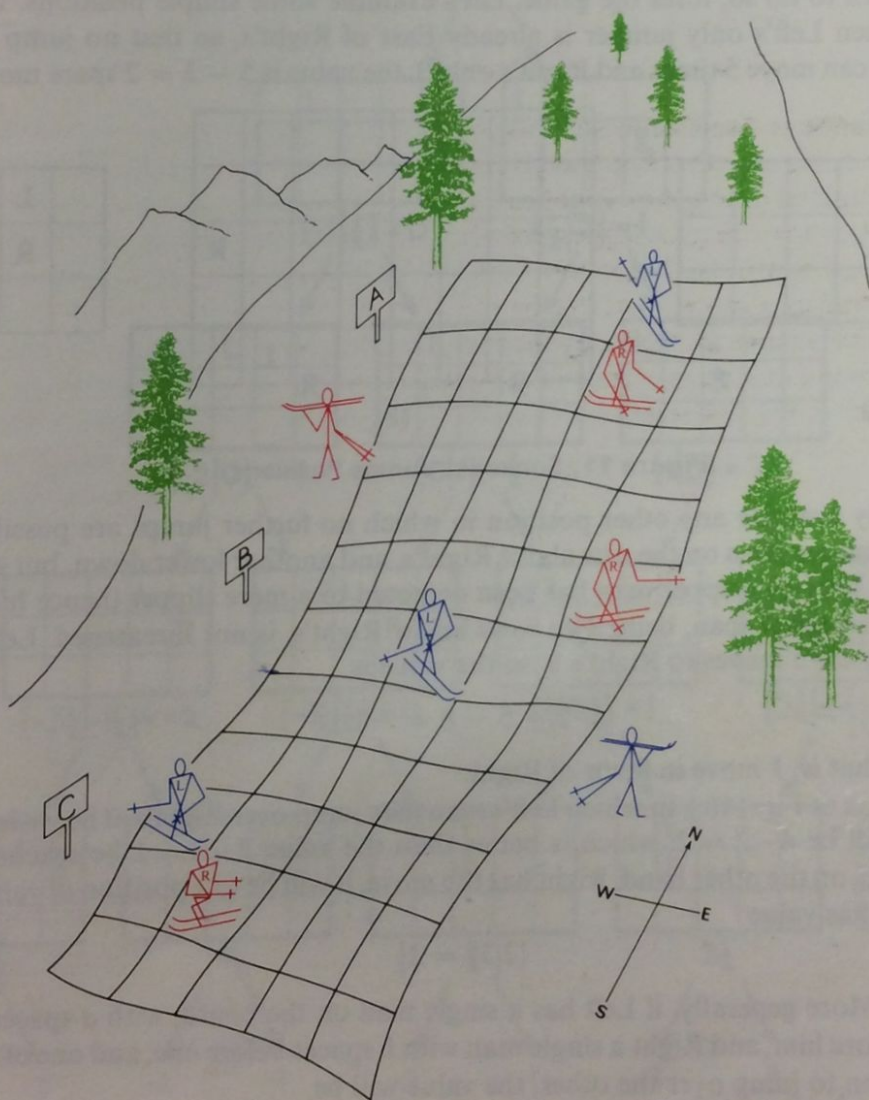


Figure 10. A Game of Ski-Jumps.

SKI-JUMPS FOR BEGINNERS

Figure 10 shows a ski-slope with some skiers in the pay of Left and Right, about to participate in our next game. In a single move, Left may move any skier a square or more Eastwards, or Right any one of his, Westwards, provided there is no other active skier in the way. Such a move may take the skier off the slope; in this case he takes no further part in the game. No two skiers may occupy the same square of the slope. Alternatively a skier on the square immediately *above* one containing a skier of the opposing team, may jump over him onto the square immediately below, provided this is empty. A man jumped over is so humiliated that he will never jump over anyone else—in fact he is demoted from being a *jumper* to an ordinary skier, or *slipper*!

No other kind of move is permitted in this game, so that when all the skiers belonging to one of the players have left the ski-slope, that player cannot move, and a player who cannot move when it is his turn to do so, loses the game. Let's examine some simple positions. Figure 11(a) shows a case when Left's only jumper is already East of Right's, so that no jump is possible. Since Left's man can move 5 times and Right's only 3, the value is $5 - 3 = 2$ spare moves for Left.

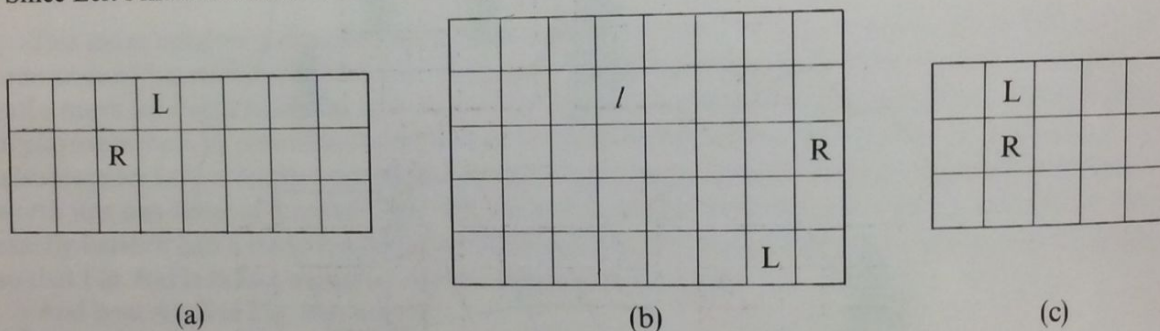


Figure 11. Some Ski-Jumps Positions.

We can similarly evaluate any other position in which no further jumps are possible. Thus in Fig. 11(b) Left has one man on the row above Right's, and another lower down, but still no jump will be possible, for Left's upper man has been demoted to a mere slipper (hence his lower case name, *l*), while his lower man, being two rows below Right's, is not threatened. Left's two men have collectively $2 + 5$ moves to Right's 8, so the value is

$$2 + 5 - 8 = -1$$

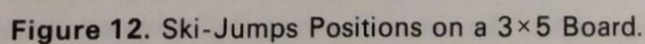
moves to Left, that is, 1 move in favor of Right.

Now let's look at Fig. 11(c), in which Left's man may jump over Right's, if he wishes. If he does so, the value will be $4 - 2 = 2$, which is better than the value $3 - 2 = 1$ he reaches by sliding one place East. If, on the other hand, Right has the move, it will be to a position of value $4 - 1 = 3$. So the position has value

$$\{2|3\} = 2\frac{1}{2}$$

moves to Left. More generally, if Left has a single man on the board, with a spaces (and hence $a + 1$ moves) before him, and Right a single man with b spaces before him, and one of the two men is now in position to jump over the other, the value will be

$$a - b + \frac{1}{2} \quad \text{or} \quad a - b - \frac{1}{2}$$



DON'T JUST TAKE THE AVERAGE!

The positions in the bottom two lines are those we have just analyzed, in which the jump is imminent or past. From any of the other positions, Left has just one move, to the position diagonally down and left from the given one, and Right similarly has a unique move, to the position diagonally rightwards. We have appended the values of all these positions, measured as usual in terms of free moves for Left, and there are some surprises. We have evaluated the rightmost position on the fourth row as

$$\{2\frac{1}{2}|4\frac{1}{2}\} = 3.$$

Surely this is wrong? Anyone can see that the average of $2\frac{1}{2}$ and $4\frac{1}{2}$ is $3\frac{1}{2}$, can't they?

Well yes, of course $3\frac{1}{2}$ is the *average*, but it turns out that the *value* is 3, nevertheless. You *don't* simply evaluate positions in games by averaging Left's and Right's best moves! Exactly how you *do* evaluate them is the main topic of this book, so we can't reveal it all at once. But we *will* explain why the second position on the fourth row has value 0, rather than $-\frac{1}{2}$, as might have been expected.

If the value were $-\frac{1}{2}$ or any other negative number, Right ought to win, no matter who starts. But in this position, if Right starts, Left can jump him immediately, after which they will each have just two moves, and Right will exhaust his before Left. In fact neither player can win this position if he starts, for if Left moves first, Right can slip leftwards past him to avoid the threatened jump, leaving Left with but one move to Right's two. A position in which the first player to move loses always has value zero.

We could have seen the same thing from the symbolic expression $\{-1\frac{1}{2}|\frac{1}{2}\}$ for the position, for since Left's best option has negative value he cannot move to it and win (if Right plays well), and since Right's best move is positive, he cannot move to win either. It does not matter exactly *how much* each of these moves favors the second player, so long as he is assured of a win. So for exactly the same reason, the game $\{-\frac{1}{2}|17\} = 0$, since the starter loses, even though 17 is much further above 0 than $-\frac{1}{2}$ is below it.

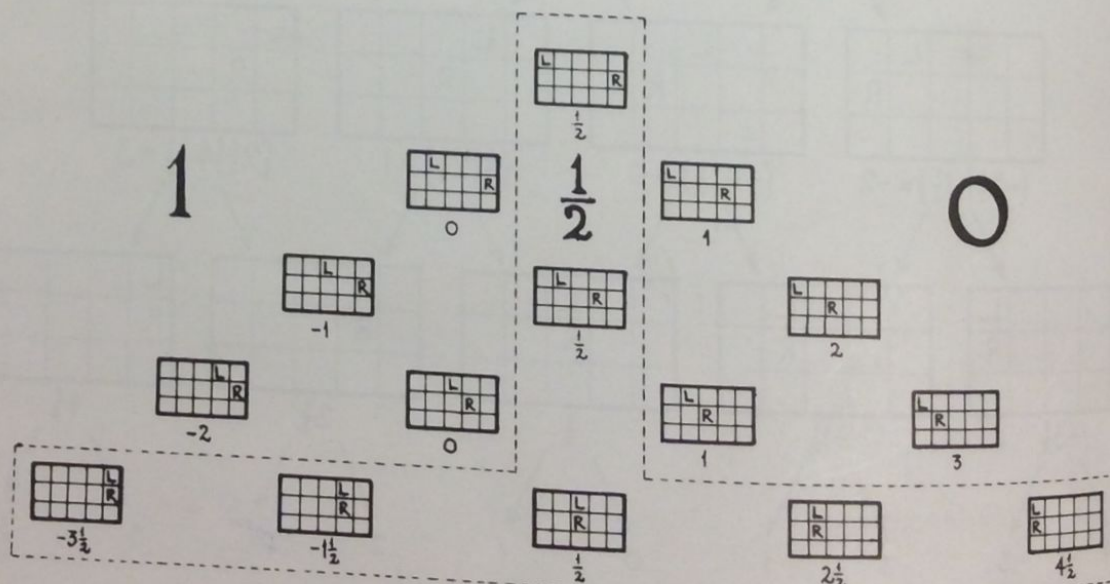


Figure 13. The Value of a Potential Jump is 1, $\frac{1}{2}$, or 0.

WHAT IS A JUMP WORTH?

We do not explain the other values here. The reader can verify for example, that $\{2\frac{1}{2}|4\frac{1}{2}\} = 3$, by playing the position $\{2\frac{1}{2}|4\frac{1}{2}\}$ together with an allowance of just 3 moves for Right, and checking that the starter loses. We can summarize the results of Fig. 12 as follows: a potential jump is worth half a move only if it is either imminent or the two players are the same distance from the central column. It is worth a whole move (just as if it were a sure thing) if the potential jumper is nearer to the central column than the jumpee and worth nothing (just as if it were impossible) otherwise (Fig. 13).

We can now predict who will win the more complicated Ski-Jumps positions of Fig. 10. Because the pairs of rows A,B,C are so far apart, moves made by the skiers in one of these pairs will not affect the play in others, so we can just add up the values for the three pairs A,B,C (Fig. 14).

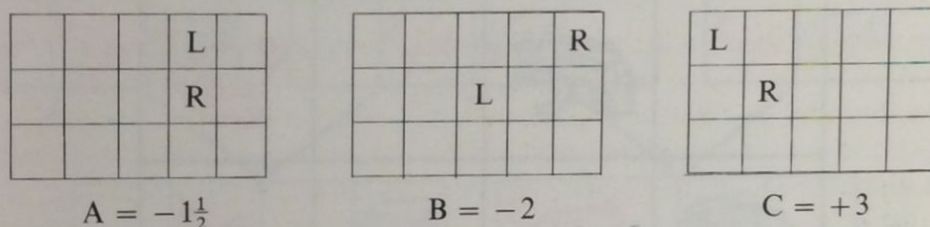


Figure 14. Values of Ski-Jumps Positions in Figure 10.

The values for A and C can be read off from Fig. 12 as $-1\frac{1}{2}$ and $+3$, while that for B is

L				
		R		

(value 2) with the roles of Left and Right reversed, and so has value -2 .

The total value is therefore

$$-1\frac{1}{2} - 2 + 3 = -\frac{1}{2},$$

and so Right is half a move ahead and should be able to win, no matter who starts. It will be harder for him if he starts himself, since then he must use up a move. What move should he make? His three choices are from

$$-1\frac{1}{2} \text{ to } -1 \text{ (in A),} \quad -2 \text{ to } -1 \text{ (in B),} \quad \text{and} \quad 3 \text{ to } 4\frac{1}{2} \text{ (in C)}$$

which lose him

$$\frac{1}{2},$$

$$1,$$

$$1\frac{1}{2}$$

moves respectively. So he can only guarantee to retain his win if he moves his A man, so as to avoid the otherwise imminent jump by Left.

TOADS-AND-FROGS

Left has trained a number of Toads (*Bufo vulgaris*) and Right a number of Frogs (*Rana pipiens*) to play the following game. Each player may persuade one of his creatures either to move one square or to jump over an opposing creature, onto an empty square. Toads move only Eastward, Frogs only to the West (toads to, frogs fro). The game is to be played according to the normal play rule that a player unable to move loses. Verify the values in Fig. 16. Who wins Fig. 15 and by how much?

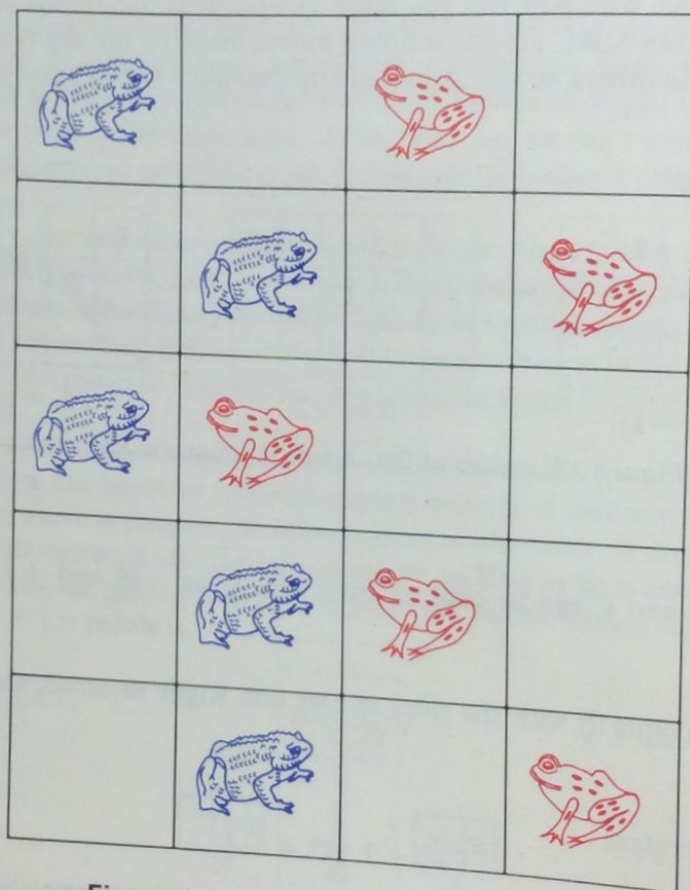


Figure 15. A Game of Toads-and-Frogs.

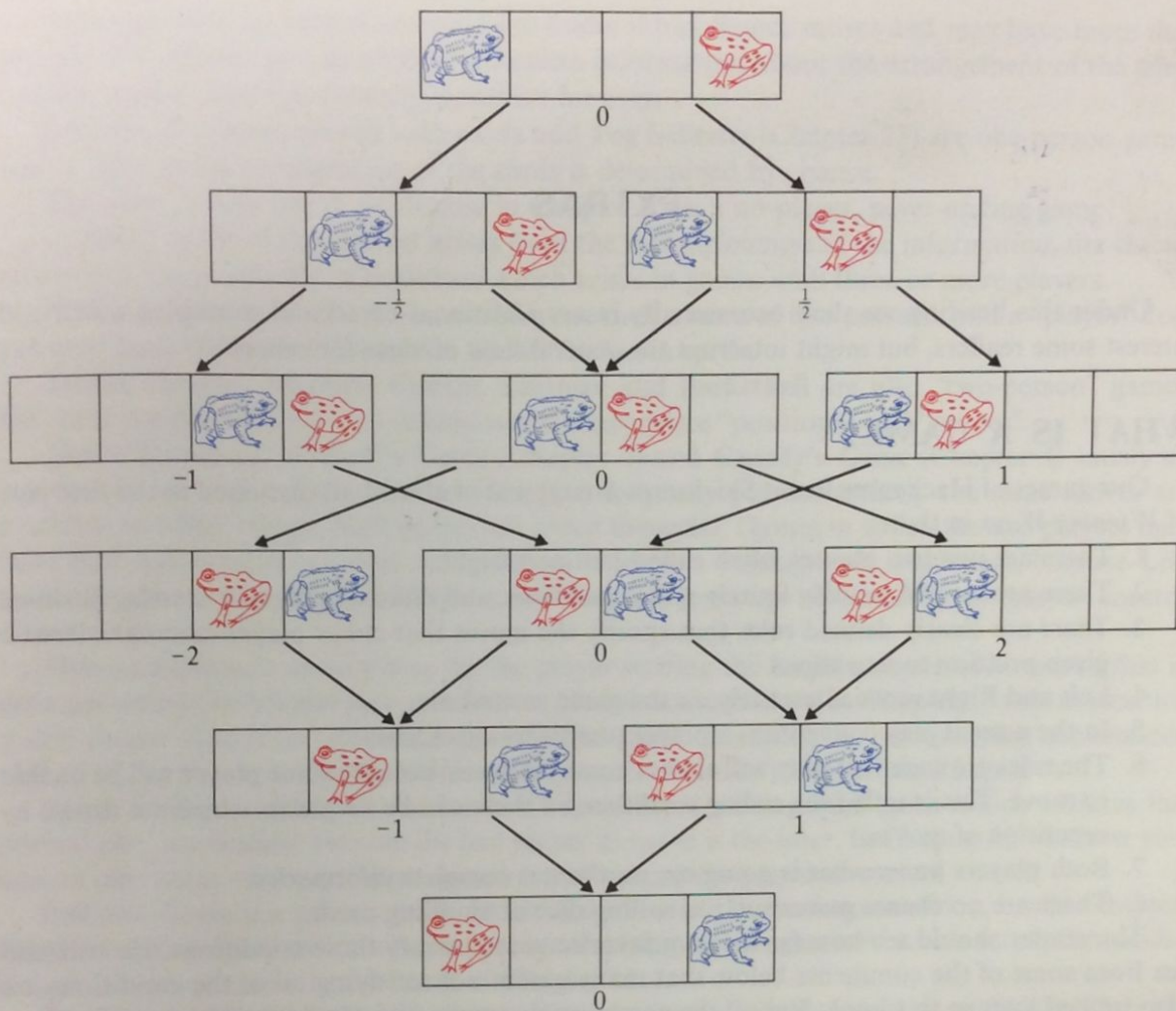


Figure 16. Values of Positions in 4-place Toads-and-Frogs.

DO OUR METHODS WORK?

Several questions will have entered the reader's mind. Can we really evaluate positions by adding up numbers of moves advantage, even when they are fractions? Is it wise to regard all positions in which the starter loses as having zero value? The answers are yes. For the pragmatic reader perhaps the best proof of this pudding will be in the eating—if he works out who has more moves advantage this way he'll be sure to pick the winner. Mathematical unbelievers must await our later discussion.

EXTRAS

Under this heading we shall occasionally insert additional detail and examples which will interest some readers, but might interrupt the general flow of ideas for others.

WHAT IS A GAME?

Our games of Hackenbush and Ski-Jumps are typical of almost all discussed in the first part of *Winning Ways* in that:

1. There are just two players, often called Left and Right.
2. There are several, usually finitely many, **positions**, and often a particular **starting position**.
3. There are clearly defined **rules** that specify the **moves** that either player can make from a given position to its **options**.
4. Left and Right move alternately, in the game as a whole.
5. In the **normal play** convention a player unable to move **loses**.
6. The rules are such that play will always come to an end because some player will be unable to move. This is called the **ending condition**. So there can be no games which are drawn by repetition of moves.
7. Both players know what is going on, i.e. there is **complete information**.
8. There are no **chance moves** such as rolling dice or shuffling cards.

The reader should see how far his own favorite games satisfy these conditions. He will also see from some of the comments below that many games not satisfying all of the conditions are also treated later in this book. But all the games we *do* treat satisfy 7 and 8.

Tic-Tac-Toe (Noughts-and-Crosses) fails 5. because a player unable to move is not necessarily the loser, since ties are possible. We will give a complete analysis in Chapter 22, and will discuss various generalizations, such as **Go-Moku**.

Chess also fails 5. and contains positions that are *tied* by stalemate (in which the last player does *not* win) and positions that are *drawn* by infinite play (of which perpetual check is a special case). We reserve the complete analysis for a later volume.

The words "tied" and "drawn" are often used interchangeably, though with slight transatlantic differences, for games which are neither won nor lost. We suggest that **drawn** be used for cases when this happens because play is drawn out indefinitely and **tied** for cases when play definitely ends but the rules do not award a win to either player.

Ludo, Snakes-and-Ladders and **Backgammon** all have complete information, but contain chance moves, since they all use dice.

Battleships, Kriegspiel, Three-Finger Morra and **Scissors-Paper-Stone** have no chance moves but the players do not have complete information about the disposition of their opponent's pieces or fingers. In both the finger games, moreover, the players move simultaneously rather than alternately.

Monopoly fails on several counts. Like Ludo, it has chance moves and may have more than two players. The players don't have complete information about the arrangement of the cards and the game could, theoretically, continue for ever.

Solitaire (Patience) played with cards and **Peg Solitaire** (Chapter 23) are one-person games and in the first the arrangement of the cards is determined by chance.

The game of **Life** which we discuss in Chapter 25, is a no-player, never-ending game!

In **Poker** much of the interest arises from the incompleteness of the information, the chance moves and the possibility of **coalitions** which arises in games with three or more players.

Bridge is peculiar in that it has two players, each a team of two persons, and a "player" does not even have complete information about "his" own cards.

Tennis, Hockey, Baseball, Cricket, Lacrosse and **Basketball** are also "two-person" games, but there are difficulties in the definitions of appropriate "positions" and "moves".

Nim (Chapter 2), **Wythoff's Game** (Chapter 3) and **Grundy's Game** (Chapter 4) satisfy all our conditions and indeed a further one, that from any position exactly the same moves are available to either player. Such games are called **impartial**. Games in which the two players may have different options we shall call **partizan**. Red-Blue Hackenbush is partizan because Left may only remove blue edges and Right only red ones; Ski-Jumps because different players control different skiers.

Dots-and-Boxes is usually won by the player scoring the larger number of boxes, so that it does not satisfy the normal play convention. However, we shall see in Chapter 16 that in practice it can almost always be treated as an impartial game, satisfying our normal play convention, part of whose theory is closely related to **Kayles** and **Dawson's Kayles** (see Chapter 4).

Sylver Coinage, which we discuss in Chapter 18, is an impartial game which violates the normal play convention because the last player to move is the *loser*. In Chapter 13 we show you how to play sums of impartial games subject to this **misère play** convention.

Fox-and-Geese is a pursuit game which doesn't satisfy the ending condition, but in Chapter 20 we are able to compare its value with those of other partizan games which *do* satisfy the condition. It is a loopy game in the sense of Chapter 11.

The French Military Hunt and other partizan pursuit games also yield to analysis in Chapter 21.

Go is not analyzed in this book, but provides an interesting example of a "hot" partizan game. Go players might find the thermographic techniques of Chapter 6 useful in their game.

WHEN IS A MOVE GOOD?

We usually call a move "good" if it will win for you, and "bad" if it will not, and throughout most of the book we regard it as sufficient analysis to find any good move, or show that none exists.

But in real life games there are many other criteria for choosing between your various options. If you're *losing*, then all your options are bad in the above sense, but in practice they're not all equal, and you might prefer one that makes the situation too complicated for your opponent to analyze (the **Enough Rope Principle**).

There are even cases where you should prefer a bad move to a good one! Your opponent might be learning how to play a game which you're already familiar with. In this case you'll probably be able to win a few times despite the bad moves you deliberately make so as not to

give your strategy away. Or one move, though theoretically the best, might gain you only a dollar, while another, which theoretically *loses* a dollar, might actually get you a hundred if your opponent fails to find the rather subtle winning reply. And of course you might be a card sharp who's playing badly now so as to win more later when the stakes are raised.

FIGURE 8(d) IS WORTH $\frac{3}{4}$

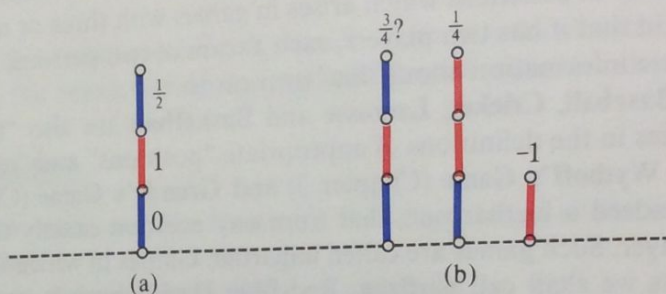


Figure 17. How we can have Three-Quarters of a Move.

The Blue-Red Hackenbush position of Fig. 8(d) may be evaluated as follows. Write against each edge (Fig. 17(a)) the value of the position when that edge is removed. Then the greatest number against a blue edge (here $\frac{1}{2}$) is Left's best option, and the least number against a red edge is Right's. So in the given case we obtain the expression

$$\{\frac{1}{2}|1\}$$

suggesting a value of $\frac{3}{4}$. So if we add $\frac{1}{4}$ and subtract 1 as in Fig. 17(b) we should obtain a zero position. Check that whoever starts loses.

Verify that the Blue-Red Hackenbush positions in Fig. 18 have the indicated values, in terms of moves advantage to Left.

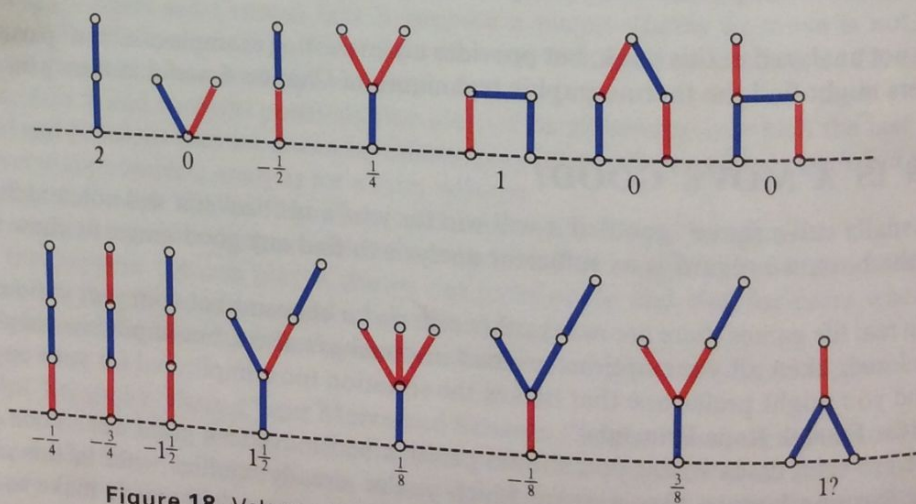


Figure 18. Values of some Blue-Red Hackenbush Positions.



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