NUMBER THEORY CHALLENGE

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Set 1/2/11

Show that

$$2x(x+1) = y(y+1)$$

has infinitely many solutions $(x, y) \in \mathbb{Z}^2$**.**

Rewrite the equation as

$$y^2 + y - (2x^2 + 2x) = 0.$$

Then we have (by the quadratic formula)

$$y = \frac{-1 + \sqrt{1 + 4(2x^2 + 2x)}}{2} = \frac{-1 + \sqrt{8x^2 + 8x + 1}}{2}.$$

So for y to be an integer, it is certainly necessary that $8x^2 + 8x + 1$ is a perfect square. It is also sufficient, as $8x^2 + 8x + 1$ will always be odd — so if it is a perfect square, then its square root will be odd, implying that $-1 + \sqrt{8x^2 + 8x + 1}$ is even and hence that y is an integer.

So we need

$$8x^2 + 8x + 1 = z^2$$

for some integer z. Write $w = 2x + 1$ — then we have

$$z^2 - 2w^2 = -1.$$

This is an equation that we have already shown to have infinitely many solutions. The solutions are parametrised by

$$z = \frac{u^{2r+1} + \tilde{u}^{2r+1}}{2}, \qquad w = \frac{u^{2r+1} - \tilde{u}^{2r+1}}{2\sqrt{2}}$$

for $r \in \mathbb{Z}$, where $u = 1 + \sqrt{2}$ is the fundamental unit of $\mathbb{Z}[\sqrt{2}]$. We have

$$x = \frac{w-1}{2}, \qquad y = \frac{z-1}{2}$$

— so we need to show that z and w are both odd for all r.

We have

$$u^{2r+1} = (1+\sqrt{2})^{2r+1} = 1 + \binom{2r+1}{1}\sqrt{2} + \binom{2r+1}{2}2 + \cdots + \binom{2r+1}{2r}\sqrt{2}^{2r} + \sqrt{2}^{2r+1},$$

$$\tilde{u}^{2r+1} = 1 - \binom{2r+1}{1} + \binom{2r+1}{2}2 - \cdots + \binom{2r+1}{2r}\sqrt{2}^{2r} + \sqrt{2}^{2r+1}$$

— so

$$z = 1 + 2\binom{2r+1}{2} + 4\binom{2r+1}{4} + \cdots + 2^i\binom{2r+1}{2i} + \cdots + 2^r\binom{2r+1}{2r}$$

and

$$w = \binom{2r+1}{1} + 2\binom{2r+1}{3} + 4\binom{2r+1}{5} + \cdots + 2^i\binom{2r+1}{2i+1} + \cdots + 2^r\binom{2r+1}{2r+1},$$

and hence w, z are odd (in each series, every term but the first is even, and the first is odd). So there are infinitely many solutions, and we have a parametrisation

$$x = r + \sum_{i=1}^{r} 2^{i-1}\binom{2r+1}{2i+1}, \quad y = \sum_{i=1}^{r} 2^{i-1}\binom{2r+1}{2i},$$

where $r \in \mathbb{N}$.