BRIEF COMMUNICATIONS

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On the existence of two-dimensional Euler flows satisfying energy-Casimir stability criteria

Djoko Wirosoetisno and Theodore G. Shepherd
Department of Physics, University of Toronto, Toronto M5S 1A7, Canada

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The energy-Casimir stability method, also known as the Arnold stability method, has been widely used in fluid dynamical applications to derive sufficient conditions for nonlinear stability. The most commonly studied system is two-dimensional Euler flow. It is shown that the set of two-dimensional Euler flows satisfying the energy-Casimir stability criteria is empty for two important cases: (i) domains having the topology of the sphere, and (ii) simply-connected bounded domains with zero net vorticity. The results apply to both the first and the second of Arnold's stability theorems. In the spirit of Andrews' theorem, this puts a further limitation on the applicability of the method.
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A central problem in fluid dynamics is the stability of basic flows. Stability theorems are useful for delineating possible regimes of instability and, when they are finite amplitude, can be used to bound the growth of disturbances. A very general method for deriving stability theorems for inviscid flows is the energy-Casimir method, which dates back to Fjørtoft.[1] The method was formalized by Arnold[2] and, crucially, shown to be valid at finite amplitude,[3] and is therefore sometimes known as the Arnold stability method. Essentially, the method establishes stability by showing that the flow in question corresponds to an energy extremum with respect to disturbances constrained by other invariants known as Casimirs. For two-dimensional Euler flow, which is the prototype system for this method, the Casimir invariants are spatial integrals of arbitrary functions of the vorticity scalar, and their conservation reflects the material conservation of vorticity.

The success of the energy-Casimir stability method has been considerably limited, however, by the fact, first observed by Andrews,[4] that for a domain with a continuous symmetry, any Arnold-stable flow (that is, a flow whose nonlinear stability can be established using this method) must also respect that symmetry. In that case, the momentum invariant associated with the symmetry is also available and, indeed, usually is crucial to proving stability with what might then be called the energy-Casimir-momentum method.[5,6]

There remains the possibility that flows without continuous symmetries (e.g., a vortex street) might be provably stable in asymmetric domains. In any case, domains with symmetry are nongeneric. It is therefore of interest to determine the range of applicability of the energy-Casimir stability method for general domains.

This question is re-examined in this paper in the context of two-dimensional Euler flow. It is shown that on a deformed sphere, no flows exist that satisfy the energy-Casimir stability criteria. This extends a previous result[7] for Arnold's first stability theorem alone. It follows that for generic such domains (without symmetry), no flows are provably stable using the Arnold method. Such flows can only exist in domains with a continuous symmetry, where a momentum invariant is available, and must furthermore themselves satisfy that symmetry. Indeed, the only known Arnold-stability results are of this sort and essentially reduce to a nonlinear version of Rayleigh's inflection-point theorem.[8,9]

It is also shown that for simply-connected bounded domains with zero net vorticity, no flows exist that satisfy the energy-Casimir stability criteria. This suggests that there is an error in the derivation[10,11] of a nonlinear stability criterion for a street of counter-rotating vortices[12] that was presumed to be contained within a finite box. A close inspection of the derivation reveals that an incorrect boundary condition was used: The counter-rotating vortex solution cannot be confined to a finite box without violating the no-normal-flow boundary condition. The present result shows that no modification of the boundary could lead to a stability result so long as at least two counter-rotating vortices were included.

The analysis in this paper is for completely general domain geometries, and is therefore presented in a manner that is free of any particular coordinate system. All that is assumed is that the domain be a smooth manifold M with a metric g. In local coordinates, the two-dimensional Euler equation can be written

$$\frac{\partial\omega}{\partial t}-\frac{\partial\omega}{\partial x^1}\frac{\partial\psi}{\partial x^2}+\frac{\partial\omega}{\partial x^2}\frac{\partial\psi}{\partial x^1}=0, \tag{1}$$

where the vorticity ω and stream function ψ scalars are related by

$$\omega=\Delta\psi=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(g^{ij}\sqrt{g}\frac{\partial\psi}{\partial x^j}\right) \tag{2}$$

where $\sqrt{g}:=\sqrt{\det g_{ij}}$. The velocity is related to the vorticity and stream function by

$$\omega=\frac{\epsilon_{ij}}{\sqrt{g}}\frac{\partial}{\partial x^i}(g_{jk}u^k),\quad u^i=-\sqrt{g}g^{ij}\epsilon_{jk}g^{kl}\frac{\partial\psi}{\partial x^l}, \tag{3}$$

where $\epsilon_{12}=-\epsilon_{21}=1$, $\epsilon_{11}=\epsilon_{22}=0$. For a steady flow, it follows from (1) that $\partial(\psi,\omega)=0$, and thus the stream function can be written as a (possibly multivalued) function of the vorticity,

$$\psi=\Phi(\omega). \tag{4}$$

Equation (1) is considered in one of two topologies: either the domain is topologically equivalent to the sphere, or it has a boundary and is simply connected in which case one must append the boundary condition of no normal flow. In the latter case, it is assumed that the domain integrated vorticity is zero. Then in either case, for the stream function ψ, scalar φ, the eigenvalues of the problem

$$\Delta\psi+\lambda\psi=0 \tag{5}$$

are non-negative:

$$0<\lambda_0<\lambda_1<\cdots<+\infty \tag{6}$$

($\lambda=0$ is also an eigenvalue for $\psi=$const).

The system in question conserves the energy

$$\mathcal{E}[u]=\frac{1}{2}\int_M\langle u,u\rangle\mu, \tag{7}$$

where $\langle\cdot,\cdot\rangle$ is the inner product induced by the metric g, and $\mu:=\sqrt{g}dx^1\wedge dx^2$ is the volume form on M. It also conserves the Casimir invariants

$$\mathcal{C}[u]=\int_M C(\omega)\mu, \tag{8}$$

where $C(\cdot)$ is an arbitrary function of its argument. In general, two-dimensional Euler flow conserves the circulation along each connected portion of the boundary, but in the case of a simply-connected domain, the circulation is equal to the domain-integrated vorticity, which is a Casimir and is by hypothesis zero in this study. Thus in both of the cases considered here, $\mathcal{E}$ may alternatively be written as

$$\mathcal{E}[u]=-\frac{1}{2}\int_M\omega\psi\mu. \tag{9}$$

The energy and Casimir invariants are robust with respect to perturbations to M, in the sense that they survive when smooth deformations are made to M without changing its topology. This is, of course, not true of momentum invariants (see discussion below).

To establish nonlinear stability using Arnold's method, one constructs an invariant that is sign-definite in the disturbance quantity when the latter is sufficiently small.[13,14] In what follows, we show that the use of the energy and Casimir invariants alone cannot give a sign-definite invariant for the domains under consideration.

We consider the stability of the time-independent basic flow U, with vorticity Ω and stream function Ψ. Since the basic flow is stationary, by (1) we have $\partial(\Psi,\Omega)=0$, so we can define the function $\Phi(\cdot)$ by $\Psi:=\Phi(\Omega)$ and the function $G(\cdot)$ by

$$G(x):=\int^x\Phi(s)ds. \tag{10}$$

Consider a perturbation u to the basic flow, so that the total velocity is $U+u$; the total vorticity and stream function are then $\Omega+\omega$ and $\Psi+\psi$.

We construct the invariant functional

$$\mathcal{A}[u;U]:=\mathcal{E}[U+u]-\mathcal{E}[U]+\int_M\{G(\Omega+\omega)-G(\Omega)\}\mu. \tag{11}$$

Since $\mathcal{A}$ is the sum of energy and Casimir invariants, less constants, it is conserved by the dynamics. Using the form (9), we can write $\mathcal{A}$ as

$$\mathcal{A}[u;U]=\int_M\left\{-\frac{1}{2}\omega\psi+G(\Omega+\omega)-G(\Omega)-\omega G'(\Omega)\right\}\mu, \tag{12}$$

where $G'(\cdot)=\Phi(\cdot)$. In order to prove the stability of the basic state U, we need to show that this quantity is sign-definite for sufficiently small u. To this end, consider the terms involving $G(\cdot)$ in the integral in Eq. (12). We can write them as

$$G(\Omega+\omega)-G(\Omega)-\omega G'(\Omega)=\int_0^\omega[\Phi(\Omega+\xi)-\Phi(\Omega)]d\xi, \tag{13}$$

and for small ω, Eq. (13) is approximately given by $\frac{1}{2}\omega^2\Phi'(\Omega)$.

From (9) we have

$$\mathcal{E}[U]=-\frac{1}{2}\int_M\Omega\Psi\mu. \tag{14}$$

When M is topologically equivalent to the sphere, Stokes' theorem requires that

$$\int_M\Omega\mu=0. \tag{15}$$

In the simply-connected bounded case, this holds by hypothesis. Either way, it implies that Ω must take the value of zero somewhere, and that the value of $\mathcal{E}$ is unchanged by the addition of a constant to Ψ. Since Ψ is determined only up to an additive constant in any case, we may set $\Psi=0$ when $\Omega=0$, or in other words, $\Phi(0)\equiv 0$.

Arnold's first and second stability theorems correspond to cases where (12) is, respectively, positive definite or negative definite, representing energy-Casimir minima or maxima. The integral of the first term, the disturbance kinetic energy, is obviously positive definite by (7) and (9). The integral of the three remaining terms can be of either sign, depending on the sign of $G''(\Omega)=\Phi'(\Omega)$. In order for (12) to be positive definite, these remaining terms have to be positive definite also, which requires $\Phi'(\Omega)>0$ for all Ω. But this is impossible, as can be seen from the following: Since $\mathcal{E}[U]>0$, $\Omega\Psi=\Omega\Phi(\Omega)$ must be negative for some value of Ω, but this is precluded by our hypothesis that $\Phi(0)=0$ and $\Phi'(\Omega)>0$ for all relevant Ω. Thus, the method of *Arnold's first theorem* does not apply to the domains under consideration.[15]

The other possibility is for (12) to be negative definite. This may be possible if the integral of the last three terms is less than the integral of $\frac{1}{2}\omega\psi$, which for an infinitesimal disturbance translates to the condition

$$0<-\int_M \omega\psi\mu<-\int_M \omega^2\Phi'(\Omega)\mu. \tag{16}$$

Since $\omega=\Delta\psi$, condition (16) is assured if $\Phi'(\Omega)<-\lambda_0^{-1}$ for all Ω, where λ_0 is the smallest positive eigenvalue of the Laplacian operator. This is the stability criterion of Arnold's second theorem. But this situation is also impossible, as the following makes clear: Since $\Omega=\Delta\Psi$, the same Poincaré inequality provides the estimate

$$0<-\int_M \Omega\Psi\mu\leqslant\frac{1}{\lambda_0}\int_M \Omega^2\mu, \tag{17}$$

which implies that somewhere in the domain we must have $\lambda_0\Psi\geqslant-\Omega$. But this last requirement means that $\Phi'(\Omega)\geqslant-\lambda_0^{-1}$ for some relevant Ω [using the mean value theorem, with $\Phi(0)=0$], which is precluded by our hypothesis that $\Phi'(\Omega)<-\lambda_0^{-1}$ for all relevant Ω. Thus, we also find that *Arnold's second theorem* fails to apply to this problem for the domains under consideration.

The results derived above show that the energy-Casimir stability method is inapplicable to two-dimensional Euler flows, in the sense that no flows exist satisfying the energy-Casimir stability criteria, in two important cases: When the domain is topologically equivalent to the sphere, and when the domain is bounded and simply connected with zero net vorticity. Any flows that do satisfy the criteria must therefore be in different flow topologies: examples include a sinh jet in an unbounded domain,[2] a sinusoidal jet in a bounded doubly-connected domain,[2] and a Kelvin–Stuart cat's-eye within a bounded simply-connected domain with non-zero net vorticity.[16] (Note that the first flow is an example of Arnold's first stability theorem, while the second and third flows are examples of Arnold's second stability theorem.)

When the domain has a continuous symmetry, there will be a momentum invariant (i.e., a linear functional of the velocity field) associated with this symmetry.[17] These momentum invariants can be combined with the energy and Casimir invariants to prove nonlinear stability,[14] in what might then be called the energy-Casimir-momentum method. However, the existence of momentum invariants is fragile, in the sense that it is destroyed by slight changes to M; for a generic Riemannian manifold, the isometry group that generates the continuous symmetries consists solely of the identity.[18] This fragility of the momentum invariants is in sharp contrast to the energy and Casimir invariants, which are robust.

The results do not imply that there are no stable flows on M when it does not support symmetries, only that there are no flows that are provably stable using the Arnold stability method. For example, the vorticity distribution given by the gravest eigenfunction of Δ on M (which is generically nondegenerate in the absence of symmetries) always gives a nonlinearly stable flow, as can be proved using the energy-enstrophy argument of Fjørtoft.[19] When the gravest eigenspace is degenerate, one may be able to prove stability modulo continuous symmetries.[20] In addition, several extensions to Arnold's method are known in which the convexity hypothesis has been somehow relaxed—one of which has already been mentioned.[15] It would be interesting to determine whether our conclusion applies to the result of Wolansky and Ghil.[21] However, the formulation of this extension is technically involved (and for want of actual flow examples to which it applies).

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