Basics on set-theoretic properties of functions

We assume basic familiarity with sets and functions. The following lists some properties of sets and functions which are constantly used in Topology III. The reader should check the validity of these statements, none of which is very difficult.

Let A, B, C be sets. Then

1. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
2. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
3. $A - (B \cup C) = (A - B) \cap (A - C)$.
4. $A - (B \cap C) = (A - B) \cup (A - C)$.

Here we write $-$ for the set-theoretic difference, that is, $A - B = \{a \in A \,|\, a \notin B\}$. We do not need that $B \subset A$, but we get $(A - B) \subset A$. Properties (3) and (4) are called *DeMorgan's laws.*

We can also look at arbitrary unions and intersections. For this let I be a set and assume that for every $i \in I$ we have a set A_i. The union of all the sets A_i is written as

$$\bigcup_{i \in I} A_i = \{a \,|\, a \in A_i \text{ for some } i \in I\}$$

and the intersection of all the sets A_i is written as

$$\bigcap_{i \in I} A_i = \{a \,|\, a \in A_i \text{ for all } i \in I\}.$$

DeMorgan's laws also work for arbitrary unions and intersection, they then read as

$$\begin{aligned} A - \bigcup_{i \in I} A_i &= \bigcap_{i \in I} A - A_i. \\ A - \bigcap_{i \in I} A_i &= \bigcup_{i \in I} A - A_i. \end{aligned}$$

Let X, Y be sets, and $f\colon X \to Y$ a function. Let $A \subset X$. The *image* of A under f is denoted by $f(A)$, and is a subset of Y defined as

$$f(A) \quad = \quad \{y \in Y \,|\, y = f(a) \text{ for some } a \in A\}.$$

Let $B \subset Y$. The *pre-image*, or *inverse image*, of B under f is denoted by $f^{-1}(B)$, and is a subset of X defined as

$$f^{-1}(B) \quad = \quad \{x \in X \,|\, f(x) \in B\}.$$

Note that if $f\colon X \to Y$ is bijective, we denote the inverse function as $f^{-1}\colon Y \to X$. Then $f^{-1}(B)$ can be both image of f^{-1} or pre-image of f. However, both are the same in this instance, so there is no ambiguity.

The pre-image behaves much nicer with respect to set-theoretic operations. In the following, $f\colon X \to Y$ is a function, $B_0, B_1 \subset Y$.

1. $B_0 \subset B_1 \Rightarrow f^{-1}(B_0) \subset f^{-1}(B_1)$.
2. $f^{-1}(B_0 \cup B_1) = f^{-1}(B_0) \cup f^{-1}(B_1)$.
3. $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$.

4. $f^{-1}(B_0 - B_1) = f^{-1}(B_0) - f^{-1}(B_1)$.

Properties 2. and 3. also hold for arbitrary unions and intersections.
For the image, we get the following, where $A_0, A_1 \subset X$.

1. $A_0 \subset A_1 \Rightarrow f(A_0) \subset f(A_1)$.
2. $f(A_0 \cup A_1) = f(A_0) \cup f(A_1)$.
3. $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$.
4. $f(A_0 - A_1) \supset f(A_0) - f(A_1)$.

Again, properties 2. and 3. hold for arbitrary unions and intersections. Also, there exist examples where equality in 3. and 4. fail.
Let $A \subset X$ and $B \subset Y$. Then

1. $f^{-1}(f(A)) \supset A$, and equality holds if f is injective.
2. $f(f^{-1}(B)) \subset B$, and equality holds if f is surjective.

Let $f: X \to Y$ and $g: Y \to Z$ be functions, $A \subset X$ and $C \subset Z$. Then

1. $(g \circ f)(A) = g(f(A))$.
2. $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$.