Topology (Math 3281)

Solutions to Problem Set 3 21.11.14

1. Define $g : \mathbb{C}^3 \to \mathbb{C}$ by $g(z_1, z_2, z_3) = z_1^5 + z_2^2 + z_3^2$, which is continuous. Therefore $Y = g^{-1}(\{0\})$ is closed. Note that Y is unbounded. However, $S^5 \subset \mathbb{C}^3$ is bounded, and also closed. Therefore $X = Y \cap S^5$ is closed as an intersection of closed sets and also bounded. Since we can identify $\mathbb{C}^3$ with $\mathbb{R}^6$, the Heine-Borel Theorem applies and we can conclude that X is compact.

2. To check continuity, let $C \subset Y$ be closed. Then

$$\begin{aligned} f^{-1}(C) &= (f^{-1}(C) \cap A) \cup (f^{-1}(C) \cap B) \\ &= (f|_A)^{-1}(C) \cup (f|_B)^{-1}(C), \end{aligned}$$

where $f|_A$ and $f|_B$ are the restrictions to A and B, respectively. By continuity of $f|_A$ and $f|_B$ we get that $(f|_A)^{-1}(C)$ is closed in A and $(f|_B)^{-1}(C)$ is closed in B. As A and B are closed, the sets $(f|_A)^{-1}(C)$ and $(f|_B)^{-1}(C)$ are closed in X, and so $f^{-1}(C)$ is closed as the union of two closed sets. Therefore f is continuous.

3. Assume that Z is not connected. Then there exists a continuous surjective map $f : Z \to \{0, 1\}$. Now, as $Z \cap Y$ is connected, we get that $f(Z \cap Y)$ is only one point. Without loss of generality, we may assume the image is $\{0\}$. Now define $F : Z \cup Y \to \{0, 1\}$ by

$$F(x) = \begin{cases} 0 & x \in Y \\ f(x) & x \in Z \end{cases}$$

Notice that F is a well defined function, because if $x \in Y \cap Z$, then $f(x) = 0$, so both defining lines agree.

Now $F|_Y$ is continuous, and $F|_Z = f$ is continuous, so F is continuous by Question 2, as Z and Y are closed subsets of $Y \cup Z$. But if f is surjective, then so is F, contradicting the fact that $Z \cup Y$ is connected. Therefore Z has to be connected as well.

An entirely symmetrical argument shows that Y is connected.

4. (a) If $I \subset J$ are ideals, then $J \subset P$ implies $I \subset P$. So if $P \in Z(J)$, then $P \in Z(I)$ which means $Z(J) \subset Z(I)$.

(b) Recall that $I \cdot J$ are the finite sums of elements of the form $i \cdot j$ with $i \in I$ and $j \in J$. Hence $I \cdot J \subset I$ and $I \cdot J$, as I and J are ideals. By part (a) we get $Z(I) \subset Z(I \cdot J)$ and $Z(J) \subset Z(I \cdot J)$, that is, we have $Z(I) \cup Z(J) \subset Z(I \cdot J)$.

Now assume that $P \in Z(I \cdot J)$, but $P \notin Z(I)$. This means that there is an element $x \in I$ with $x \notin P$. As $I \cdot J \subset P$, we get that $x \cdot y \in P$ for all $y \in J$. By the prime ideal property of P, we have either $x \in P$ or $y \in P$. As $x \notin P$, this means $y \in P$ for all $y \in J$, or in other words, $J \subset P$. This means $P \in Z(J)$. Hence $Z(I \cdot J) \subset Z(I) \cup Z(J)$.

(c) Note that a prime ideal is never the full ring R, so $Z(R) = \emptyset$. Hence $Spec(R) = Spec(R) - Z(R) \in \tau$. Also, $Z(\{0\}) = Spec(R)$, so $\emptyset \in \tau$. To show that finite intersections of open sets are open, it is enough to show that finite unions of the $Z(I_1)$ are also of the form $Z(J)$. But this follows directly from part (b). Finally, if I_j is an ideal for all $j \in \mathfrak{J}$, we need to show that $\bigcap_{j\in\mathfrak{J}} Z(I_j) = Z(I)$ for some ideal I. For this, define

$$I = \left\{ \sum_{j\in\mathfrak{J}} x_j \;\middle|\; x_j \in I_j \text{ with only finitely many } x_j \neq 0 \right\},$$

which is easily seen to be an ideal, and $I_j \subset I$ for all $j \in \mathfrak{J}$. By part (a) we get

$$Z(I) \subset \bigcap_{j\in\mathfrak{J}} Z(I_j).$$

Now let $P \in \bigcap_{j\in\mathfrak{J}} Z(I_j)$, that is, $I_j \subset P$ for all $j \in \mathfrak{J}$. Then any finite sum of elements of the I_j is contained in P, which means that $I \subset P$, or $P \in Z(I)$. The proves the required other inclusion. Therefore arbitrary unions of open sets are open and τ is a topology.

(d) Note that the prime ideals of $\mathbb{Z}$ are ideals $p\mathbb{Z}$ with p a prime number, and $\{0\}$. Arbitrary ideals are of the form $n\mathbb{Z}$ with $n \in \mathbb{Z}$, and we can assume $n \geq 0$. As any such n is the product of finitely many prime numbers, $Z(n\mathbb{Z})$ only contains $p\mathbb{Z}$ if p divides n. In particular, $Z(n\mathbb{Z})$ is finite for $n \geq 1$. So if we are given an open covering of $Spec(\mathbb{Z})$, then one of these open sets $Spec(\mathbb{Z}) - Z(n\mathbb{Z})$ covers $2\mathbb{Z}$, and the complement $Z(n\mathbb{Z})$ only has finitely many points left. Hence $Spec(\mathbb{Z})$ is compact by the same argument that gave compactness in Example 5.2 (3).