Topology (Math 3281)

Homework Problem Set 1 10.10.14

This set of homeworks will be collected on Friday 24.10.14.

1. Let (M, d) be a metric space, and $A \subset M$. Define $d_A : A \times A \to [0, \infty)$ by $d_A(a, b) = d(a, b)$ for all $a, b \in A$.
   (a) Show that (A, d_A) is a metric space.
   (b) Show that the inclusion map $i : A \to M$ is continuous.

2. Let $\mathbb{R}^2$ be given the metrics

$$\begin{aligned} d_2(x, y) &= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \\ d_1(x, y) &= |x_1 - y_1| + |x_2 - y_2| \end{aligned}$$

   and let $B_i(x; r)$ the open ball around $x \in \mathbb{R}^2$ of radius $r > 0$ with respect to the metric d_i, $i = 1, 2$.

   Find functions $R_1, R_2 : (0, \infty) \to (0, \infty)$ such that

$$\begin{aligned} B_2(x; R_2(r)) &\subset B_1(x; r), \\ B_1(x; R_1(r)) &\subset B_2(x; r) \end{aligned}$$

   for all $x \in \mathbb{R}^2$ and $r > 0$.

   Use this to show that the identity on $\mathbb{R}^2$ is continuous both as

$$\mathrm{id}\colon (\mathbb{R}^2, d_1) \to (\mathbb{R}^2, d_2) \text{ and } \mathrm{id}\colon (\mathbb{R}^2, d_2) \to (\mathbb{R}^2, d_1).$$

3. Define $d : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ by

$$d(x, y) = \sqrt{|x_1 - y_1| + |x_2 - y_2|}.$$

   (a) Show that d is a metric on $\mathbb{R}^2$.
   (b) Show that $d(x, y) = d(x, z) + d(z, y)$ if and only if $z = x$ or $z = y$.