1. (a) By definition we have

\[
\delta^2 \varphi(x_0, \ldots, x_{p+2}) = \sum_{i=0}^{p+2} (-1)^i \delta \varphi(x_0, \ldots, \hat{x}_i, \ldots, x_{p+2})
\]

\[
= \sum_{i=0}^{p+2} (-1)^i \left( \sum_{j=0}^{i-1} (-1)^j \varphi(x_0, \ldots, \hat{x}_j, \ldots, \hat{x}_i, \ldots, x_{p+2}) + \sum_{j=i+1}^{p+2} (-1)^{j-1} \varphi(x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{p+2}) \right)
\]

Now notice that every summand \(\varphi(x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{p+2})\) with \(i < j\) appears exactly twice, but with opposite sign. Hence the sum is 0.

(b) We have

\[
\delta D \varphi(x_0, \ldots, x_p) = \sum_{i=0}^{p} (-1)^i \varphi(\bar{x}, x_0, \ldots, \hat{x}_i, \ldots, x_p)
\]

and

\[
D \delta \varphi(x_0, \ldots, x_p) = \varphi(x_0, \ldots, x_p) + \sum_{i=0}^{p} (-1)^{i+1} \varphi(\bar{x}, x_0, \ldots, \hat{x}_i, \ldots, x_p)
\]

so \(\delta D + D \delta = \text{id}: C^p(X; G) \to C^p(X; G)\) for \(p \geq 1\). So if \(\varphi \in C^p(X; G)\) is a cocycle with \(p \geq 1\), we get \(\varphi = \delta(D \varphi)\) which means it is also a coboundary. Therefore \(H^p(C^*(X; G)) = 0\) for \(p \geq 1\). Now if \(\varphi \in C^0(X; G)\) is a cocycle, then \(\varphi(x) = \varphi(\bar{x})\) for all \(x \in X\). This means that \(\varphi: X \to G\) is constant. As there are no coboundaries, we get \(H^0(C^*(X; G)) = G\) identifying \(g \in G\) with the cocycle given by the constant function \(\varphi(x) = g\).

(c) Let \(\varphi \in C^p(X; G)\) be locally zero and \((U_i)_{i \in I}\) the corresponding open cover of \(X\). Then \(\delta \varphi\) is locally zero using the same open
cover. If \( \psi \in C^p(X;G) \) is locally zero with respect to the open cover \((V_j)_{j \in J}\), form a third open cover \(W_{i,j} = U_i \cap V_j\). Then \( \varphi + \psi \) is locally zero with respect to this cover. Hence \( C^*_p(X;G) \) is a subcomplex.

(d) Note that \( C^p(\ast;G) \equiv G \) and the only locally zero function is the zero function. Hence \( \overline{C}(\ast;G) = C^p(\ast;G) \), and we can use the calculation from part (b) to get \( H^0(\ast;G) = G \) and \( H^p(\ast;G) = 0 \) for \( p \neq 0 \).

(e) Note that \( \overline{C}^0(X;G) = C^0(X;G) \) as only the zero function is locally zero. Let \( \varphi: X \to G \) represent a cocycle. This means that \( \delta \varphi \) is locally zero. Let \((U_i)_{i \in I}\) be the open cover such that \( \delta \varphi(x_0, x_1) = \varphi(x_1) - \varphi(x_0) = 0 \) for all \( x_0, x_1 \in U_i \), any \( i \in I \). This means that \( \varphi: X \to G \) is continuous when \( G \) is given the discrete topology. But if \( X \) is connected, then \( \varphi \) is constant. As \( \varphi \) could be any value, and there are no coboundaries, we get \( H^0(X;G) = G \).

(f) We need a space that is connected, but not path connected. Take the closure of \( X = \{(x, \sin(\pi/x)) | x \in (0,1)\} \) in \( \mathbb{R}^2 \). Then \( H^0(X;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \neq \mathbb{Z} = H^0(X;\mathbb{Z}) \).

(g) Any function \( f: X \to Y \) induces a cochain map \( f^*: C^p(Y;G) \to C^p(X;G) \) by

\[
f^*(\varphi)(x_0, \ldots, x_p) = \varphi(f(x_0), \ldots, f(x_p))
\]

which behaves well with composition and identities. And if \( f \) is continuous and \( \varphi \in C^p(Y;G) \) is locally zero with respect to the open cover \((U_i)_{i \in I}\) of \( Y \), then \((f^{-1}(U_i))_{i \in I}\) is an open cover of \( X \) showing that \( f^*(\varphi) \) is locally zero. Hence we get an induced cochain map \( f^*: C^p(Y;G) \to C^p(X;G) \), leading to the required homomorphisms on cohomology.

(h) Consider the cochain map \( i^*: \overline{C}^\ast(X;G) \to \overline{C}^\ast(A;G) \) induced by the inclusion \( i: A \to X \). This is surjective: let \( \varphi + C^0_p(A;G) \in \overline{C}^p(A;G) \), with \( \varphi: A^{p+1} \to G \). Extend \( \varphi \) somehow to \( \tilde{\varphi}: X^{p+1} \to G \) (e.g. using 0 on points not in \( A^{p+1} \)). Then \( i^*(\tilde{\varphi} + C^0_p(X;G)) = \varphi + C^0_p(A;G) \). Now let \( \overline{C}^\ast(X, A;G) = \ker i^* \) which gives rise to a short exact sequence of cochain complexes. By the Snake Lemma we get the required long exact sequence with \( \overline{H}^\ast(X, A;G) \) the cohomology of \( \overline{C}^\ast(X, A;G) \).
(i) For \( \varphi \in C^p(X; \mathbb{K}) \) and \( \psi \in C^q(X; \mathbb{K}) \), define \( \varphi \cup \psi \in C^{p+q}(X; \mathbb{K}) \) by

\[
\varphi \cup \psi(x_0, \ldots, x_{p+q}) = \varphi(x_0, \ldots, x_p) \cdot \psi(x_p, \ldots, x_{p+q}).
\]

We get \( \delta(\varphi \cup \psi) = \delta \varphi \cup \psi + (-1)^q \varphi \cup \delta \psi \) by a calculation as for the cup-product in singular cohomology.

Now assume that \( \varphi \) is locally zero with respect to the open cover \( (U_i)_{i \in I} \). Then \( \varphi \cup \psi \) is also locally zero with respect to this cover. It follows that the cup-product induces a product \( \cup : H^p(X; \mathbb{K}) \times H^q(X; \mathbb{K}) \to H^{p+q}(X; \mathbb{K}) \) which turns cohomology into a ring.

The identity is defined by using \( 1 \in C^0(X; \mathbb{K}) \) given by \( 1(x) = 1_\mathbb{K} \) for all \( x \in X \).