This set of problems will be discussed in the Problem Class on 05.02.15.

1. Let $X$ be a topological space and $G$ an abelian group. For $p \geq 0$, let $C^p(X; G)$ be the abelian group of functions $\varphi: X^{p+1} \to G$ and define $\delta: C^p(X; G) \to C^{p+1}(X; G)$ by

$$\delta(\varphi)(x_0, \ldots, x_{p+1}) = \sum_{i=0}^{p+1} (-1)^i \varphi(x_0, \ldots, \hat{x}_i, \ldots, x_{p+1})$$

where the hat indicates that $x_i$ is omitted.

(a) Show that $\delta^2 = 0$.

(b) Assume that $X$ is nonempty and choose $\bar{x} \in X$. Define $D: C^p(X; G) \to C^{p-1}(X; G)$ for $p \geq 1$ by

$$D(\varphi)(x_0, \ldots, x_{p-1}) = \varphi(\bar{x}, x_0, \ldots, x_{p-1}).$$

Calculate $\delta D + D \delta$, and use this to show that $H^0(C^\ast(X; G)) = G$ and $H^p(C^\ast(X; G)) = 0$ for $p \neq 0$.

(c) An element $\varphi \in C^p(X; G)$ is called locally zero, if there is an open covering $(U_i)_{i \in I}$ of $X$ such that $\varphi$ vanishes on any $(p+1)$-tuple coming from $U_{i}^{p+1}$. Show that the locally zero functions form a subcomplex $C^\ast_0(X; G)$ of the cochain complex $C^\ast(X; G)$.

(d) Let $\bar{C}^\ast(X; G) = C^\ast(X; G)/C^\ast_0(X; G)$ denote the quotient complex and $\bar{H}^\ast(X; G)$ the cohomology of this cochain complex. Calculate $\bar{H}^p(\ast; G)$ for all $p \geq 0$ for a point $\ast$.

(e) Assume $X$ is connected, show that $\bar{H}^0(X; G) \cong G$.

(f) Give an example of a space $X$ for which $H^\ast(X; \mathbb{Z}) \neq \bar{H}^\ast(X; \mathbb{Z})$.

(g) Show that continuous maps $f: X \to Y$ induce maps on cohomology groups $f^*: H^\ast(Y; G) \to H^\ast(X; G)$ which behave well with composition and identities.

(h) Define cohomology groups $\bar{H}^\ast(X, A; G)$ for pairs of spaces $(X, A)$ such that there is a long exact sequence for the pair.

(i) Let $\mathbb{K}$ be a commutative ring with identity. For $\varphi \in C^p(X; \mathbb{K})$ and $\psi \in C^q(X; \mathbb{K})$ define a cup-product $\varphi \cup \psi \in C^{p+q}(X; \mathbb{K})$ which turns $\bar{H}^\ast(X; \mathbb{K})$ into a ring with identity.
The groups $\tilde{H}^*(X; G)$ are called the \textit{Alexander-Spanier cohomology} of $X$, and they satisfy the Eilenberg-Steenrod axioms for a cohomology theory. Excision is easier than for singular cohomology, but Homotopy Invariance is harder. These groups agree with singular cohomology on CW complexes, but not necessarily on more general spaces (as witnessed by part (f)).