1. (a) Addition is defined as in Hom case, scalar multiplication is defined as \((\lambda \cdot f)(v) = \lambda f(v)\). Checking that it is indeed a vector space is standard.

(b) Given a basis of \(V\) and one of \(W\), every linear map is determined by the values on the basis of \(V\), and the coefficients in the basis of \(W\). That is, it’s determined by an \((m \times n)\)-matrix. Note this is just Linear Algebra.

(c) We can write \(F = \bigoplus_{i \in I} \mathbb{Z}\) and \(V = \bigoplus_{i \in I} \mathbb{F}\). As in the case of \(\text{Hom}\), we get

\[
\text{Hom}_\mathbb{F}\left(\bigoplus_{i \in I} \mathbb{F}, W\right) \cong \prod_{i \in I} \text{Hom}_\mathbb{F}(\mathbb{F}, W)
\]

and \(\text{Hom}_\mathbb{F}(\mathbb{F}, W) \cong W\). Also, \(\text{Hom}(G, W)\) can be given a vector space structure for any abelian group \(G\) (as in the case \(W = \mathbb{F}\)), and for \(G = \mathbb{Z}\) this agrees with \(W\). The isomorphisms between direct sum and direct product decompositions work the same over \(\mathbb{Z}\) and \(\mathbb{F}\), so we have this isomorphism.

(d) As vector spaces, both \(C^*_\mathbb{F}(X)\) and \(C^*(X; \mathbb{F})\) are isomorphic by (c), and the coboundary formula is the same in both cases. Therefore we have an isomorphism on the level of cochain complexes which induces an isomorphism on cohomology.

(e) An abelian group homomorphism \(\varphi: \mathbb{Q} \to \mathbb{Q}\) is determined by \(\varphi(1)\). To see this first note that if \(q > 0\) is an integer, then

\[
\varphi(1) = \varphi(1/q) + \cdots + \varphi(1/q)
\]

where we have \(q\) summands. This determines \(\varphi(1/q)\). Similarly,

\[
\varphi(p/q) = \varepsilon \varphi(1/q) + \cdots + \varphi(1/q)
\]

where we have \(|p|\) summands and \(\varepsilon \in \{\pm 1\}\) is the sign of \(p\). Finally, \(\varphi(1)\) can be any rational number and we get a homomorphism. Therefore \(\text{Hom}(\mathbb{Q}, \mathbb{Q}) \cong \mathbb{Q}\), and this also follows for \(\text{Hom}_\mathbb{Q}(\mathbb{Q}, \mathbb{Q})\) by (b).
(f) We still have $\text{Hom}_C(C, C) \cong C$, a 1-dimensional $C$ vector space. But $\text{Hom}(C, C)$ is much bigger. Note that $\bar{\varphi}: C \to C$ given by $\bar{\varphi}(z) = \bar{z}$ sending $z$ to its complex conjugate, and this homomorphism is linearly independent of the identity homomorphism. In fact, one can think of $C$ as a $\mathbb{Q}$-vector space with an uncountable basis, $C \cong \bigoplus_{i \in I} \mathbb{Q}$, and then $\text{Hom}(C, C) \cong \text{Hom}(\bigoplus_{i \in I} \mathbb{Q}, C) \cong \prod_{i \in I} \text{Hom}(\mathbb{Q}, C) \cong \prod_{i \in I} C$ is a $C$-vector space with an uncountable basis. This also happens for $\mathbb{R}$, or algebraic fields between $\mathbb{Q}$ and $\mathbb{R}$ (except that one can get finite dimensional $\mathbb{Q}$-vector spaces for some algebraic field extensions).

2. (a) We have the short exact sequence $0 \to \mathbb{Z}_n \to C_n \to B_{n-1} \to 0$. As $B_{n-1}$ is a vector space, and therefore has a basis, we can split the sequence to get $C_n \cong \mathbb{Z}_n \oplus B_{n-1}$. Similarly, we have the short exact sequence $0 \to B_n \to \mathbb{Z}_n \to H_n(C) \to 0$ which splits as $H_n(C)$ has a basis as a vector space.

(b) We first note that this is well defined: If $\varphi = \delta \psi$, then $\varphi(\sigma) = \psi(\partial \sigma) = 0$ as $\sigma$ is a cycle. Similarly, if $\sigma$ is a boundary, then the cocycle $\varphi$ vanishes on it. Therefore we get the function $\Phi$, which is clearly linear.

(c) We need to show that $\Phi$ is injective and surjective. To see that $\Phi$ is injective, let $\varphi$ be a cocycle with $\varphi(\sigma) = 0$ for all cycles $\sigma$. We need that $\varphi$ is a coboundary. By part (a) we have $C_{n-1} \cong H_{n-1}(C) \oplus B_{n-1} \oplus B_{n-2}$. Also, $C_n \cong \mathbb{Z}_n \oplus B_{n-1}$. Note that $\varphi$ vanishes on $\mathbb{Z}_n$. Define $\psi: C_{n-1} \to F$ to vanish on $H_{n-1}(C) \oplus B_{n-2}$, and on $B_{n-1}$ define it to be $\varphi$. Then $\delta(\psi) = \varphi$, because $\partial: C_n \cong \mathbb{Z}_n \oplus B_{n-1} \to C_{n-1}$, vanishes on $\mathbb{Z}_n$, and is inclusion on $B_{n-1}$. Therefore $\varphi$ is a coboundary.

To see that $\Phi$ is surjective, let $\psi: H_n(C) \to F$ be linear. This induces a linear map $\bar{\psi}: \mathbb{Z}_n \to F$ which vanishes on $B_n \subset \mathbb{Z}_n$. As $C_n \cong \mathbb{Z}_n \oplus B_{n-1}$ we can extend this to a linear map $\bar{\varphi}: C_n \to F$. Since $\varphi$ vanishes on $B_n \subset \mathbb{Z}_n$, this is a cocycle. Now $\Phi([\varphi] = \bar{\psi}$ by construction.

3. Let $\varphi \in C^0(X, A; \mathbb{Z})$ be a cocycle. As $\varphi: C_0(X, A) \to \mathbb{Z}$ is a homomorphism, it is determined by its values on points of $X$ and it vanishes on the points of $A$. The cocycle condition means that $\varphi$ has the same
value for points in the same path component. Cocycles are therefore functions \( \varphi : \pi(X, A) \to \mathbb{Z} \), where \( \pi(X, A) \) denotes the set of path components of \( X \) which do not have points of \( A \). As there are no coboundaries apart from 0, we therefore get

\[
H^0(X, A; \mathbb{Z}) \cong \prod_{x \in \pi(X, A)} \mathbb{Z}.
\]

For \( A = \emptyset \), we just get the product over the path components of \( X \). In order to get \( H^0 \not\cong H_0 \) we need a space with infinitely many path components, to get the direct sum different from the direct product. So, for example, \( X = \mathbb{Z} \) will do.