1. We know that $H^i(L(p,q); \mathbb{Z}/p) \cong \mathbb{Z}/p$ for $i = 0, 1, 2, 3$ and 0 else, and we get the same for homology with $\mathbb{Z}/p$ coefficients.

Denote by $x \in H^1(L(p,q); \mathbb{Z}/p)$ a generator. Then $x \cup x = -x \cup x$, and as 2 does not divide $p$, we get that this is 0. Now denote $y \in H^2(L(p,q); \mathbb{Z}/p)$ also a generator (note that if $p$ is prime, then every non-zero element generates $\mathbb{Z}/p$, but if $p$ is not prime, this is not the case). By Poincaré duality, we have that $[L(p,q)] \cap x \in H_2(L(p,q); \mathbb{Z}/p)$ is a generator, and therefore

$$\langle [L(p,q)], x \cup y \rangle = \langle [L(p,q)] \cap x, y \rangle$$

is a generator of $\mathbb{Z}/p$. This implies that $x \cup y \in H^3(L(p,q); \mathbb{Z}/p)$ is a generator.

2. Since $H_4(\mathbb{C}P^2)$ is generated by $[\mathbb{C}P^2]$, we have $f_*[\mathbb{C}P^2] = k[\mathbb{C}P^2]$ for some $k \in \mathbb{Z}$. Let $x \in H^2(\mathbb{C}P^2; \mathbb{Z})$ be a generator, so that $x^2 = x \cup x$ is a generator of $H^4(\mathbb{C}P^2; \mathbb{Z})$. We can assume that $\langle [\mathbb{C}P^2], x^2 \rangle = 1 \in \mathbb{Z}$, for otherwise we can replace $[\mathbb{C}P^2]$ with its negative. Now

$$f^*(x) = n \cdot x$$

for some $n \in \mathbb{Z}$, again as $x$ is a generator of $H^2(\mathbb{C}P^2; \mathbb{Z})$. Therefore

$$k = \langle k \cdot [\mathbb{C}P^2], x^2 \rangle$$

$$= \langle f_*([\mathbb{C}P^2]), x^2 \rangle$$

$$= \langle [\mathbb{C}P^2], f^*(x \cup x) \rangle$$

$$= \langle [\mathbb{C}P^2], f^*x \cup f^*x \rangle$$

$$= \langle [\mathbb{C}P^2], n \cdot x \cup n \cdot x \rangle$$

$$= n^2 \langle [\mathbb{C}P^2], x^2 \rangle$$

$$= n^2$$

which is what we had to show.

3. Bilinearity follows from the properties of the cup-product and of $\langle \cdot, \cdot \rangle$. Symmetry follows because $x \cup y = y \cup x$ as $x$ and $y$ have even degree. To
see non-degeneracy, let $0 \neq x$. As $H^{2k}(M; \mathbb{Q}) \cong \text{Hom}(H_{2k}(M), \mathbb{Q})$, we get that $px \in H^{2k}(M; \mathbb{Z})$ for some $p \in \mathbb{Q} - \{0\}$. We might as well assume $p = 1$ will work. We need to find $y \in H^{2k}(M; \mathbb{Q})$ with $s(x, y) \neq 0$. Let $X \in H_{2k}(M; \mathbb{Z})$ the Poincaré dual of $x$, that is, the element with $X = [M] \cap x$. By Poincaré duality, $X \neq 0$, and we also have $nX \neq 0$ for all $n > 0$ as otherwise $nx = 0$ which is not possible as $x \in H^{2k}(M; \mathbb{Q})$ is the element of a vector space over $\mathbb{Q}$. Let $y \in H^{2k}(M; \mathbb{Q})$ satisfy $y(X) \neq 0$. Then

\[
\langle [M], (x \cup y) \rangle = \langle [M] \cap x, y \rangle = \langle X, y \rangle
\]

and the last line is non-zero, as it is basically evaluation of $y$ on the homology class $X$ under the identification $H^{2k}(M; \mathbb{Q}) \cong \text{Hom}(H_{2k}(M), \mathbb{Q})$.

4. Let us first assume that $\partial M$ is connected. As $M$ is compact, connected and orientable with non-empty boundary, we have $H_n(M) \cong 0$, and $H_n(M, \partial M) \cong \mathbb{Z}$. By the long exact sequence we get an injective map $H_n(M, \partial M) \cong \mathbb{Z} \to H_{n-1}(\partial M)$ which means that $H_{n-1}(\partial M)$ cannot be 0. Therefore $\partial M$ is orientable by Theorem 4.7.

If $\partial M$ is not connected, there are finitely many components $M_1, \ldots, M_k$. Now $\partial_*([M]) \in H_{n-1}(\partial M)$ is represented by

\[
\sum \varepsilon_{1r} \sigma_{1r} + \cdots + \sum \varepsilon_{kr} \sigma_{kr}
\]

where $\sigma_{ir}$ are exactly the $n - 1$-simplices in $M_i$ for all $i = 1, \ldots, k$, and the $\varepsilon_{ir} \in \{\pm 1\}$ are appropriate coefficients. Since the $M_i$ are all disjoint, $\partial(\sum \varepsilon_{1r} \sigma_{1r})$ and $\partial(\sum \varepsilon_{jr} \sigma_{jr})$ have no common $n - 2$ simplices for $i \neq j$. As

\[
\partial(\sum \varepsilon_{1r} \sigma_{1r} + \cdots + \sum \varepsilon_{kr} \sigma_{kr}) = \partial \partial(\sum \varepsilon_{r} \tau_{r}) = 0
\]

where $\tau_j$ are all the $n$-simplices in $M$, we get

\[
\partial(\sum \varepsilon_{ir} \sigma_{ir}) = 0
\]

so each $M_i$ has a fundamental class, which means that each $M_i$ is orientable. Notice that $H_{n-1}(\partial M) \cong \mathbb{Z}^k$ and $\partial_*([M])$ represents the diagonal element $(1, \ldots, 1) \in \mathbb{Z}^k$. 
5. Assume there is a retraction \( r : M \to \partial M \), that is, a map with \( r \circ i = \text{id}_{\partial M} \). Then \( i_* : H_{n-1}(\partial M; \mathbb{Z}/2) \to H_{n-1}(M; \mathbb{Z}/2) \) is injective. But the long exact sequence looks like

\[
0 \to H_n(M, \partial M; \mathbb{Z}/2) \to H_{n-1}(\partial M; \mathbb{Z}/2) \to H_{n-1}(M; \mathbb{Z}/2)
\]

and since \( H_n(M, \partial M; \mathbb{Z}/2) \neq 0 \) by Theorem 4.7, \( i_* \) cannot be injective, contradiction. Notice that we only use \( \mathbb{Z}/2 \) coefficients because we do not assume \( M \) to be orientable. If \( M \) is orientable, we can also argue with \( \mathbb{Z} \) coefficients.

6. Let \( r : X \to A \) be the retraction, and let \( f : A \to A \). Consider \( i \circ f \circ r : X \to X \), where \( i \) is inclusion. Since \( X \) has the FPP, there is a \( x \in X \) with \( i \circ f \circ r(x) = x \). Since \( i \) is just inclusion, this means \( f(r(x)) = x \) and so \( x \in A \). But if \( x \in A \), we get \( r(x) = x \) and \( x = f(r(x)) = f(x) \), which means that \( x \) is a fixed point of \( f \). It follows that every \( f : A \to A \) has a fixed point, which means \( A \) has the FPP.

7. We have inclusions \( i_X : X \to X \vee Y \), \( i_Y : Y \to X \vee Y \) and retractions \( r_X : X \vee Y \to X \), \( r_Y : X \vee Y \to Y \).

Assume that \( X \vee Y \) has the FPP. As \( X \) and \( Y \) are retracts of \( X \vee Y \), we get that both \( X \) and \( Y \) have the FPP.

So assume that both \( X \) and \( Y \) have the FPP. Let \( f : X \vee Y \to X \vee Y \) be a map. Write \( * = [x_0] = [y_0] \in X \vee Y \) for the common base point. Then \( f(*) \in X \) or \( f(*) \in Y \). Assume that \( f(*) \in X \). Then consider \( f_X : X \to X \) given by \( f_X = r_X : f \circ i \). As \( X \) has the FPP, we get that there is a \( x \in X \) with \( f_X(x) = x \). If \( f(x) \in X \vee Y - X \), then \( r_X(f(x)) = * \), so for this to be the fixed point we would need \( x = * \). But note that \( f(*) \in X \). Therefore \( f(x) \in X \) which implies \( f_X(x) = f(x) \). As \( x \) is a fixed point of \( f_X \), we now get that \( x \) is a fixed point of \( f \). If \( f(*) \in Y \), we can use the same argument using the FPP of \( Y \). Either way, we get that \( f \) has a fixed point, so \( X \vee Y \) has the FPP.